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Letter

# Estimating the strong coupling from $\tau$ decay using accelerating series convergence

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ABSTRACT

We apply the Euler transformation to accelerate the convergence of the QCD perturbative series with the aim to determine the strong coupling $\alpha_s$ in terms of the total $\tau$-decay rate $r_\tau$. The variation of the result with the order of the QCD perturbation theory is small and comparable with the uncertainties of $r_\tau$. We also present an estimate of a range of the yet unknown 5th and 6th order coefficients $k_5$ and $k_6$ of the Adler function.

## 1. Introduction

The value of the strong coupling constant $\alpha_s$ depends on the scale at which it is determined. At the scale of the $\tau$ mass, $m_\tau$, it can be determined very precisely from the total hadronic $\tau$-decay rate (see for example the review Ref. [1] and references therein) for two reasons: the $\tau$ mass is large enough so that a perturbative approach seems to be justified and its theoretical expression is known to order $\alpha_s^4$ owing to the heroic calculation of the QCD current correlator by Baikov, Chetyrkin and Kühn [2]. Higher-order calculations have reached a state where it is unlikely that the next order will become available soon. It is therefore important to gain insight in the behaviour of the perturbative series and possibly obtain some hint on how good the extraction of $\alpha_s$ from the total $\tau$-decay rate can be if only a few first terms of the perturbative series are known. In this note we investigate the applicability of a well-known technique of accelerating the convergence of the QCD perturbative series, namely the Euler transformation [3–5].

## 2. The hadronic decay of the $\tau$-lepton

Taking into account radiative corrections, the $\tau$-decay rate into non-strange hadrons for the vector and axial-vector components, $V$ and $A$, can be written as

$$R_{\tau,V/A} = \frac{N_C}{2}\left|V_{ud}\right|^2 S_{EW}\left(1 + r_\tau + \delta'_{EW} + \delta^{(2,m_q)}_{ud,V/A} + \sum_{D\geq 4}\delta^{(D)}\right). \tag{1}$$

Here, $S_{EW} = 1.01907 \pm 0.0003$ [6] is a factor describing logarithmically enhanced electroweak corrections calculated in Refs. [7,8], $\delta'_{EW} = 0.0010$ [9] takes into account residual non-logarithmic electroweak corrections, $\delta^{(2,m_q)}_{ud,V/A}$ is the dimension $D = 2$ perturbative quark mass correction (smaller than 0.1% for $u$, $d$ quarks) and $\delta^{(D)}$ are higher-dimension contributions from condensates in the operator product expansion (OPE) and possible contributions from genuine duality violation. OPE corrections and non-perturbative contributions dominate the uncertainty. An estimate of these contributions has been obtained from a fit to the ALEPH data [10,11] with the result $\delta_{NP} = -0.0064 \pm 0.0013$. Based on the recent analysis of Ref. [12], we will use

$$r_\tau = 0.2027 \pm 0.0028 \tag{2}$$

in our numerical results. We also note the reference value for the strong coupling at the $\tau$ mass given by the Particle Data Group (PDG) [6]:

$$\alpha_s(m_\tau) = 0.312 \pm 0.015\,. \tag{3}$$

Note that we are particularly interested in the higher-order corrections from perturbative QCD which are comprised in $r_\tau$. These are determined by the current-current correlator and can be written as a power series in the strong coupling constant $\alpha_s$ which we describe in the next section.

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## 3. Perturbative corrections

The hadronic branching ratio of the $\tau$-lepton is related to the spectral function, i.e. the imaginary part of the current-current correlator. We consider the vector plus axial-vector correlator

$$4\pi^2\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx}\langle 0|T(J_\mu(x)J_\nu(0))|0\rangle = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2)\,, \quad (4)$$

where the currents $J_\mu(x) = \frac{1}{2}(V_\mu(x) + A_\mu(x))$ with $V_\mu(x) = \bar{u}(x)\gamma_\mu d(x)$ and $A_\mu(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)$ are constructed from the light-quark field operators $u(x)$ and $d(x)$. Using Cauchy's theorem and omitting non-perturbative and OPE corrections one finds

$$R_\tau(s_0) = -6\pi i\,|V_{ud}|^2 S_{EW}\oint_{|s|=s_0}\frac{ds}{s_0}\left(1-\frac{s}{s_0}\right)^2\left(1+2\frac{s}{s_0}\right)\Pi(s)\,, \quad (5)$$

where the integration is over a circle of radius $|s| = s_0$, given by the $\tau$ mass, $s_0 = m_\tau^2$, $m_\tau = 1.77686 \pm 0.00012$ GeV [6].

