Betwixt and Between 2-Factor Hamiltonian and Perfect-Matching-Hamiltonian Graphs

Federico Romaniello

Dipartimento di Matematica,
Informatica ed Economia
Università degli Studi della Basilicata
Italy

federico.romaniello@unibas.it

Jean Paul Zerafa*

St. Edward's College, Triq San Dwardu
Birgu (Città Vittoriosa)
BRG 9039, Cottonera, Malta;
Department of Technology and Entrepreneurship Education
University of Malta, Malta;
Department of Computer Science
Faculty of Mathematics, Physics and Informatics
Comenius University, Mlynská Dolina
842 48, Bratislava, Slovakia

zerafa.jp@gmail.com

Submitted: Jan 24, 2022; Accepted: Mar 25, 2023; Published: Apr 7, 2023

Abstract

A Hamiltonian graph is 2-factor Hamiltonian (2FH) if each of its 2-factors is a Hamiltonian cycle. A similar, but weaker, property is the Perfect-Matching-Hamiltonian property (PMH-property): a graph admitting a perfect matching is said to have this property if each one of its perfect matchings (1-factors) can be extended to a Hamiltonian cycle. It was shown that the *star product* operation between two bipartite 2FH-graphs is necessary and sufficient for a bipartite graph admitting a 3-edge-cut to be 2FH. The same cannot be said when dealing with the PMH-property, and in this work we discuss how one can use star products to obtain graphs (which are not necessarily bipartite, regular and 2FH) admitting the PMH-property with the help of *malleable* vertices, which we introduce here. We show

*The author was partially supported by VEGA 1/0743/21, VEGA 1/0727/22, and APVV-19-0308.

that the presence of a malleable vertex in a graph implies that the graph has the PMH-property, but does not necessarily imply that it is 2FH. It was also conjectured that if a graph is a bipartite cubic 2FH-graph, then it can only be obtained from the complete bipartite graph $K_{3,3}$ and the Heawood graph by using star products. Here, we show that a cubic graph (not necessarily bipartite) is 2FH if and only if all of its vertices are malleable. We also prove that the above conjecture is equivalent to saying that, apart from the Heawood graph, every bipartite cyclically 4-edge-connected cubic graph with girth at least 6 having the PMH-property admits a perfect matching which can be extended to a Hamiltonian cycle in exactly one way. Finally, we also give two necessary and sufficient conditions for a graph admitting a 2-edge-cut to be: (i) 2FH, and (ii) PMH.

Mathematics Subject Classifications: 05C45, 05C70, 05C76

1 Introduction

Graphs considered in the sequel are connected (unless otherwise stated) and are allowed to have multiedges but no loops. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively, and the number of neighbours of a vertex v is denoted by $\deg(v)$. For some integer $t \geqslant 1$, a *t-factor* of a graph G is a t-regular spanning subgraph of G (not necessarily connected). In particular, a *perfect matching* of a graph is the edge set of a 1-factor, and a connected 2-factor of a graph is a *Hamiltonian cycle*. For $k \geqslant 3$, a *cycle* of length k (or a k-cycle), denoted by $(v_1, \ldots, v_k)$, is a sequence of mutually distinct vertices $v_1, v_2, \ldots, v_k$ with corresponding edge set $\{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$. For other definitions not explicitly stated here we refer the reader to [5]. A graph G admitting a perfect matching is said to have the *Perfect-Matching-Hamiltonian property* (for short, the PMH-property) if every perfect matching M of G can be *extended* to a Hamiltonian cycle of G, that is, there exists a perfect matching N of G such that $M \cup N$ induces a Hamiltonian cycle of G. For simplicity, a graph admitting the PMH-property is said to be PMH or a PMH-graph. This property was introduced in the 1970s by Las Vergnas [15] and Häggkvist [11] and for recent results in this area we suggest the following non-exhaustive list [1, 2, 3, 4, 6, 10, 9, 12]. If we restrict ourselves to the class of 3-regular graphs (cubic graphs), there is already a known and well-studied class which are naturally PMH (as we shall see in Theorem 4). This is the class of cubic 2-factor Hamiltonian graphs. The term *2-factor Hamiltonian* (2FH) was coined by Funk *et al.* in [8], where the authors study Hamiltonian graphs with the property that all their 2-factors are Hamiltonian. In their work, the authors prove that if a graph G is a bipartite t-regular 2FH-graph, then G is either a cycle or $t = 3$.

Before proceeding, we define what a star product is. Let G_1 and G_2 be two graphs each containing a vertex of degree 3, say, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let x_1, y_1, z_1 be the neighbours of v_1 in G_1, and x_2, y_2, z_2 be the neighbours of v_2 in G_2. A *star product* on v_1 and v_2, denoted by $G_1(x_1y_1z_1) * G_2(x_2y_2z_2)$, is a graph operation that consists in constructing the new graph $(G_1 - v_1) \cup (G_2 - v_2) \cup \{x_1x_2, y_1y_2, z_1z_2\}$. The 3-edge-cut $\{x_1x_2, y_1y_2, z_1z_2\}$ is referred to as the *principal 3-edge-cut* of the resulting graph (see for instance [7]). Different graphs can be obtained by a star product on v_1 and v_2, for example, $(G_1 - v_1) \cup (G_2 - v_2) \cup \{x_1z_2, y_1y_2, z_1x_2\}$, but, unless otherwise stated, if it is irrelevant how the adjacencies in the principal 3-edge cut look like, we use the notation $G_1(v_1) * G_2(v_2)$ and we say that it is a graph obtained by a star product on v_1 and v_2. For simplicity, we shall also say that the resulting graph has been obtained by applying a star product *between* G_1 and G_2. Since a star product between a graph G and the unique cubic graph on two vertices results in G itself, in the sequel we shall tacitly assume that when considering a star product between two graphs, neither one of the two graphs is the cubic graph on two vertices.

Proposition 1. [8] *Let $G = G_1(v_1) * G_2(v_2)$ be a bipartite graph which is obtained by a star product on $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, both of degree* 3. *Then, G is* 2FH *if and only if G_1 and G_2 are both* 2FH.

We note that in the above proposition, G_1 and G_2 are not necessarily cubic graphs, and only need to admit a vertex of degree 3 each, denoted above by v_1 and v_2, respectively. Moreover, we remark that, in the above proposition, the hypothesis that G is bipartite is needed, because although the complete graph K_4 is a 2FH-graph, the graph obtained by applying a star product between two copies of K_4 is not 2FH (and neither PMH). By using Proposition 1, the authors construct an infinite family of bipartite cubic 2FH-graphs by taking repeated star products of $K_{3,3}$ and the Heawood graph. For example, for each $i \in \{1, 2, 3\}$, let G_i be a copy of $K_{3,3}$ or the Heawood graph, and let $v_i \in V(G_i)$. The graph $(G_1(v_1) * G_2(v_2)) * G_3(v_3)$ is a graph obtained by repeated star products of $K_{3,3}$ and the Heawood graph. In [8], the authors also conjecture that these are the only bipartite cubic 2FH-graph, and this conjecture is still widely open.

