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A New Analytic Approximation of Luminosity Distance in Cosmology Using the Parker–Sochacki Method

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Abstract: The luminosity distance d_L is possibly the most important distance scale in cosmology and therefore accurate and efficient methods for its computation is paramount in modern precision cosmology. Yet in most cosmological models the luminosity distance cannot be expressed by a simple analytic function in terms of the redshift z and the cosmological parameters, and is instead represented in terms of an integral. Although one can revert to numerical integration techniques utilizing quadrature algorithms to evaluate such an integral, the high accuracy required in modern cosmology makes this a computationally demanding process. In this paper, we use the Parker–Sochacki method (PSM) to generate a series approximate solution for the luminosity distance in spatially flat Λ CDM cosmology by solving a polynomial system of nonlinear differential equations. When compared with other techniques proposed recently, which are mainly based on the Padé approximant, the expression for the luminosity distance obtained via the PSM leads to a significant improvement in the accuracy in the redshift range $0 \leq z \leq 2.5$. Moreover, we show that this technique can be easily applied to other more complicated cosmological models, and its multistage approach can be used to generate analytic approximations that are valid on a wider redshift range.

Keywords: FLRW cosmology; luminosity distance; Parker–Sochacki method



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1. Introduction

The luminosity distance in the spatially flat Λ CDM cosmology is expressed exactly in terms of an elliptic integral. Evaluation of this integral via numerical quadratures tend to be computationally heavy when high accuracy is required in applications that involve ultra precise measurements of cosmological parameters such as the calculation of H_0 in view of the problem of the Hubble constant tension [1,2]. Different strategies for obtaining analytical approximations for this integral have been proposed during the last two decades, and these can be divided into three types. The first type involves obtaining an algebraic fitting formula expressed in terms of elementary functions for the integral which compares quite well with the numerical solution over a wide redshift range and for different values of the matter density parameter Ω_m . This started with the work of Ue Li Pen [3] who obtained a formula which has a relative error of less than 0.4% for $0.2 < \Omega_m < 1$. A similar approach was used by Wickramasinghe and Ukwatta [4] and Liu et al. [5] who obtained alternative fitting formulae that run faster and have smaller relative errors than that obtained by Pen [3]. Finally, the formula obtained by Adachi Kasai [6] which is based on the Padé approximation has even smaller relative error for the range of density parameter $0.3 \leq \Omega_m \leq 1$ and redshift range $0.03 \leq z \leq 1000$. The second strategy is to identify exact analytic expressions for the luminosity distance which are often expressed in terms of transcendental functions, such as elliptic integrals [5,7–9] or hypergeometric functions [10]. Being exact expressions, this approach leads to a better accuracy, but the presence of such transcendental functions makes these more computationally demanding to evaluate. The third strategy involves using truncated Taylor series approximations [11–13] for the luminosity distance in a relatively small redshift range. This technique is easy to use but

has divergence problems for higher values of z , and the fact that nowadays supernova data goes as far as $z \approx 2.35$ [14] this can pose a significant problem. Thus several recent works suggest the use of the Padé approximant [15–17] to represent the luminosity distance, due to its better convergence over a larger redshift range. Instead of obtaining power series or Padé approximations to the luminosity integral directly, some authors have also considered the associated differential equation with corresponding initial conditions and obtained approximate analytical solutions for the luminosity distance using various methods, such as the homotopy perturbation method (HPM) [18] and its variants [19], the variational iteration method (VIM) [20] and the Daftardar-Jafari Method (DJM) [21]. Unlike the fitting formulae developed earlier, which are valid over a wide redshift range, the series approximations obtained using this strategy converge on a much limited redshift range and the relative error for the approximated luminosity can be larger for higher values of z meaning that one has to reach a compromise between accuracy and redshift convergence interval, although the accuracy and convergence can be somewhat improved by using a combination of the Padé approximant with the HPM [19]. However, these methods have the advantage that they can be easily applied to any cosmological model, unlike the earlier fitting formulae or exact expressions which only hold for Λ CDM cosmology. This is particularly important when comparing different dark energy cosmological models with observational data.