The result for the correlator renormalized at a scale $\mu$ is of the form

$$4\pi^2\Pi(s) = -\sum_{n=0}^{\infty}a^n\sum_{i=1}^{n+1}c_{ni}L^i\,, \qquad a \equiv \frac{\alpha_s(\mu)}{\pi}\,, \qquad L \equiv \ln\frac{-s}{\mu^2}\,, \quad (6)$$

where $\alpha_s(\mu)$ is the $\overline{\mathrm{MS}}$ renormalized running coupling. Non-logarithmic terms, i.e. with coefficients $c_{n0}$ are not included, as these terms are related to external renormalization and do not contribute to measurable quantities. We also have $c_{n,n+1} = 0$ for $n \ge 1$. Only the coefficients $c_{n1}$ have to be calculated from $(n+1)$-loop diagrams; the other coefficients $c_{ni}$ with $i \ge 2$ are related by the renormalization group equation (RGE) to $c_{n1}$. The following shorter notation will be used for the independent coefficients:

$$c_{n1} = k_n$$

with $k_0 = k_1 = 1$. The non-trivial coefficients have been calculated in the $\overline{\mathrm{MS}}$ scheme, $k_2$ and $k_3$ by Bardeen et al. [13] and $k_4$ by Baikov et al. [2]. For three flavours, $n_f = 3$, the result is

$$k_2 = \frac{299}{24} - 9\zeta_3 = 1.63982\,,$$
$$k_3 = \frac{58057}{288} - \frac{779}{4}\zeta_3 + \frac{75}{2}\zeta_5 = 6.37101\,,$$
$$k_4 = 49.076\,.$$

For the current-current correlator one finds explicitly the power series in the coupling $a$ up to order $O(a^6)$ [1,2]:

$$\begin{aligned}
-4\pi^2\Pi(s) = {} & L + aL + a^2\left(k_2L - \frac{1}{2}b_0L^2\right) \\
& + a^3\left(k_3L - \left(\frac{1}{2}b_1 + b_0k_2\right)L^2 + \frac{1}{3}b_0^2L^3\right) \\
& + a^4\left(k_4L - \left(\frac{1}{2}b_2 + b_1k_2 + \frac{3}{2}b_0k_3\right)L^2 \right.\\
& \left. + \left(\frac{5}{6}b_0b_1 + b_0^2k_2\right)L^3 - \frac{1}{4}b_0^3L^4\right) \\
& + a^5\left(k_5L - \left(\frac{1}{2}b_3 + b_2k_2 + \frac{3}{2}b_1k_3 + 2b_0k_4\right)L^2 \right.\\
& + \left(b_0b_2 + \frac{1}{2}b_1^2 + \frac{7}{3}b_0b_1k_2 + 2b_0^2k_3\right)L^3 \\
& \left. - \left(\frac{13}{12}b_0^2b_1 + b_0^3k_2\right)L^4 + \frac{1}{5}b_0^4L^5\right) \\
& + a^6\left(k_6L - \left(\frac{1}{2}b_4 + b_3k_2 + \frac{3}{2}b_2k_3 + 2b_1k_4 + \frac{5}{2}b_0k_5\right)L^2 \right.\\
& + \left(\frac{7}{6}\left(b_1b_2 + b_0b_3\right) + \frac{4}{3}\left(b_1^2 + 2b_0b_2\right)k_2 + \frac{9}{2}b_0b_1k_3 \right.\\
& \left. + \frac{10}{3}b_0^2k_4\right)L^3 \\
& - \left(\frac{35}{24}b_0b_1^2 + \frac{3}{2}b_0^2b_2 + \frac{47}{12}b_0^2b_1k_2 + \frac{5}{2}b_0^3k_3\right)L^4 \\
& \left. + \left(\frac{77}{60}b_0^3b_1 + b_0^4k_2\right)L^5 - \frac{1}{6}b_0^5L^6\right).
\end{aligned} \quad (7)$$

The coefficients $b_i$ of the QCD $\beta$-function are as given in Refs. [2,14]

$$\begin{aligned}
b_0 &= 2.75 - 0.166\,667\,n_f = 2.25\,, \\
b_1 &= 6.375 - 0.791\,667\,n_f = 4\,, \\
b_2 &= 22.3203 - 4.368\,92\,n_f + 0.094\,0394\,n_f^2 = 10.059\,896\,, \\
b_3 &= 114.23 - 27.1339\,n_f + 1.582\,38\,n_f^2 + 0.005\,8567\,n_f^3 = 47.228\,040\,, \\
b_4 &= 524.56 - 181.8\,n_f + 17.16\,n_f^2 - 0.225\,86\,n_f^3 - 0.001\,7993\,n_f^4 \\
&= 127.322\,.
\end{aligned} \quad (8)$$

The last numerical value in each line is given for three flavors, $n_f = 3$.