Conjecture 2 (Funk *et al.*, 2003 [8])**.** A bipartite cubic 2FH-graph can be obtained from the complete bipartite graph $K_{3,3}$ and the Heawood graph by repeated star products.

2 Malleable vertices

Let ∂v be the set of edges incident to a vertex v.

Definition 3. Let G be a graph admitting a perfect matching and let v be a vertex of G having degree $t \geqslant 2$. The vertex v is said to be *t-malleable* (or just *malleable*) if for every perfect matching M of G, there exist Hamiltonian cycles $H_1, \ldots, H_{t-1}$ all extending M, such that $\partial v - M \subset \cup_{i=1}^{t-1} E(H_i)$.

Therefore, if G admits a t-malleable vertex v, given a perfect matching M of G, there exist $t-1$ distinct Hamiltonian cycles, such that each Hamiltonian cycle extends M and contains a different edge of $\partial v - M$, implying that the $t-1$ Hamiltonian cycles cover all edges incident to v (since every Hamiltonian cycle contains the edge in $\partial v \cap M$). Moreover, if a graph admits a malleable vertex, then it clearly is PMH. In particular, if $|V(G)| > 2$ and $v \in V(G)$ is malleable, then the number of neighbours of v must be equal to $\deg(v)$, that is, there cannot be any multiedges incident to v. Although the definition of malleable vertices seems quite strong, in even cycles and cubic graphs, the presence of a malleable vertex is equivalent to saying that the graph is 2FH.

2.1 Even cycles and cubic graphs

A (connected) 2-regular graph admitting a 2-malleable vertex, must be bipartite, otherwise it does not admit a perfect matching. One can easily see that cycles on an even number of vertices are 2FH and all the vertices are 2-malleable. So consider cubic graphs.

Theorem 4. *A cubic graph G is* 2FH *if and only if G admits a* 3*-malleable vertex.*

Proof. ($\Rightarrow$) Let u be a vertex of G and let M be a perfect matching of G. Moreover, let $\overline{M} = E(G) - M$, that is, the edge set of the complementary 2-factor of M. Since G is 2FH, $\overline{M}$ gives a Hamiltonian cycle, and since G is of even order, $E(\overline{M}) = N_1 \cup N_2$, where N_1 and N_2 are edge-disjoint perfect matchings of G. Once again, since G is 2FH, $M \cup N_1$ and $M \cup N_2$ are both Hamiltonian cycles of G. Thus, u is a 3-malleable vertex.

($\Leftarrow$) Let v be a 3-malleable vertex of G and let M_1 be a perfect matching of G. We are required to show that $\overline{M_1}$ (the edge set of the complementary 2-factor of M_1) gives a Hamiltonian cycle. Since G contains a 3-malleable vertex, it is PMH, and so there exists a perfect matching M_2 such that $M_1 \cup M_2$ gives a Hamiltonian cycle of G. Let $M_3 = E(G) - (M_1 \cup M_2)$ and let $\partial v = \{e_1, e_2, e_3\}$, such that $e_i \in M_i$, for each $i \in \{1, 2, 3\}$. Since v is 3-malleable, there exists a Hamiltonian cycle of G which extends M_3 and contains the edge e_2. Since $M_1 \cup M_2$ forms a Hamiltonian cycle and $(M_1 \cup M_2) \cap M_3 = \emptyset$, the only perfect matching of $G - M_3$ containing e_2 is M_2, and so $M_2 \cup M_3$ (which is equal to $\overline{M_1}$) forms a Hamiltonian cycle, as required. $\square$

Since the vertex u in the first part of the above proof was arbitrary, the next result clearly follows.

Proposition 5. *Let G be a cubic graph admitting a* 3*-malleable vertex. Then, G is* 2FH *and all its vertices are* 3*-malleable.*

Consequently, Theorem 4 can be restated as follows: a cubic graph is 2FH if and only if all its vertices are malleable. In other words, either all or none of the vertices of a cubic graph are 3-malleable. Figure 1 depicts a perfect matching of the cube $\mathcal{Q}_3$ which can only be extended to a Hamiltonian cycle in exactly one way, and so, there is no vertex in $\mathcal{Q}_3$ which is 3-malleable. In fact, the cube is not 2FH (although it is PMH).

In general, if a cubic PMH-graph G (not necessarily bipartite) admits a perfect matching M which extends to a Hamiltonian cycle in exactly one way (that is, there exists a unique perfect matching N for which $M \cup N$ gives a Hamiltonian cycle), then the vertices of G are not malleable, and so the graph is not 2FH (by Theorem 4). The converse of this statement is also true.

Lemma 6. *Let G be a cubic* PMH*-graph (not necessarily bipartite). The graph G is not* 2FH *if and only if it admits a perfect matching which can be extended to a Hamiltonian cycle in exactly one way.*

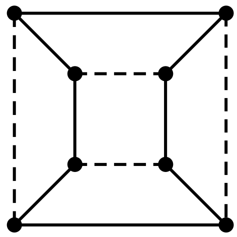

Figure 1: $\mathcal{Q}_3$ does not admit any 3-malleable vertex since the dashed edges can be extended to a Hamiltonian cycle in exactly one way.

Proof. By the comment prior to the statement of the lemma, it suffices to prove the forward direction. Since G is not 2FH, by Theorem 4, no vertex in G is malleable. Let $v \in V(G)$ and let $\partial v = \{e_1, e_2, e_3\}$. Since v is not malleable, there exists a perfect matching of G, say M_1, such that all perfect matchings M_2 of G for which $M_1 \cup M_2$ is a Hamiltonian cycle, intersect $\partial v - M_1$ in the same edge. Let M_2 be such a perfect matching and, without loss of generality, assume that e_1 and e_2 belong to M_1 and M_2, respectively. Since $M_1 \cup M_2$ is a Hamiltonian cycle, $E(G) - (M_1 \cup M_2)$ is a perfect matching, say M_3, containing the edge e_3. Since G is PMH, there exists a perfect matching N of $G - M_3$, such that $N \cup M_3$ is a Hamiltonian cycle of G. Since $G - M_3$ is a connected even cycle, the perfect matching N is either equal to M_1 or to M_2. By our assumption, N cannot be equal to M_1, because otherwise there exists a Hamiltonian cycle extending M_1 which contains the edge e_3. Therefore, N must be equal to M_2. Consequently, M_3 is a perfect matching of G which can be extended to a Hamiltonian cycle of G in exactly one way. $\square$