In this paper, we follow the latter strategy and apply the Parker–Sochacki method (PSM) [22,23] to obtain an approximate solution of the ordinary differential equation satisfied by the luminosity distance function. PSM is an extension of the Picard iteration [24] which is in turn an algorithm for solving linear systems of differential equations. It is a simple yet powerful technique for computing iteratively the coefficients of the Maclaurin’s series solution of a polynomial system derived from a nonlinear initial value problem (IVP) or a boundary value problem (BVP). Over the last two decades, interest in PSM has increased dramatically and it has been applied to various problems of technological and scientific importance (see for examples Refs. [25–37]) including stiff differential equations whose solution cannot be obtained efficiently with existing numerical solvers [38]. This would be the first example of the application of PSM in cosmology. The main advantage of the PSM over the other methods mentioned above, is that the radius of convergence R of the Maclaurin’s series solution to the polynomial system can be obtained easily and a priori in terms of the parameters in the system. This would allow the domain of the problem to be divided into subdomains over which the PSM can be used individually to obtain convergent solutions, such that the end result is a piecewise convergent solution. This allows one to extend the domain of convergence of the approximate analytic solution without compromising the accuracy of the solution. It has been shown [39] that such an approach compares relatively well with the Runge–Kutta method of order four.

This paper is structured as follows. In the next section, we introduce the Parker–Sochacki method. Then in Section 3 we obtain the differential equation satisfied by the luminosity distance function in Λ CDM cosmology and generate an approximate solution for $d_L(z)$ using the PSM. This will be followed by a comparison of this solution with the corresponding numerical solution and with the approximate solutions obtained by using the other methods mentioned above. The PSM is also used to obtain the luminosity distance in a dynamical dark energy model, which is chosen to be the Chevallier, Polarski and Linder (CPL) model [40–42] with the equation of state (EOS) function $\omega(z) = \omega_0 + \omega_1 z / (1 + z)$, where ω_0 and ω_1 are constants. This is followed with a discussion of the results and a conclusion. In this paper, we will use geometrized units such that $8\pi G = c = 1$.

2. The Parker–Sochacki Method

The Parker–Sochacki method is a modification of the conventional power series method for solving non-linear ordinary differential equations which may include transcendental functions. This involves expressing the given differential equation into a polynomial system of differential equations (also called polynomial projection) by introducing appro-

appropriate new variables to represent the non-polynomial functions in the original differential equation. The system is then solved using a standard power series method. The concept of polynomial projection may be better explained by referring to a simple example (taken from Ref. [43]). So consider the following first order nonlinear IVP

$$y'(t) = \frac{\sin(y(t)e^{-t^2})}{\sqrt{t}}, \quad y(1) = 2. \tag{1}$$

This can be expressed in a polynomial system of differential equations by introducing the variables

$$x_1 = y, \quad x_2 = \sin(ye^{-t^2}), \quad x_3 = \cos(ye^{-t^2}), \quad x_4 = e^{-t^2}, \quad x_5 = t, \quad x_6 = t^{-1/2}, \tag{2}$$

such that (1) is recast in the form

$$\begin{aligned} x_1' &= x_2x_6, \\ x_2' &= -2x_1x_3x_4x_5 + x_2x_3x_4x_6, \\ x_3' &= 2x_1x_2x_4x_5 - x_2^2x_4x_6, \\ x_4' &= -2x_4x_5, \\ x_5' &= 1, \\ x_6' &= -\frac{1}{2}x_6^3, \end{aligned} \tag{3}$$

with the initial conditions $x_1(1) = 2, x_2(1) = \sin(2e^{-1}), x_3(1) = \cos(2e^{-1}), x_4(1) = e^{-1}, x_5(1) = 1, x_6(1) = 1$. So the solution $y(t)$ to the given differential equation in (1) is now embedded in the polynomial system (3). This is called a projection for the solution $y(t)$. Now by expressing each of the six variables $x_i, i = 1 \dots 6$ in terms of a truncated Maclaurin series in t and substituting in (3), one can apply the simple Picard iteration to solve the polynomial system and so get an approximate series solution for $y(t)$. In general, given a nonlinear IVP of the form

$$\mathbf{y}' = f(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \leq t \leq t_f, \tag{4}$$

where $t_0 \in \mathbb{R}$ and $\mathbf{y}_0 \in \mathbb{R}^n$, one can transform this to a projectively polynomial system by the introduction of auxiliary variables as seen in the above example, to get

$$\mathbf{x}' = \mathbf{g}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t_0 \leq t \leq t_f, \tag{5}$$

where $\mathbf{x}_0 \in \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^m \rightarrow \mathbb{R}^m$. Taking $t_0 = 0$ without loss of generality and using the truncated Maclaurin's expansions $\mathbf{x}(t) = \sum_{i=0}^p \mathbf{x}_i t^i$ and $\mathbf{g}(\mathbf{x}) = \sum_{i=0}^p \mathbf{g}_i t^i$ and substituting in the above polynomial system, gives