For the calculation of $R_\tau$, one needs to evaluate integrals of the form

$$I(q,k) = \frac{1}{2\pi i}\oint_{|s|=s_0}s^q\left(\log\frac{-s}{\mu^2}\right)^k ds$$

and their expressions can be extracted from results given in Refs. [15, 16] as follows

$$I(q,k) = s_0^{q+1}\sum_{p=0}^{k}\sum_{l=0}^{k-p}(-1)^{\frac{p-1}{2}}\frac{[1-(-1)^p]}{2}\frac{k!}{p!\,l!}\frac{(-1)^{k-p-l}}{(q+1)^{k-p-l+1}}\pi^{p-1}\left(\log\frac{s_0}{\mu^2}\right)^l \quad \text{for} \quad q \ne -1$$

and

$$I(-1,k) = \sum_{p=0}^{k}\frac{1+(-1)^p}{2}(-1)^{p/2}\frac{\pi^p k!}{(k-p)!\,(p+1)!}\left(\log\frac{s_0}{\mu^2}\right)^{k-p} \quad \text{for} \quad q = -1\,.$$

Using these results and setting $\mu^2 = s_0 = m_\tau^2$, we obtain the following power expansion of $r_\tau$ in terms of $a = \alpha_s(m_\tau)/\pi$ up to order $O(a^6)$:

$$\begin{aligned}
r_\tau = {} & a + 5.20232\,a^2 + 26.3659\,a^3 + 127.079\,a^4 + (307.787 + k_5)\,a^5 \\
& + (-5646.6 + 17.8125\,k_5 + k_6)\,a^6\,.
\end{aligned} \quad (9)$$

This result is obtained with the prescription known as fixed-order perturbation theory (FOPT), i.e. by keeping the renormalization scale $\mu^2 = s_0$ fixed along the contour of integration. We do not consider the alternative approach known as contour-improved perturbation theory since this has been shown recently to be inconsistent with the standard way to treat non-perturbative effects [17–20].

## 4. Numerical results

Predictions of Eq. (9) for $r_\tau$ at different orders are shown in Fig. 1 using the PDG value as input for the strong coupling, $\alpha_s(m_\tau) = 0.312 \pm 0.015$. All terms in the power expansion of $r_\tau$ are positive (except possibly the 5th and 6th order term depending on the values of the unknown coefficients $k_{5,6}$). Therefore the value of $r_\tau$ as well as its uncertainty resulting from the error of $\alpha_s(m_\tau)$ increase when including the next higher order. The values shown by blue points in Fig. 1 indicate the effect of the coefficients $k_5$ and $k_6$ on the higher-order predictions when their values are changed from $k_5 = 0$ to $k_5 = 277$ and $k_6 = 0$ to $k_6 = 3460$ as estimated in Ref. [21] (see below).

The expression for the $\tau$-decay constant given in the previous section in Eq. (9), can be inverted to obtain the strong coupling as a power series in $r_\tau$:

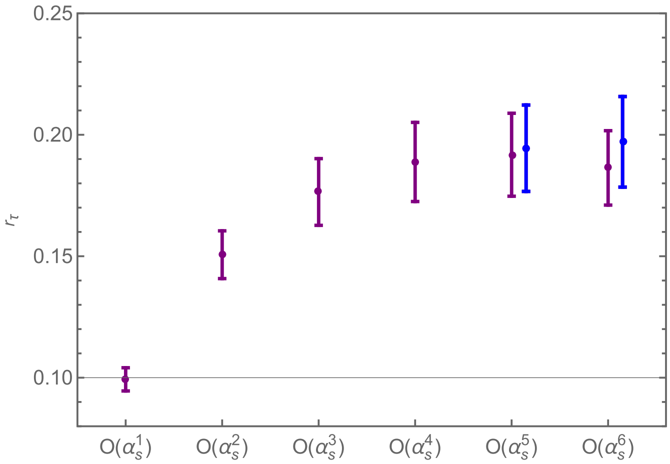


**Fig. 1.** $r_\tau$ at different orders of $\alpha_s$ using the PDG value $\alpha_s(m_\tau) = 0.312 \pm 0.015$. The estimates shown by purple points are calculated with $k_5 = k_6 = 0$. The estimates in blue for the 5th and 6th order are obtained with $k_5 = 277$, $k_6 = 3460$ [21].