Before proceeding, the following notions dealing with the cyclic connectivity of a graph require defining. An edge-cut X is said to be *cycle-separating* if at least two components of $G - X$ contain cycles. A (connected) graph G is said to be cyclically k-edge-connected if G admits no set with less than k edges which is cycle-separating. Consider once again Conjecture 2. As stated in [8], a smallest counterexample to this conjecture must be cyclically 4-edge-connected (see [14]), and such a counterexample must have girth at least 6 (see [13]). The authors of [8] state that to prove this conjecture it suffices to show that the Heawood graph is the only bipartite cyclically 4-edge-connected cubic 2FH-graph of girth at least 6. However, thinking about cubic 2FH-graphs through malleable vertices and Lemma 6 suggests another way how one can look at Conjecture 2. In fact, a smallest counterexample to this conjecture can be searched for in the class of bipartite cubic PMH-graphs (recall that Conjecture 2 deals with bipartite cubic graphs). By Lemma 6, Conjecture 2 of Funk *et al.* can be restated equivalently in terms of a strictly weaker property than 2-factor Hamiltonicity: the PMH-property.

Conjecture 7. Every bipartite cyclically 4-edge-connected cubic PMH-graph with girth at least 6, except the Heawood graph, admits a perfect matching which can be extended to a Hamiltonian cycle in exactly one way.

2.2 Non-cubic graphs admitting a malleable vertex

Even though Section 2.1 may suggest otherwise, the existence of a malleable vertex in a graph does not necessarily imply that the graph is 2FH. In fact, we note that for every $t > 3$, there exists a bipartite t-regular graph whose vertices are all t-malleable, but the graph itself is not 2FH (recall that in [8] it was shown that there are no bipartite t-regular 2FH-graphs for $t > 3$). Consider, for example, the complete bipartite graphs $K_{t,t}$ for every $t > 3$. Also, for every odd $t > 3$, the vertices of the complete graph K_{t+1} are all t-malleable, but the graph is not 2FH.

Graphs admitting a malleable vertex which are not 2FH are not necessarily regular. In fact, consider the graph $\mathcal{Y}_{2n+1}$ obtained by adding a new vertex v_0 to the complete graph K_{2n+1}, for some $n \geqslant 2$, such that v_0 is adjacent to exactly three vertices of K_{2n+1} (see Figure 2).

Proposition 8. *The graph $\mathcal{Y}_{2n+1}$ is* PMH *but not* 2FH*. Moreover, the vertex v_0 is* 3*-malleable.*

Proof. Let $V(K_{2n+1}) = \{v_1, \ldots, v_{2n+1}\}$ and, without loss of generality, let the neighbours of v_0 in $\mathcal{Y}_{2n+1}$ be v_1, v_2, v_3. Then, the two disjoint cycles (v_0, v_1, v_2) and $(v_3, v_4, \ldots, v_{2n+1})$ form a 2-factor, making the graph not 2FH. We also claim that the vertex v_0 is a 3-malleable vertex. In fact, let M be a perfect matching of $\mathcal{Y}_{2n+1}$ and, without loss of generality, assume that $v_0v_1 \in M$. If one can show that $\mathcal{Y}_{2n+1} - v_0v_2$ and $\mathcal{Y}_{2n+1} - v_0v_3$ each admit a Hamiltonian cycle extending M, then this would imply that v_0 is a 3-malleable vertex. Without loss of generality, consider $\mathcal{Y}_{2n+1} - v_0v_2$. Since $\mathcal{Y}_{2n+1} - v_0v_2$ contains a copy of the complete graph K_{2n+1}, there exists a Hamiltonian path of $\mathcal{Y}_{2n+1} - \{v_0\}$ with endvertices v_1 and v_3 which contains all the edges of $M - \{v_0v_1\}$. This latter path together with the edges v_0v_1 and v_0v_3 gives a Hamiltonian cycle of $\mathcal{Y}_{2n+1} - v_0v_2$ extending M. By a similar reasoning, $\mathcal{Y}_{2n+1} - v_0v_3$ admits a Hamiltonian cycle extending M. Since M was arbitrary, the vertex v_0 is 3-malleable. □

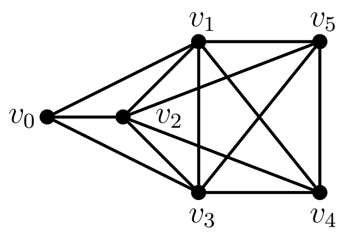

Figure 2: The graph $\mathcal{Y}_5$.

The above construction also provides us with graphs which are not 2FH, admit a malleable vertex, but not all of its vertices are as such (unlike even cycles and cubic graphs). In fact, let M be a perfect matching of $\mathcal{Y}_{2n+1}$ containing the edges v_0v_1 and v_2v_3. Any Hamiltonian cycle of $\mathcal{Y}_{2n+1}$ extending M cannot contain v_1v_2 or v_1v_3, and so, in particular, the vertex v_1 is not $(2n + 1)$-malleable.

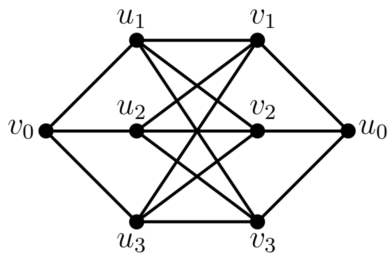

Figure 3: The graph $\mathcal{B}_3$.

We can also obtain examples of graphs which are not 2FH and admit a malleable vertex which are bipartite and less dense (with respect to the number of edges) in the following way. For every $n \geqslant 3$, let $\mathcal{B}_n$ be the bipartite graph with partite sets $\{u_0, u_1, \ldots, u_n\}$ and $\{v_0, v_1, \ldots, v_n\}$, such that

$$E(\mathcal{B}_n) = \{u_0v_1, u_0v_2, u_0v_3\} \cup \{v_0u_1, v_0u_2, v_0u_3\} \cup \{u_iv_j : \text{for any } i, j \in [n]\},$$

where $[n] = \{1, \ldots, n\}$.

Proposition 9. *For every $n \geqslant 3$, the graph $\mathcal{B}_n$ is* PMH *but not* 2FH*. Moreover, the vertices u_0 and v_0 are 3-malleable.*

Proof. Consider the cycles:

(i) (u_0, v_1, u_2, v_2) and (v_0, u_1, v_3, u_3), when $n = 3$; and

(ii) (u_0, v_1, u_2, v_2) and $(v_0, u_1, v_3, u_4, v_4, \ldots, v_n, u_3)$, when $n > 3$.