$$\mathbf{x}_{i+1} = \frac{\mathbf{g}_i}{i+1}, \tag{6}$$

where the coefficients \mathbf{g}_i are obtained by the repeated use of the simple Cauchy product. By using (6), one can then obtain the truncated series solution

$$\mathbf{x}(t) = \sum_{i=0}^p \mathbf{x}_i t^i, \tag{7}$$

of the polynomial system in (18). We now introduce a few notations. The definition of the norm of the vector \mathbf{x} is $\|\mathbf{x}\| = \max_{1 \leq i \leq m} |x_i|$, where $\mathbf{x} \in \mathbb{R}^m$. If \mathbf{g} is a vector polynomial function and $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}$, then the components of \mathbf{g} are given by

$$\mathbf{g}_i(x_1, \dots, x_m) = \sum_{|\alpha| \leq k} a_{\alpha,i} X^\alpha, \tag{8}$$

where $|\alpha| = \alpha_1 + \dots + \alpha_m$, and $k = \text{deg}(g_i)$ is the degree of g_i . We also define

$$\Sigma g_i = \sum_{|\alpha| \leq k} |a_{\alpha,i}|, \tag{9}$$

and $\Sigma \mathbf{g} = \max(\Sigma g_1, \dots, \Sigma g_m)$, $\text{deg}(\mathbf{g}) = \max(\text{deg}(g_1), \dots, \text{deg}(g_m))$. Then the convergence for the above series solution in (7) obtained by the PSM is given by the following Taylor approximation theorem [44].

Theorem 1. *If \mathbf{x} satisfies (5), $k = \text{deg}(\mathbf{g}) \geq 2$, $\alpha = \max(1, \|x_0\|)$, $M = (k - 1)\Sigma \mathbf{g}\alpha^{k-1}$, $t_0 = 0$, and $|t| < 1/M$ then*

$$\|\mathbf{x}(t) - \sum_{i=0}^p \mathbf{x}_i t^i\| \leq \frac{\alpha |Mt|^{p+1}}{1 - |Mt|}. \tag{10}$$

In most cases the radius of convergence $|t| < 1/M$ of the series solution as stated in the above theorem is less than the interval of integration $t_0 \leq t \leq t_f$. Therefore as proposed in [44] one can apply the PSM on smaller subintervals such that the convergence of the solution is preserved on each subinterval. In other words, the interval $t_0 \leq t \leq t_f$ is divided into subintervals by introducing the nodes $t_i = t_0 + ih$ where $h = 1/M$ and $i = 0, 1, \dots, N = \frac{t_f - t_0}{h}$, such that the union of the non-overlapping sub intervals $[t_{k-1}, t_k]$ gives $\cup [t_{k-1}, t_k] = [t_0, t_f]$. Then, when applying the PSM on each subinterval the value of the function at the end-point of a given subinterval becomes the initial condition for the next subinterval. Combining the solutions obtained by the PSM for each subinterval yields a highly accurate piecewise smooth solution for the entire integration domain $t_0 \leq t \leq t_f$. Moreover, by increasing the order of the approximation p in (7) one can also increase the accuracy of the approximation in each subinterval. This multistage approach for the implementation of the PSM has been used to obtain accurate series approximations of various non-linear and stiff ordinary differential equations (see for example Ref. [38]).

In the next section, we obtain the non-linear differential equation satisfied by the luminosity function and solve this for two different cosmological models by using the PSM via the multistage approach as described above.

3. Differential Equation for the Luminosity Distance in a Flat Universe

According to recent observations our universe is spatially flat [45,46] and is homogeneous and isotropic on the large scale, such that it can be described by the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{11}$$

where $a(t)$ is the scale factor. One of the most fundamental distance scales in cosmology is the luminosity distance which is defined by $d_L = \sqrt{L/(4\pi F)}$, where L is the absolute luminosity of the source and F is the associated flux measured by the observer. In a spatially flat universe this is given by [47]

$$d_L = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{\sqrt{W(z')}}}, \tag{12}$$

where $W(z) = (H(z)/H_0)^2$ depends on the Hubble parameter $H(z)$ as a function of the redshift z and H_0 is the current value of the Hubble constant such that $W(0) = 1$. Even in the simple case of the Λ CDM universe the above integral cannot be integrated explicitly and normally one has to revert to numerical integration or use any of the approximations mentioned in the Introduction. Introducing the function

$$u(z) = \frac{H_0 d_L}{c(1+z)}, \tag{13}$$

the above integral can be written as

$$u(z) = \int_0^z \frac{dz'}{\sqrt{W(z')}}. \tag{14}$$

The function $u(z)$ then satisfies the non-linear differential equation

$$u''(z) + \frac{1}{2}W'(z)(u'(z))^3 = 0; \quad u(0) = 0, \quad u'(0) = 1, \tag{15}$$

where $'$ denotes differentiation with respect to z . In the next section, we obtain analytic approximations to this differential equation by using the multistage PSM described in the previous section for two cosmological models, namely the Λ CDM and the Chevallier–Polarski–Linder (CPL) models.