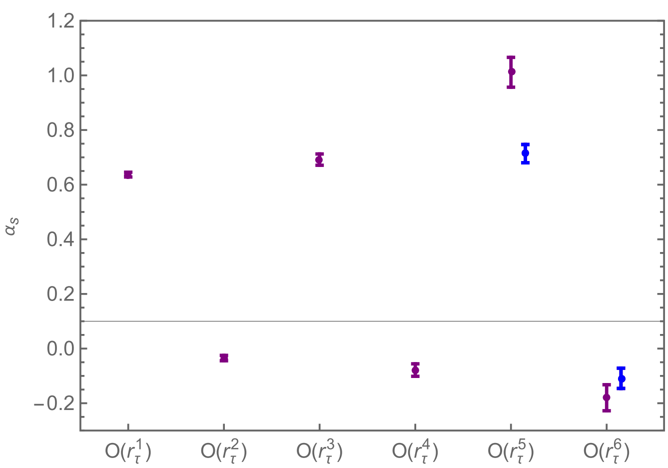


**Fig. 2.** $\alpha_s$ at different orders of $r_\tau$ using $r_\tau = 0.2027 \pm 0.0028$ [12] in Eq. (10), and assuming $k_5 = k_6 = 0$ (purple points) or $k_5 = 277$, $k_6 = 3460$ (blue points).

$$a = r_\tau - 5.20232\, r_\tau^2 + 27.7624\, r_\tau^3 - 145.241\, r_\tau^4 + (1013.89 - k_5)\, r_\tau^5 + (-5467.2 + 18.6037\, k_5 - k_6)\, r_\tau^6 . \quad (10)$$

Using this inverted power series in Eq. (10), we can determine $\alpha_s$ from the experimental value of $r_\tau$. The series for $a(r_\tau)$ is approximately geometric with alternating signs and coefficients that grow approximately as $(-5)^k$, $k = 0, \ldots 3$. Since $r_\tau \simeq 1/5$, the resulting order-by-order determination of the strong coupling does not seem to be convergent, at least not up to order $r_\tau^6$, as can be seen in Fig. 2. The error bars shown in the figure are due to the uncertainty estimate of $r_\tau$, see Eq. (2), which includes the experimental uncertainty of $R_\tau$ and an estimate of the non-perturbative contributions. They appear much smaller than the order-to-order variations of the power expansion of $a(r_\tau)$. It seems obvious from these results that the truncation of the perturbative series dominates the total uncertainty of $\alpha_s$ determined from the hadronic $\tau$ decay.

## 5. Acceleration of the series expansion of the strong coupling using Euler transformation

From the discussion above it seems desirable to improve the series expansion of the strong coupling constant. A powerful method, well-known in the mathematical literature (see e.g. Refs. [3,4]), which can be used to this end is the Euler transformation [5]. For a convergent series, the Euler transform is again convergent and it converges to the same limit as the original series. The Euler transform can be used for the fast acceleration of convergence of infinite series and for analytic continuation. In particular, this can be the case for a series which is close to a geometric one.

The Euler transformation of a series of the form

$$S = \sum_{k=0}^{\infty} x_k = 1 + x_1 + x_2 + \ldots \quad (11)$$

is defined as

$$S_E = \sum_{k=0}^{\infty} x_k^E$$

with

$$x_k^E = \frac{1}{2^{k+1}} \left( \binom{k}{0} x_0 + \binom{k}{1} x_1 + \ldots + \binom{k}{k} x_k \right) .$$

The properties of this transformation can be exemplified for a geometric series, a close approximation of the power expansion of $a(r_\tau)$. Setting $x_k = x^k$ in Eq. (11), it is straightforward to show that the Euler transformation of the series

$$S = \sum_{k=0}^{\infty} x^k \quad (12)$$

is

$$S_E = \frac{1}{2} \sum_{k=0}^{\infty} 2^{-k} (1+x)^k . \quad (13)$$

Obviously both series converge to the same value, $S = S_E = 1/(1-x)$. However, while the original geometric series is convergent for all complex $x$ with $|x| < 1$, the new series converges in a circle of radius 2 centred at $-1$, i.e. $|1+x| < 2$. One can say in this case that the Euler transformation provides the analytic continuation of the series into a larger domain. However, it depends on the value of $x$ whether the original series, or the Euler transformed version, converges faster.