In each case a disconnected 2-factor of $\mathcal{B}_n$ is formed, where, in particular, v_4 is followed by u_5 when $n > 4$. Consequently, $\mathcal{B}_n$ is not 2FH. Next, we show that u_0 and v_0 are 3-malleable, which implies that $\mathcal{B}_n$ is PMH, for every $n \geqslant 3$. Let M be a perfect matching of $\mathcal{B}_n$, and without loss of generality, assume that $\{u_0v_1, v_0u_1\} \subset M$. Due to the symmetry of $\mathcal{B}_n$, without loss of generality, we can further assume that exactly one of the following occurs:

(i) $\{u_2v_2, u_3v_3\} \subset M$;

(ii) $u_2v_2 \in M$ and $u_3v_3 \notin M$; and

(iii) $u_2v_2, u_2v_3, u_3v_3, u_3v_2$ do not belong to M.

We note that the last two instances only occur when $n > 3$. Let $M' = M - \{u_0v_1, v_0u_1\}$. The graph $\mathcal{B}' = \mathcal{B}_n - \{u_0, u_1, v_0, v_1\}$ is isomorphic to the complete bipartite graph $K_{n-1,n-1}$ and M' is one of its perfect matchings. Since every vertex in $\mathcal{B}'$ is $(n-1)$-malleable, there exist Hamiltonian cycles H_1' and H_2' of $\mathcal{B}'$, both extending M' in such a way that $u_2v_3 \in E(H_1')$ and $u_3v_2 \in E(H_2')$.

In the first case, the following set of edges gives a Hamiltonian cycle of $\mathcal{B}_n$ which extends M and contains u_2v_0 and u_0v_3: $(E(H_1') - \{u_2v_3\}) \cup \{u_2v_0, v_0u_1, u_1v_1, v_1u_0, u_0v_3\}$.

In the second case, $(E(H_2') - \{u_3v_2\}) \cup \{u_3v_0, v_0u_1, u_1v_1, v_1u_0, u_0v_2\}$ is the edge set of a Hamiltonian cycle of $\mathcal{B}_n$ which extends M and contains u_3v_0 and u_0v_2. Hence, u_0 and v_0 are both 3-malleable. $\square$

We note that the 3-malleability of u_0 and v_0 cannot be proved by using Las Vergnas' Theorem [15], and that Theorem 2 in [18] can only be used for the cases when $n = 3$ or 4. Furthermore, the graph $\mathcal{B}_n$ can be turned into a non-bipartite non-regular graph which is not 2FH and admits a 3-malleable vertex by adding the edge $v_{n-1}v_n$.

Finally, we also remark that if a graph is 2FH, it does not mean that all its vertices are malleable (as in the case of even cycles and cubic graphs). An example of such a graph is $K_{3,3}$ with an edge e added between two vertices of the same partite set—we denote this graph by $K_{3,3} + e$. Since there is no perfect matching of $K_{3,3} + e$ which contains e, the graph $K_{3,3} + e$ is 2FH (since $K_{3,3}$ is 2FH). However, the endvertices of the edge e are not 4-malleable.

The reason why the construction of the graphs $\mathcal{B}_n$ and $\mathcal{Y}_{2n+1}$ was given is because the two classes of graphs contain 3-malleable vertices and so can be used in the general results proven in Section 3.2 to obtain PMH-graphs with arbitrarily large maximum degree by using star products.

3 Star products and PMH-graphs

In this section, we study what happens when we look at star products between PMH-graphs which are not necessarily bipartite and 2FH as in [8]. We find general ways how one can obtain PMH-graphs (not necessarily cubic) from smaller graphs by using star products. This is done by the help of malleable vertices. Although there is a clear connection between 2FH-graphs and PMH-graphs, an analogous result to Proposition 1 for PMH-graphs is not possible, as the following section on cubic graphs shows.

3.1 Cubic graphs revisited

Proposition 10. *Let G_1 and G_2 be two cubic graphs, and let $u \in V(G_1)$ and $v \in V(G_2)$.*

- *(i) If $G_1(u) * G_2(v)$ is* PMH, *then G_1 and G_2 are* PMH.
- *(ii) The converse of (i) is not true.*

Proof. (i) First assume that $G_1(u) * G_2(v)$ is PMH and let $X = \{u_1v_1, u_2v_2, u_3v_3\}$ be the principal 3-edge-cut of $G_1(u) * G_2(v)$, where u_1, u_2, u_3 are the neighbours of u in G_1, and v_1, v_2, v_3 are the neighbours of v in G_2. Let M be a perfect matching of G_1, and without loss of generality, assume that $u_1u \in M$. Let M' be a perfect matching of $G_1(u) * G_2(v)$ containing u_1v_1 and $M - \{u_1u\}$. We remark that such a perfect matching exists, since, in particular, every edge of a bridgeless cubic graph is contained in a perfect matching (see [17]). Furthermore, since $G_1(u) * G_2(v)$ is PMH, M' (and every other perfect matching of this graph) intersects X in exactly one edge, and there exists a Hamiltonian cycle H of $G_1(u) * G_2(v)$ extending M' and containing exactly one of the edges u_2v_2 and u_3v_3.

Assume $u_2v_2 \in E(H)$. This means that H induces a path in G_1 having end-vertices u_1 and u_2, passes through all the vertices in $V(G_1) - \{u\}$ and contains $M - \{u_1u\}$. This path together with the edges u_1u and u_2u forms a Hamiltonian cycle of G_1 extending M. Hence, G_1 is PMH, and by a similar reasoning, one can show that G_2 is also PMH.

(ii) Let G_1 and G_2 be two copies of the cube, and let $u \in V(G_1)$ and $v \in V(G_2)$. Both G_1 and G_2 are PMH (by [6]), but $G_1(u) * G_2(v)$ is not. In fact, consider the perfect matching of $G_1(u) * G_2(v)$ shown in Figure 4. One can clearly see that it cannot be extended to a Hamiltonian cycle. □

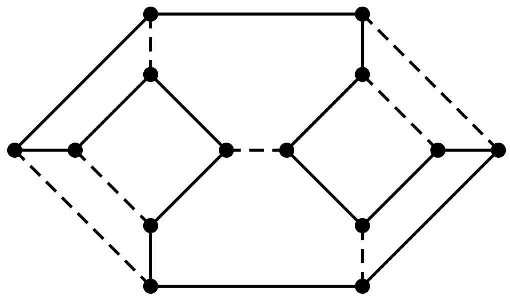

Figure 4: A star product between two copies of the cube. The dashed edges cannot be extended to a Hamiltonian cycle.

The second part of the above proof shows that unlike Proposition 1, a star product between two bipartite PMH-graphs does not guarantee that the resulting graph is PMH.