4. Luminosity Distance in Cosmology

4.1. Λ CDM Cosmological Model

For the Λ CDM model the function $W(z)$ in the non-linear differential Equation (15) is given in terms of the cosmological parameter Ω_m by

$$W(z) = \Omega_m(1+z)^3 + (1-\Omega_m). \tag{16}$$

By introducing the additional auxiliary variables

$$V(z) = u'(z), \quad S(z) = W'(z) = 3\Omega_m(1+z)^2, \quad Q(z) = (1+z), \tag{17}$$

the differential equation in (15) can be incorporated in the following polynomial system which will be solved using the PSM as explained earlier

$$\begin{aligned} u' &= V \\ V' &= -\frac{1}{2}SV^3 \\ S' &= 6\Omega_m Q \\ Q' &= 1, \end{aligned} \tag{18}$$

where $u(0) = 0, V(0) = 1, S(0) = 3\Omega_m, Q(0) = 1$. Note that the required function $u(z)$ which is related to the luminosity distance function $d_L(z)$ by (13) is contained in the above polynomial system. Using the series expansions for the auxiliary variables

$$u(z) = \sum_{i=0}^p u_i z^i, \quad V(z) = \sum_{i=0}^p V_i z^i, \quad S(z) = \sum_{i=0}^p S_i z^i, \quad Q(z) = \sum_{i=0}^p Q_i z^i, \tag{19}$$

and substituting in the above polynomial system, leads to the following recursive relations for the coefficients

$$\begin{aligned} u_{i+1} &= \frac{V_i}{i+1} \\ V_{i+1} &= -\frac{1}{2(i+1)} \sum_{j=0}^i S_{i-j} \sum_{p=0}^j \left(\sum_{k=0}^p V_k V_{p-k} \right) V_{j-p} \\ S_{i+1} &= 6\Omega_m \frac{Q_i}{i+1} \\ Q_i &= \begin{cases} 0 & : i > 1 \\ 1 & : i = 1 \end{cases}. \end{aligned} \tag{20}$$

According to Theorem 1 the interval of convergence for the above series solution to the polynomial system (18) is given by $|z| < 1/M$ where $M = (k-1)\Sigma g a^{k-1}$. In our

case $k = 4$, $\Sigma \mathbf{g} = 6\Omega_m$, and $\alpha = 3\Omega_m$ such that $M = 6(3\Omega_m)^4$. So for example taking $\Omega_m = 0.28$ we get $1/M = 0.33$ and so we choose a slightly tighter bound $|z| < 0.25$ to make sure that the series approximations converge on this interval. Then computing the coefficients by using the above recursive relations gives a series approximation valid on the interval $[0, 0.25]$. As described further in the Appendix A, this process can be carried on until the full redshift range is covered, yielding the piecewise convergent approximate solution shown in Table 1. For computational simplicity we have chosen to limit ourselves to third-order approximations and limit the full redshift range to $0 \leq z \leq 2.5$. The order of the approximation and the redshift range can be increased as required. Figure 1a shows the luminosity distance function $H_0 d_L(z)/c = (1+z)u(z)$ for the Λ CDM model obtained via the multistage PSM in the red-shift range $0 \leq z \leq 2.5$, for $\Omega_m = 0.28$. The piecewise convergent luminosity distance has an excellent agreement with the solution obtained via numerical integration of the integral in (12).

Table 1. Piecewise convergent approximate solution for $u(z)$ with $\Omega_m = 0.28$.

z	$u(z)$
$0 \leq z \leq 0.25$	$z - 0.21z^2 - 0.0518z^3$
$0.25 \leq z \leq 0.50$	$0.2361 + 0.8885(z - 0.25) - 0.2301(z - 0.25)^2 - 0.0035(z - 0.25)^3$
$0.50 \leq z \leq 0.75$	$0.4438 + 0.7753(z - 0.5) - 0.2202(z - 0.5)^2 + 0.0272(z - 0.5)^3$
\vdots	\vdots
$2.25 \leq z \leq 2.5$	$1.3133 + 0.3112(z - 2.25) - 0.0245(z - 2.25)^2 - 0.0336(z - 2.25)^3$