Thus, we apply the Euler transformation to the series obtained above for $a(r_\tau)$, see Eq. (10). The Euler-improved version of $a$ is again a power series in $r_\tau$, though with coefficients which change at each order, namely

$$\begin{aligned}
a_1^E &= 0.5\, r_\tau ,\\
a_2^E &= 0.75\, r_\tau - 1.30058\, r_\tau^2 ,\\
a_3^E &= 0.875\, r_\tau - 2.60116\, r_\tau^2 + 3.47029\, r_\tau^3 ,\\
a_4^E &= 0.9375\, r_\tau - 3.5766\, r_\tau^2 + 8.67573\, r_\tau^3 - 9.07754\, r_\tau^4 ,\\
a_5^E &= 0.96875\, r_\tau - 4.22689\, r_\tau^2 + 13.8812\, r_\tau^3 - 27.2326\, r_\tau^4 \\
&\quad + 0.03125\,(1013.89 - k_5)\, r_\tau^5 ,\\
a_6^E &= 0.984375\, r_\tau - 4.63332\, r_\tau^2 + 18.219\, r_\tau^3 - 49.9265\, r_\tau^4 \\
&\quad + 0.109375\,(1013.89 - k_5)\, r_\tau^5 \\
&\quad - 0.015625\,(5467.1 - 18.6037\, k_5 + k_6)\, r_\tau^6 ,
\end{aligned} \quad (14)$$

where $a_n^E$ includes the coefficients of Eq. (10) up to the order $n$. In Fig. 3 we plot the corresponding numerical results for $a_n^E$. The result looks indeed much better, i.e. the variations from one to the next perturbative order turn out to be much smaller. For example, the difference of the central values of $a$ between the third and the fourth order is reduced from about 0.5% to about 0.09%, namely about a factor 6 smaller after performing the Euler transformation. Correspondingly, if this variation is used to estimate a theory uncertainty, one would find a much smaller value.

It is often argued that an estimate of the truncation error of the perturbative series of a quantity can be obtained by assuming that the

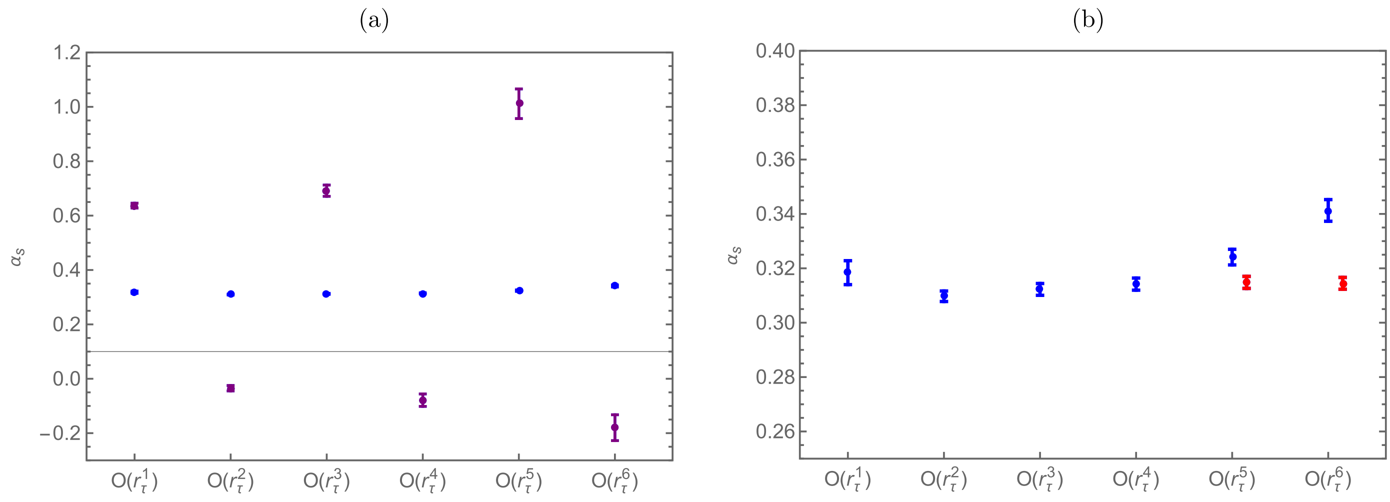


**Fig. 3.** (a) plot of $\alpha_s$ at different orders of $r_\tau$ using $r_\tau = 0.2027 \pm 0.0028$ and assuming $k_5 = k_6 = 0$. Purple points are obtained from Eq. (10), blue points are calculated from the Euler transformed series. (b) plot of the same results with an enlarged scale to make the size of the error bars visible. Shifted (red) points at 5th and 6th order are obtained for values of $k_5 = 277$ and $k_6 = 3460$.