Corollary 11. *If G is a cubic* PMH-*graph having a* 3-*edge-cut, then G can be obtained by an appropriate star product between two cubic* PMH-*graphs G_1 and G_2.*

The above corollary (and also Conjecture 7) are the main reasons why in [2], the study of cubic graphs which are PMH or just even-2-factorable was restricted to graphs having girth at least 4. In [2], a graph G is defined to be *even-2-factorable* (for short E2F) if each of its 2-factors consist only of even cycles. When G is cubic, G is E2F if and only if each of its perfect matchings can be extended to a 3-edge-colouring (see Figure 5). We note that if a cubic graph is PMH then it is even-2-factorable as well, but the converse is not necessarily true.

As in Corollary 11, a cubic graph having girth 3 which is also even-2-factorable (not necessarily PMH), can be obtained by applying a star product between an even-2-factorable cubic graph and the complete graph K_4 (see [2] for more details). Applying a star product between a graph and K_4 is also known as applying a Y-extension, which can be seen as expanding a vertex into a triangle (see Figure 6). We remark that the results given in the sequel do not necessarily yield PMH-graphs having girth 3, as Remark 17 shows.

Despite the discouraging statement of Proposition 10, one can still obtain PMH-graphs from smaller PMH-graphs by using a star product (or repeated star products) and 3-malleable vertices, as we shall see in the following section.

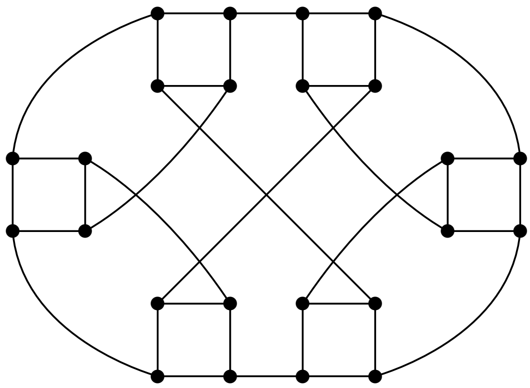

Figure 5: An example of a papillon graph: an even-2-factorable cubic graph given in [2].

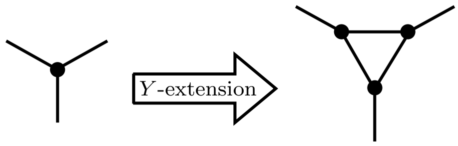

Figure 6: Y-extension.

3.2 Obtaining PMH-graphs from smaller graphs

Before proceeding we give the following definition. Following the notation in [16], an edge-cut in a graph G admitting a perfect matching is said to be *tight* if every perfect matching of G intersects it in exactly one edge (not necessarily the same).

Lemma 12. *Let G_1 be a* PMH*-graph admitting a vertex u of degree* 3 *and let G_2 be a graph admitting a* 3*-malleable vertex v. The principal* 3*-edge-cut of $G_1(u) * G_2(v)$ is tight if and only if $G_1(u) * G_2(v)$ is* PMH.

Proof. Since G_1 and G_2 are both PMH-graphs, $G_1 - u$ and $G_2 - v$ are both of odd order. This implies that a perfect matching of $G_1(u)*G_2(v)$ cannot intersect its principal 3-edge-cut in 2 edges. Hence, if $G_1(u)*G_2(v)$ is PMH, its principal 3-edge-cut is tight, as required. Consequently, it suffices to prove the forward direction. Let $X = \{u_1v_1, u_2v_2, u_3v_3\}$ be the principal 3-edge-cut of $G_1(u) * G_2(v)$, and assume that X is tight. Let M be a perfect matching of $G_1(u) * G_2(v)$, and let u_1, u_2, u_3 and v_1, v_2, v_3 be the neighbours of $u \in V(G_1)$ and $v \in V(G_2)$, respectively. Without loss of generality, assume that $M \cap X = \{u_1v_1\}$. Consequently, M respectively induces perfect matchings M_1 and M_2 in G_1 and G_2, such that $u_1u \in M_1 \subset E(G_1)$, $v_1v \in M_2 \subset E(G_2)$, $M_1 - \{u_1u\} \subset M$ and $M_2 - \{v_1v\} \subset M$. Since G_1 is PMH, M_1 can be extended to a Hamiltonian cycle H_1 of G_1. Without loss of generality, we assume that $u_2u \in E(H_1)$. Since v is a 3-malleable vertex, M_2 can be extended to a Hamiltonian cycle H_2 of G_2 whose edge set intersects v_2v. Consequently, $(E(H_1) - \{u_1u, u_2u\}) \cup (E(H_2) - \{v_1v, v_2v\}) \cup \{u_1v_1, u_2v_2\}$ is a Hamiltonian cycle of $G_1(u) * G_2(v)$ extending M, as required. $\square$

This lemma shall be needed in the sequel when considering star products between PMH-graphs. The graph G_1 (similarly G_2) in Lemma 12 can either be bipartite or not, and in what follows we shall consider star products in two instances:

(i) between PMH-graphs with at least one being non-bipartite (Section 3.2.1); and

(ii) between non-bipartite PMH-graphs (Section 3.2.2).

We then finish this section with some examples of cubic PMH-graphs having small order (see Section 3.2.3).

3.2.1 Star products between PMH-graphs with at least one being bipartite

Whilst a star product between two bipartite 2FH-graphs yields a 2FH-graph, Figure 4 shows that a star product between two bipartite PMH-graphs is not necessarily PMH. The example given in the figure is a star product between two copies of $\mathcal{Q}_3$, where the graph $\mathcal{Q}_3$ is itself PMH, but does not admit any 3-malleable vertex. The following proposition shows that the presence of a 3-malleable vertex in at least one of the two graphs between which a star product is applied guarantees the PMH-property in the resulting graph, given that at least one of the two initial graphs is bipartite.

Proposition 13. *Let G_1 be a* PMH-*graph admitting a vertex u of degree* 3 *and let G_2 be a graph admitting a* 3-*malleable vertex v. If at least one of G_1 and G_2 is bipartite, then $G_1(u) * G_2(v)$ is* PMH.

Proof. Since at least one of G_1 and G_2 is bipartite, the principal 3-edge-cut of $G_1(u)*G_2(v)$ is tight. The result follows by Lemma 12. □

Corollary 14. *Let G_1 and G_2 be two bipartite graphs having the* PMH-*property such that u is a vertex of degree* 3 *in G_1 and v is a* 3-*malleable vertex in G_2. Then, $G_1(u) * G_2(v)$ is a bipartite* PMH-*graph.*

We can extend the above corollary further. Let G_0 be a bipartite PMH-graph admitting 2 vertices of degree 3, say u_1 and u_2. Furthermore, let G_1 and G_2 be bipartite PMH-graphs each admitting a 3-malleable vertex, say $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. By the previous corollary, $G_0(u_1) * G_1(v_1)$ is PMH. This graph is also bipartite, and so, reapplying a star product on the vertex corresponding to u_2 in $G_0(u_1) * G_1(v_1)$ and the vertex v_2 in G_2 gives a bipartite PMH-graph once again (by Corollary 14). For simplicity, we shall say that the resulting graph has been obtained by applying a star product on u_i and v_i, for each $i \in \{1, 2\}$. By repeating this argument we can state the following more general result.