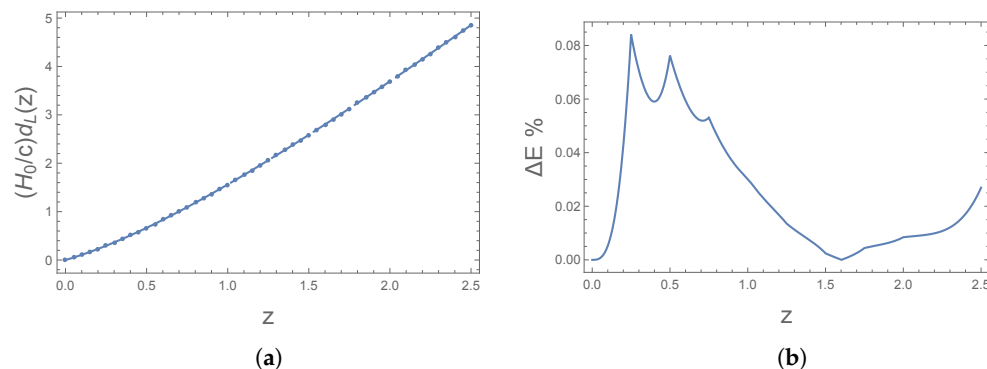


Figure 1. (a) The luminosity distance function $(H_0/c)d_L(z)$ (solid curve) vs. the redshift z for the Λ CDM model with $\Omega_m = 0.28$. The corresponding numerical solution obtained by integrating (12) is also shown as plotted points for discrete values of z . (b) The absolute relative percentage error of the luminosity distance ΔE vs. the redshift z .

To determine the accuracy of the approximate luminosity distance we define the absolute relative percentage error by

$$\Delta E = \left| \frac{d_L^{\text{approx}} - d_L^{\text{num}}}{d_L^{\text{num}}} \right| \times 100\%, \tag{21}$$

where d_L^{approx} is the luminosity distance values calculated by the PSM and d_L^{num} are the values obtained via numerical integration of the integral in (12). This is shown in Figure 1b.

4.2. CPL Cosmological Model

For the CPL model [40–42] the function $W(z)$ in (15) is given by

$$W(z) = \Omega_m(1+z)^3 + (1 - \Omega_m)(1+z)^{3(1+\omega_0+\omega_1)} e^{-3\omega_1 \frac{z}{1+z}}, \tag{22}$$

which now contains the additional parameters ω_0 and ω_1 . The case $\omega_0 = -1, \omega_1 = 0$ corresponds to the Λ CDM. Introducing the auxiliary variables

$$\begin{aligned} V(z) &= u'(z); S(z) = W'(z); P(z) = e^{-3\omega_1 z/(1+z)}; R(z) = (1+z)^{(3\omega_0+3\omega_1)}; \\ Q(z) &= (1+z); Z(z) = 1/Q(z), \end{aligned} \tag{23}$$

the differential equation in (15) can be projected into the polynomial system

$$\begin{aligned} u' &= V(z) \\ V' &= -\frac{1}{2}S(z)V^3(z) \\ S' &= 6\Omega_m Q(z) + 3(1 + \omega_0 + \omega_1)(2 + 3\omega_0 + 3\omega_1)(1 - \Omega_m)P(z)Q(z)R(z) \\ &\quad - 6\omega_1(2 + 3\omega_0 + 3\omega_1)(1 - \Omega_m)PR + 9\omega_1^2(1 - \Omega_m)P(z)R(z)Z(z) \\ P' &= -3\omega_1 P(z)Z(z)^2 \\ R' &= 3(\omega_0 + \omega_1)R(z)Z(z) \\ Z' &= -Z^2(z) \\ Q' &= 1, \end{aligned} \tag{24}$$

with the initial conditions $u(0) = 0, V(0) = 1, S(0) = 3(1 + \omega_0(1 - \Omega_m)), P(0) = 1, R(0) = 1, Z(0) = 1, Q(0) = 1$. Using the series expansions

$$\begin{aligned} u(z) &= \sum_{i=0}^p u_i z^i, V(z) = \sum_{i=0}^p V_i z^i, S(z) = \sum_{i=0}^p S_i z^i, P(z) = \sum_{i=0}^p P_i z^i, R(z) = \sum_{i=0}^p R_i z^i, \\ Q(z) &= \sum_{i=0}^p Q_i z^i, Z(z) = \sum_{i=0}^p Z_i z^i. \end{aligned} \tag{25}$$

and substituting in the above polynomial system, gives the following recursive relations for the coefficients