next unknown higher-order term should be expected to be of the same size as the last known term in the series. We impose a similar condition by assuming

$$a_n^E = a_{n-1}^E \tag{15}$$

to determine an expected value of the unknown coefficients $k_n$. The truncation error could then be estimated by assigning a 100% uncertainty to $k_n$. The unknown $k_5$ enters at 5th order in $a_5^E$ albeit with a very small coefficient. Estimating $k_5$ from the condition that $a_5^E(k_5) = a_4^E$ results in $k_5 = 276$. Note that our prediction is very close to the value of $k_5 = 277$ obtained in Ref. [21] where stability of Padé approximants of the Borel-transformed Adler function was studied. Using our estimate for $k_5$ one can go one step further and determine $k_6$ from the condition $a_6^E(k_6) = a_5^E$ with the result $k_6 = 3249$. Also this estimate is in surprisingly good agreement with the value obtained in Ref. [21] where $k_6 = 3460 \pm 690$ was found. The prescription of Eq. (15) to determine the higher-order coefficients is corroborated by the fact that also the known $k_4$ can be estimated this way. We find $k_4 = 54.95$ from $a_4^E(k_4) = a_3^E$. Our result differs by only about 10% from the correct value $k_4 = 49.086$. We note that if $k_4$ was predicted in this manner from the original series in Eq. (10), one would find $k_4 = -78.0$. This result is completely off and the corresponding estimate of a truncation error would be much larger.

## 6. Estimate of the strong coupling

At 4th order and using only the known higher-order coefficients our result for the strong coupling is $\alpha_s = 0.3142 \pm 0.0022$ determined from the Euler-transformed power expansion of $\alpha_s(r_\tau)$. The error is propagated from the input $r_\tau = 0.2027 \pm 0.0028$. The uncertainty includes in this case only errors from the experimental determination of $R_\tau$ and an estimate of non-perturbative contributions as described above. At the next order we find instead $\alpha_s = 0.3149 \pm 0.0023_{\Delta r_\tau} \pm 0.0093_{\Delta k_5}$ where a truncation error is added, resulting from our estimate of $k_5$ with a 100% uncertainty, i.e. using $k_5 = 276 \pm 276$. Going one step further to the 6th order, fixing $k_5 = 276$ and using $k_6 = 3249 \pm 3249$, we find $\alpha_s = 0.3153 \pm 0.0023_{\Delta r_\tau} \pm 0.0110_{\Delta k_6}$. We can see that the Euler transformation leads to a determination of $\alpha_s$ with small changes when going from the 4th to the 5th and the 6th order. However, the truncation error remains dominating.

## 7. Summary

In order to determine the strong coupling constant we have investigated the inverted power series of perturbative QCD for the total hadronic $\tau$-decay rate to express $\alpha_s$ in terms of $r_\tau$. In principle, using $\alpha_s(r_\tau)$ to rewrite other observables (as e.g. the cross section for $e^+e^- \to$ hadrons) as a function of $r_\tau$ one can hope to perform more direct comparisons of a variety of QCD predictions [22]. However, the power expansion of $\alpha_s(r_\tau)$ is badly behaved and cannot be used directly for this purpose. Therefore we studied the application of the Euler transformation to accelerate the convergence of this series. We have also determined estimates of the unknown 5th and 6th order coefficients $k_5$ and $k_6$ of the Adler function and used them to calculate estimates of the strong coupling. We found that the determination of $\alpha_s$ from the Euler transformed series is stable with small variations when going from one to the next higher order. The theoretical uncertainty from the estimated range of the unknown coefficients, however, remains large compared with the experimental uncertainty of the $\tau$-decay rate.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Karl Schilcher reports financial support was provided by Alexander von Humboldt Foundation. Cristiana Sebu reports financial support was provided by University of Malta Research Excellence Fund Programme. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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