Theorem 15. *Let G_0 be a bipartite* PMH-*graph admitting t vertices of degree* 3, *for some $t \in \{1, \ldots, |V(G_0)|\}$, say $u_1, \ldots, u_t$. Let $\mathcal{I} \subseteq \{i : deg(u_i) = 3\}$. For each $i \in \mathcal{I}$, let G_i be a bipartite graph admitting a* 3-*malleable vertex v_i. The bipartite graph obtained by applying a star product on u_i and v_i, for each $i \in \mathcal{I}$, is* PMH.

We remark that Theorem 15 is best possible, in the sense that we cannot exchange the roles of the u_is and the v_is. In fact, if we assume that the t vertices $u_1, \ldots, u_t$ of G_0 are 3-malleable, and that, for each $i \in \mathcal{I}$, the graphs G_i are PMH-graphs with the vertex v_i being just a degree 3 vertex (and not 3-malleable), the same conclusion about the resulting graph cannot be obtained, as the following example in the class of cubic graphs shows. Let G_0 be the graph $K_{3,3}$, and let G_1 and G_2 be two copies of the graph $\mathcal{Q}_3$. Let u_1 and u_2 be two vertices in G_0 belonging to the same partite set, and let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. By Corollary 14, $G_0(u_1) * G_1(v_1)$ is PMH. However, reapplying a star product on the vertex corresponding to u_2 in $G_0(u_1) * G_1(v_1)$ and the vertex v_2 of G_2 (that is, the final graph is obtained by applying a star product on u_i and v_i, for each $i \in \{1, 2\}$) does not yield a PMH-graph. Indeed, the dashed perfect matching portrayed in Figure 7 cannot be extended to a Hamiltonian cycle.

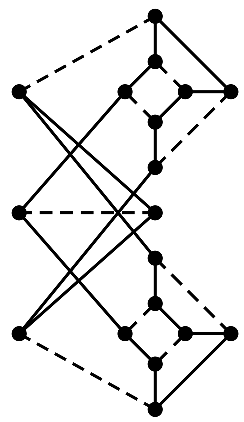

Figure 7: The dashed edges cannot be extended to a Hamiltonian cycle. The above graph is $(K_{3,3} * \mathcal{Q}_3) * \mathcal{Q}_3$.

We thus move onto the next section and look at a star product between two non-bipartite PMH-graphs.

3.2.2 Star products between non-bipartite PMH-graphs

In Section 3.2.1, Proposition 13 already tells us that a star product between two PMH-graphs with exactly one being bipartite results in a PMH-graph (given that one of them admits a 3-malleable vertex). But what happens when both are non-bipartite? As already stated before, the graph obtained after applying a star product between two copies of the complete graph K_4 is not PMH, even though the graphs we started with, that is, the two copies of K_4, are both PMH. Given that K_4 is also 2FH, the previous example tells us that the presence of 3-malleable vertices alone does not guarantee the PMH-property in the resulting graph when both the PMH-graphs we start with are non-bipartite. In order to attain a general result about PMH-graphs obtained by applying a star product between two non-bipartite PMH-graphs, we extend Proposition 13 in a similar way as in Theorem 15.

Theorem 16. *Let G_0 be a bipartite* PMH*-graph of order $2n$ and with bipartition $U = \{u_i : i \in [n]\}$ and $V = \{v_i : i \in [n]\}$, for some $n > 1$. Let $\mathcal{I} \subseteq \{i : deg(v_i) = 3\}$. For each $i \in \mathcal{I}$, let G_i be a graph admitting a 3-malleable vertex z_i. The resulting graph G obtained by applying a star product on v_i and z_i, for each $i \in \mathcal{I}$, is* PMH.

Proof. For each $i \in \mathcal{I}$, let X_i be the (principal) 3-edge-cut of G arising from a star product on v_i and z_i. This means that if $e \in X_i$, then one of the endvertices of e belongs to U and the other endvertex belongs to $V(G_i - z_i)$. If $\mathcal{I} = \emptyset$, then G is equal to G_0, and consequently, G is PMH. So we can assume that $\mathcal{I} \neq \emptyset$. Let M be a perfect matching of G, and let $E_{\mathcal{I}}$ be the collection of edges in M having one endvertex in U and one endvertex in some $V(G_i - z_i)$, for $i \in \mathcal{I}$. Since G_0 is a bipartite PMH-graph, each X_i is tight, and so, $|M \cap X_i| = 1$ for each $i \in \mathcal{I}$. Consequently, $|E_{\mathcal{I}}| = |\mathcal{I}|$. Let $E_{\mathcal{I}} = \{e_i : i \in \mathcal{I}\}$, such that for each $i \in \mathcal{I}$, $e_i = x_iy_i$ for some $x_i \in U$ and $y_i \in V(G_i - z_i)$. Moreover, for every $i \in \mathcal{I}$, let $f_i = x_iv_i$. The set of edges $M_0 = \{f_i : i \in \mathcal{I}\} \cup M - (E_{\mathcal{I}} \bigcup \cup_{i\in\mathcal{I}} E(G_i - z_i))$ is a perfect matching of G_0, and since G_0 is PMH, there exists a Hamiltonian cycle H_0 of G_0 extending M_0. Without loss of generality, assume that H_0 is equal to $(u_1, v_1, u_2, \ldots, v_n)$, where u_2 is followed by v_2, and v_n is preceded by u_n. Without loss of generality, assume further that $M_0 = \{u_iv_i : i \in [n]\}$. In particular, this implies that for each $i \in \mathcal{I}$, $x_i = u_i$.