$$\begin{aligned} u_{i+1} &= \frac{V_i}{i+1} \\ V_{i+1} &= -\frac{1}{2(i+1)} \sum_{j=0}^i S_{i-j} \sum_{p=0}^j \left(\sum_{k=0}^p V_k V_{p-k} \right) V_{j-p} \\ S_{i+1} &= \frac{1}{i+1} \left[6\Omega_m Q_i + 3(1 + \omega_0 + \omega_1)(2 + 3\omega_0 + 3\omega_1)(1 - \Omega_m) \sum_{j=0}^i \left(\sum_{k=0}^j P_k Q_{j-k} \right) R_{i-j} \right. \\ &\quad \left. - 6\omega_1(2 + 3\omega_0 + 3\omega_1)(1 - \Omega_m) \sum_{j=0}^i P_j R_{i-j} + 9\omega_1^2(1 - \Omega_m) \sum_{j=0}^i \left(\sum_{k=0}^j P_k R_{j-k} \right) Z_{i-j} \right] \\ P_{i+1} &= -\frac{3\omega_1}{i+1} \sum_{j=0}^i \left(\sum_{k=0}^j Z_k Z_{j-k} \right) P_{i-j} \\ R_{i+1} &= \frac{3(\omega_0 + \omega_1)}{i+1} \sum_{j=0}^i R_j Z_{i-j} \\ Z_{i+1} &= -\frac{1}{i+1} \sum_{j=0}^i Z_j Z_{i-j} \\ Q_i &= \begin{cases} 0 & : i > 1 \\ 1 & : i = 1 \end{cases}. \end{aligned} \tag{26}$$

Again by Theorem 1 the interval of convergence is given by $|z| < 1/M$ where $M = (k - 1)\Sigma \mathbf{g}^{\alpha^{k-1}}$. We use the values of the parameters for the CPL model obtained in Ref. [48], namely $\Omega_m = 0.311, \omega_0 = -0.937,$ and $\omega_1 = 0.064$. Then $k = 4, \Sigma \mathbf{g} = 2.619,$

$\alpha = 1.06322$ such that $1/M = 0.106$. Hence we start by computing the coefficients of the series approximations by using the above recursive relations in the subinterval $[0, 0.1]$ and again we consider third order approximations. These are shown in the Appendix A. As in the case of the Λ CDM, the process can be continued to cover the required redshift range, thereby yielding the piecewise convergent approximation shown in Table 2. Figure 2a below shows the luminosity distance function for the CPL cosmological model obtained via the multistage PSM in the red-shift range $0 \leq z \leq 1.2$, for $\Omega_m = 0.311$, $\omega_0 = -0.937$, and $\omega_1 = 0.064$. Again, as in the previous case the obtained piecewise convergent luminosity distance has an excellent agreement with the solution obtained via numerical integration of the integral in (12). This is evidenced by the very small relative percentage error obtained from (21) as shown in Figure 2b.

Table 2. Piecewise convergent approximate solution for $u(z)$ with $\Omega_m = 0.311$, $\omega_0 = -0.937$, and $\omega_1 = 0.064$.

z	$u(z)$
$0 \leq z \leq 0.1$	$z - 0.2658z^2 - 0.0164z^3$
$0.1 \leq z \leq 0.2$	$0.0973 + 0.9466(z - 0.1) - 0.2673(z - 0.1)^2 + 0.0055(z - 0.1)^3$
$0.2 \leq z \leq 0.3$	$0.1893 + 0.8935(z - 0.2) - 0.2631(z - 0.2)^2 + 0.0222(z - 0.2)^3$
\vdots	\vdots
$1.1 \leq z \leq 1.2$	$0.8088 + 0.5194(z - 1.1) - 0.1481(z - 1.1)^2 + 0.0390(z - 1.1)^3$