Before continuing, we remark that operations in the indices of the vertices u_i are taken modulo n, with complete residue system $\{1, \ldots, n\}$. Let $j \in \mathcal{I}$, and let the neighbours of u_j and u_{j+1} belonging to $V(G_j - z_j)$ be α_j and ω_j. We note that α_j is equal to what we previously denoted by y_j. Given that z_j is a 3-malleable vertex of G_j, there exists a Hamiltonian cycle H_j of G_j extending the perfect matching $(M \cap E(G_j - z_j)) \cup \{z_j\alpha_j\}$, such that $z_j\alpha_j$ and $z_j\omega_j$ belong to $E(H_j)$. Let P_j be the path obtained after deleting the vertex z_j from the cycle H_j. This process is repeated for every other integer in $\mathcal{I}$. We note that α_j and ω_j are the endvertices of P_j, and, in particular, by our assumption on M_0, we have $u_j\alpha_j \in M$, for every $j \in \mathcal{I}$. By recalling that the edge set of the Hamiltonian cycle H_0 is $\cup_{i=1}^{n}\{u_iv_i, v_iu_{i+1}\}$ and the above considerations, one can deduce that the following edge set induces a Hamiltonian cycle of G extending M:

$$\cup_{i\in[n]-\mathcal{I}}\{u_iv_i, v_iu_{i+1}\} \bigcup \cup_{j\in\mathcal{I}} \left(\{u_j\alpha_j, \omega_ju_{j+1}\} \cup E(P_j)\right),$$

as required. □

When $|\mathcal{I}| > 1$, say $|I| = 2$, and G_1 and G_2 are chosen to be non-bipartite, the above theorem shows that there do exist non-bipartite graphs such that when a star product is applied between them, the resulting graph is PMH. This follows because the graph obtained after appropriately applying a star product between G_0 and G_1 is non-bipartite.

We also remark that the reason why we cannot apply a star product on two adjacent vertices in G_0 is because the resulting graph is not necessarily PMH, as Figure 8 shows. In fact, applying a Y-extension to two adjacent vertices of $K_{3,3}$ results in a graph which does not have the PMH-property. Recall that Y-extensions can be explained in terms of a star product between a graph and K_4.

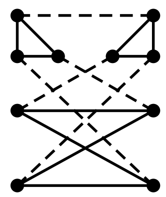

Figure 8: The dashed edges cannot be extended to a Hamiltonian cycle.

3.2.3 Examples of cubic PMH-graphs having small order

In [2], it is stated that the papillon graph on 8 vertices is the smallest (with respect to the number of vertices) non-bipartite cubic graph with girth at least 4 which is even-2-factorable (E2F) and not PMH (recall that if a cubic graph is PMH then it is E2F). However, in what follows we present other non-bipartite cubic graphs on 4, 6, or 8 vertices which are PMH, and consequently E2F—these have girth strictly less than 4. Apart from K_4, there is another cubic graph on 4 vertices which is PMH, in particular, let G be the unique bipartite cubic graph on 4 vertices (see Figure 9). By using the procedure outlined in Theorem 16, with $G_0 = G$, $\mathcal{I} = \{1, 2\}$ and $G_1 = G_2 = K_4$, we obtain a non-bipartite cubic PMH-graph on 8 vertices. This is equivalent to applying a Y-extension to two vertices belonging to the same partite set of G. Note that applying a Y-extension to a single vertex in G also gives a PMH-graph, which is the unique non-bipartite cubic PMH-graph on 6 vertices (the second graph in Figure 9).

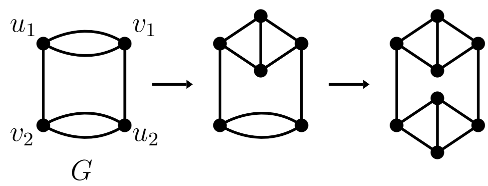

Figure 9: Using Theorem 16 to obtain non-bipartite cubic PMH-graphs.

Having said this, the graphs obtained by the methods given in the previous sections do not necessarily have girth 3, as the following remark shows.

Remark 17. An easy way to obtain cubic PMH-graphs with girth at least 4 is the following. Let F be the graph obtained by applying a Y-extension to a bipartite cubic 2FH-graph F_0 (see, for example, Figure 10). Let G be a bipartite cubic PMH-graph having no multiedges and let v be a vertex of F lying on its triangle. Since F is a cubic 2FH-graph, v is 3-malleable, and so, for any $u \in V(G)$, the graph $G(u) * F(v)$ is PMH by Proposition 13. Moreover, $G(u) * F(v)$ has girth 4. In fact, G and F_0 do not have any multiedges and, since they are both bipartite, a cycle of length 3 in $G(u) * F(v)$ can only occur if the edges of the cycle intersect (twice) the principal 3-edge-cut of $G(u) * F(v)$, which is impossible. Graphs obtained using this method are not necessarily 2FH. In fact, by letting $G = Q_3$ and $F_0 = K_{3,3}$, the resulting graph depicted in Figure 11 is not 2FH, since the complementary 2-factor of the dashed perfect matching does not form a Hamiltonian cycle.

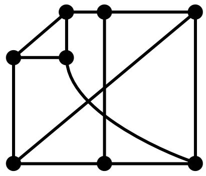

Figure 10: Applying a Y-extension to $F_0 = K_{3,3}$ from Remark 17.

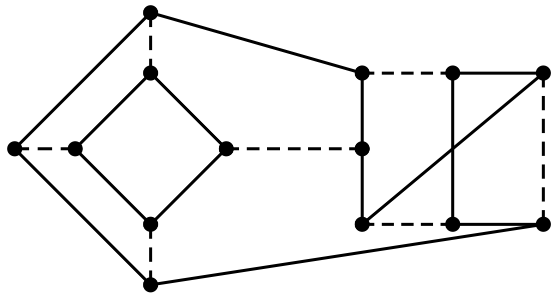

Figure 11: A non-bipartite cubic PMH-graph having girth 4 which is not 2FH.

Although the above examples are cubic, we recall that the results in Section 3.2 can generate PMH-graphs (both bipartite and non-bipartite) with arbitrarily large maximum degree. This can be done by using graphs admitting a 3-malleable vertex as the ones portrayed after the proof of Theorem 4 in Section 1.

We also remark that despite the encouraging general methods obtained above, there are PMH-graphs admitting a 3-edge-cut, that is, obtained by using a star product (see Corollary 11), which cannot be described by the methods portrayed so far. Such an example is given in Figure 12. The graph denoted by $G_1 * G_2$ (obtained by an appropriate star product on $u \in V(G_1)$ and $v \in V(G_2)$) is PMH. The graphs G_1 (bipartite) and G_2 (non-bipartite) are both PMH-graphs as well, however, G_1 and G_2 do not have any 3-malleable vertices and so, the reason why the resultant graph is PMH is not because of the above results, in particular, Theorem 16 (see also Lemma 12). The three graphs

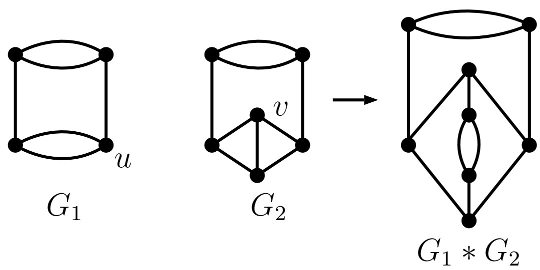

Figure 12: The graphs G_1, G_2 and $G_1 * G_2$ are all PMH.

given in Figure 12 contain 2-edge-cuts which we discuss in the next section with regards to PMH- and 2FH-graphs. In particular, although the PMH-property in the third graph given in Figure 12 cannot be explained by the previous theorems dealing with the star product, the reason behind it being PMH can be explained by Theorem 18 which gives a necessary and sufficient condition for a graph admitting a 2-edge-cut to be PMH.