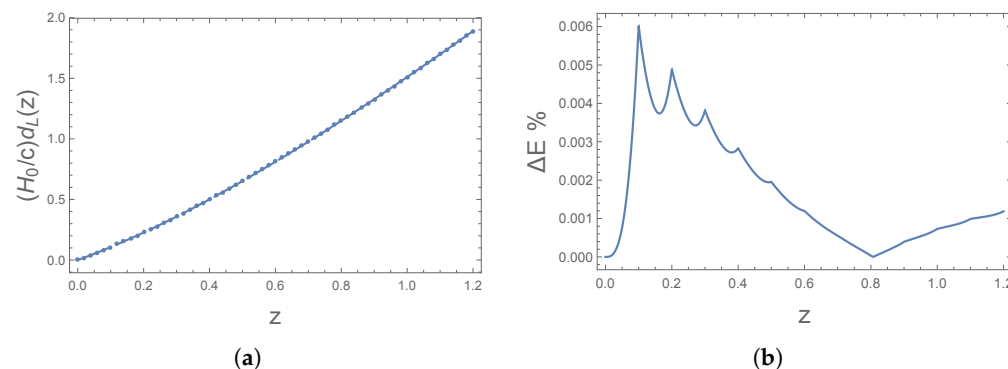


Figure 2. (a) The luminosity distance function $(H_0/c)d_L(z)$ (solid curve) vs. the redshift z for the CPL model with $\Omega_m = 0.311$, $\omega_0 = -0.937$, and $\omega_1 = 0.064$. The corresponding numerical solution obtained by integrating (12) is also shown as plotted points for discrete values of z . (b) The absolute relative percentage error of the luminosity distance ΔE vs. the redshift z for the same redshift range.

5. Discussion and Conclusions

In this paper, we have used the Parker–Sochacki method to obtain an analytical approximation of the luminosity distance for the Λ CDM and CPL cosmological models. This is done by solving the nonlinear differential equation satisfied by the luminosity distance function by first projecting it into a polynomial system and then implementing the PSM in a multistage fashion for the redshift subintervals so that the result is a piecewise convergent luminosity distance function on the entire required redshift range, which has an excellent agreement with the numerical solutions. The overall accuracy is controlled by the length of these subintervals and the order of the approximation used which can be increased for a better overall accuracy. As mentioned in the Introduction, there are other methods proposed in the literature for obtaining analytical approximations of the luminosity function by solving the differential Equation (15). For example, Shchigolev applied the variational iteration method (VIM) [18], the homotopy perturbation method (HPM) [20] and the Daftardar–Jafari method (DFM) [21] to obtain analytical approximations for $d_L(z)$ for the Λ CDM model. These methods yield fairly accurate approximations for low red-shifts $z < 1$, but the accuracy is nowhere near that obtained by using the PSM

as can be seen by comparing Figure 1 in Ref. Shchigolev [18,20,21] with Figure 1 in this paper. The relative percentage errors of the luminosity distance approximations for the PSM are still significantly less than those obtained by applying a variant of the HPM, called the HPM–Padé technique which was proposed recently by Yu et al. [19] as an improvement to the approximations generated by the HPM in [20]. In fact, for the Λ CDM model with the same value of the cosmological parameter $\Omega_m = 0.28$ and the same redshift range $0 \leq z \leq 2.5$, the HPM–Padé yields a maximum relative percentage error $\Delta E \sim 4\%$ for the best approximation over this redshift range. In our case $\Delta E < 0.1\%$, and this is comparable or even better than some of the earlier fitting formulae (such as those developed by Pen [3], Wickramasinghe and Ukwatta [4], Liu et al. [5] and Baes et al. [10]). More importantly, unlike previous approximations obtained via the VIM, HPM, DFM, and HPM–Padé, the relative percentage error ΔE in the approximated luminosity distance does not increase with redshift. The multi-stage approach in the implementation of the PSM makes it possible to apply it for even higher redshift ranges without compromising accuracy.

One can argue that unlike the various analytic approximations obtained by solving (15), the direct fitting formulae mentioned above are much easier to apply and have a shorter computation time, and they are valid over a very wide range of redshift. This is undoubtedly correct. However, a major drawback of these simple formulae is that they were specifically developed for the Λ CDM model and cannot be used for other cosmological models. On the other hand, as shown in this paper and in previous articles, analytic approximations for the luminosity distance obtained by solving the differential Equation (15) can be easily applied to various cosmological models; even more complex models than the CPL model analyzed here. This is particularly important in modern cosmology when testing and investigating new dark energy models in alternative theories of gravity. The need for an efficient and precise calculation of the luminosity distance in this case is paramount to uncover any subtle differences with the standard Λ CDM cosmology [49].

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Appendix A. Approximate Solutions

In this appendix, we obtain the series approximations to the polynomial system of equations for the Λ CDM and CPL cosmological models given by (18) and (24) respectively, by dividing the entire redshift range into subintervals as described in Section 4. The series expansions for the variable $u(z)$ (which is directly related to the luminosity via (13)) in these subintervals gives the piecewise convergent approximation shown in Tables 1 and 2.

Appendix A.1. Λ CDM Cosmological Model

From the recursive relations (20) for the first subinterval $z \in [0, 0.25]$, it follows that

$$\begin{aligned}
 u^{(1)}(z) &= z - 0.21z^2 - 0.0518z^3 \\
 V^{(1)}(z) &= 1 - 0.42z - 0.1554z^2 + 0.2040z^3 \\
 S^{(1)}(z) &= 0.84(1 + z)^2 \\
 Q^{(1)}(z) &= 1 + z,
 \end{aligned}
 \tag{A1}$$

where the superscript (1) in the above expressions indicates that these series approximations are valid on the first subinterval. The PSM algorithm is applied in a multistage approach as explained in Section 2. Therefore on the second subinterval $[0.25, 0.50]$ the PSM algorithm is applied again by solving the same polynomial system in (18), but now with the initial

conditions $u^{(2)}(0) = u^{(1)}(0.25)$, $V^{(2)}(0) = V^{(1)}(0.25)$, $S^{(2)}(0) = S^{(1)}(0.25)$ and $Q^{(2)}(0) = Q^{(1)}(0.25)$. This yields:

$$\begin{aligned} u^{(2)}(z) &= 0.2361 + 0.8885(z - 0.25) - 0.2301(z - 0.25)^2 - 0.0035(z - 0.25)^3 \\ V^{(2)}(z) &= 0.8885 - 0.4603(z - 0.25) - 0.0106(z - 0.25)^2 + 0.1653(z - 0.25)^3 \\ S^{(2)}(z) &= 1.3125 + 2.1(z - 0.25) + 0.84(z - 0.25)^2 \\ Q^{(2)}(z) &= 1 + z. \end{aligned} \tag{A2}$$

This process is repeated until the required full redshift range $[0, 2.5]$ is covered. Listing the obtained $u^{(i)}(z)$ as shown in Table 1 gives a piecewise convergent approximation for the luminosity distance function $d_L(z)$ valid on this redshift range.

Appendix A.2. CPL Cosmological Model

On the first subinterval $z \in [0, 0.1]$, the recursive relations in (26) give

$$\begin{aligned} u^{(1)}(z) &= z - 0.2658z^2 - 0.0164z^3 \\ V^{(1)}(z) &= 1 - 0.5316z - 0.0493z^2 + 0.2454z^3 \\ S^{(1)}(z) &= 1.0632 + 1.8927z + 0.8016z^2 + 0.2330z^3 \\ P^{(1)}(z) &= 1 - 0.192z + 0.2104z^2 - 0.2300z^3 \\ R^{(1)}(z) &= 1 - 2.619z + 4.7391z^2 - 7.2966z^3 \\ Z^{(1)}(z) &= 1 - z + z^2 - z^3 \\ Q^{(1)}(z) &= 1 + z. \end{aligned} \tag{A3}$$

As was done in the previous case the PSM is applied in a multistage approach, and so for the next subinterval $[0.1, 0.2]$, the polynomial system in (24) is solved with the initial conditions $u^{(2)}(0) = u^{(1)}(0.1)$, $V^{(2)}(0) = V^{(1)}(0.1)$, $S^{(2)}(0) = S^{(1)}(0.1)$, $P^{(2)}(0) = P^{(1)}(0.1)$, $R^{(2)}(0) = R^{(1)}(0.1)$, $Z^{(2)}(0) = Z^{(1)}(0.1)$ and $Q^{(2)}(0) = Q^{(1)}(0.1)$. This gives

$$\begin{aligned} u^{(2)}(z) &= 0.0973 + 0.9466(z - 0.1) - 0.2673(z - 0.1)^2 + 0.0055(z - 0.1)^3 \\ V^{(2)}(z) &= 0.9466 - 0.5347(z - 0.1) + 0.0164(z - 0.1)^2 + 0.1924(z - 0.1)^3 \\ S^{(2)}(z) &= 1.2607 + 2.0588(z - 0.1) + 0.8550(z - 0.1)^2 + 0.1328(z - 0.1)^3 \\ P^{(2)}(z) &= 0.9827 - 0.1559(z - 0.1) + 0.1540(z - 0.1)^2 - 0.1520(z - 0.1)^3 \\ R^{(2)}(z) &= 0.7782 - 1.8526(z - 0.1) + 3.0473(z - 0.1)^2 - 4.2648(z - 0.1)^3 \\ Z^{(2)}(z) &= 0.909 - 0.8263(z - 0.1) + 0.7511(z - 0.1)^2 - 0.6827(z - 0.1)^3 \\ Q^{(2)}(z) &= 1 + z. \end{aligned} \tag{A4}$$

The process is repeated until required redshift range is covered. Then the functions $u^{(i)}(z)$ represent the piecewise convergent approximation for the luminosity distance function $d_L(z)$ as shown in Table 2.

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