4 2-edge-cuts in PMH- and 2FH-graphs

Let G_1 and G_2 be two graphs (not necessarily regular), and let e_1 and e_2 be two edges such that $e_1 = x_1y_1 \in E(G_1)$ and $e_2 = x_2y_2 \in E(G_2)$. A 2*-cut connection* on e_1 and e_2 is a graph operation that consists of constructing the new graph $(G_1 - e_1) \cup (G_2 - e_2) \cup \{x_1x_2, y_1y_2\}$, and denoted by $G_1(x_1y_1)\#G_2(x_2y_2)$. The 2-edge-cut $\{x_1x_2, y_1y_2\}$ is referred to as the *principal* 2*-edge-cut* of the resulting graph. It is clear that another possible graph obtained by a 2-cut connection on e_1 and e_2 is $G_1(x_1y_1)\#G_2(y_2x_2)$. Unless otherwise stated, if it is not important which of these two graphs is obtained, we use the notation $G_1(e_1)\#G_2(e_2)$ and we say that it is a graph obtained by a 2-cut connection on e_1 and e_2. As in the case of star products, when this occurs, we say that the resulting graph has been obtained by applying a 2-cut connection *between* G_1 and G_2. Given that graphs on an odd number of vertices do not admit a perfect matching, they cannot be studied with regards to the PMH-property. Since we shall be looking at the PMH-property of $G_1(e_1)\#G_2(e_2)$, $|V(G_1)|$ and $|V(G_2)|$ are either both odd or both even. This is the reason why in the next theorem we shall assume that the graphs G_1 and G_2 are both of even order, as we are not only interested in whether $G_1(e_1)\#G_2(e_2)$ admits the PMH-property, but also whether G_1 and G_2 admit it. As we shall see, G_1 and G_2 will have a stronger property to guarantee the PMH-property in $G_1(e_1)\#G_2(e_2)$.

Theorem 18. *Let $G = G_1(e_1)\#G_2(e_2)$ be a graph obtained by applying a* 2*-cut connection on $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, such that G_1 and G_2 both admit a perfect matching. Then, G is* PMH *if and only if, for each $i \in \{1, 2\}$, every perfect matching in G_i can be extended to a Hamiltonian cycle of G_i which contains e_i.*

Proof. ($\Rightarrow$) Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be such that the principal 2-edge-cut of G is $X = \{x_1x_2, y_1y_2\}$, and let M_1 be a perfect matching of G_1. Since G is PMH, G contains a perfect matching M, such that $M_1 - \{e_1\} \subset M$, and, in particular, there exists a perfect matching N of G such that $M \cup N$ gives a Hamiltonian cycle of G. If $e_1 \in M_1$, then $X \subset M$ and $N \cap X = \emptyset$. Consequently, the set of edges $N_1 = N \cap E(G_1)$ is a perfect matching of G_1, and $M_1 \cup N_1$ gives a Hamiltonian cycle of G_1 containing e_1. Otherwise, if $e_1 \notin M_1$, then $M \cap X = \emptyset$ and $X \subset N$. Consequently, the set of edges $N_1 = (N \cap E(G_1)) \cup \{e_1\}$ is a perfect matching of G_1, and $M_1 \cup N_1$ gives a Hamiltonian cycle of G_1 containing e_1, once again. By a similar argument one can show that every perfect matching of G_2 can be extended to a Hamiltonian cycle of G_2 containing e_2.

($\Leftarrow$) Conversely, assume that M is a perfect matching of G. Notwithstanding whether M contains the edges in $X = \{x_1x_2, y_1y_2\}$ or not, M induces two perfect matchings $M_1 \in E(G_1)$ and $M_2 \in E(G_2)$ such that $M_i - \{e_i\} \subset M$, for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. Note that $M \cap X = X$ if and only if $e_i \in M_i$. By our assumption, G_i admits a Hamiltonian cycle H_i which extends M_i and contains e_i, and so, since G_i is of even order, it admits a perfect matching N_i such that $M_i \cup N_i = E(H_i)$. Consequently, the edge set $M_1 \cup N_1 \cup M_2 \cup N_2 \cup X - \{e_1, e_2\}$ gives a Hamiltonian cycle of G containing M. Thus, G is PMH as required. □

The above theorem explains the reason behind the PMH-property in all the three graphs shown in Figure 12, not only the third. The graph G_1 is obtained by applying a 2-cut connection between two copies of the cubic graph on two vertices. The graph G_2 is obtained by applying a 2-cut connection between the cubic graph on two vertices and the graph K_4. The third graph denoted by $G_1 * G_2$ in Figure 12 is obtained by applying an appropriate 2-cut connection between the cubic graph on two vertices and the graph G_2.

We also note that the condition in Theorem 18 that every perfect matching in G_i has to be extended to a Hamiltonian cycle of G_i containing the edge e_i is required because, for example, a 2-cut connection between the cube $\mathcal{Q}_3$ (which is PMH) and any other appropriate PMH-graph does not yield a PMH-graph, since any perfect matching of the resulting graph containing the dashed edges cannot be extended to a Hamiltonian cycle, as can be seen in Figure 13.

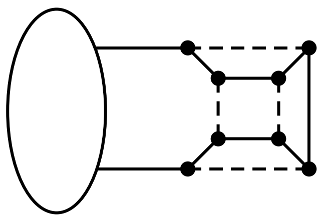

Figure 13: A perfect matching containing the dashed edges cannot be extended to a Hamiltonian cycle.

Next, we give a similar result to Theorem 18 but for 2FH-graphs. Before proceeding we note that, in general, if every 2-factor of a graph G is a Hamiltonian cycle containing a particular edge $e \in E(G)$, then every 2-factor of G containing e is a Hamiltonian cycle. However, the converse of this statement is not necessarily true, because G can admit 2-factors which do not contain the edge e. An example of such a graph is $K_{3,3}$.

Theorem 19. *Let G_1 and G_2 be two Hamiltonian graphs such that $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. The graph $G = G_1(e_1)\#G_2(e_2)$ is* 2FH *if and only if every 2-factor of G_1 is a Hamiltonian cycle containing e_1 and every 2-factor of G_2 containing e_2 is a Hamiltonian cycle (or vice-versa).*

Proof. ($\Rightarrow$) Let the principal 2-edge-cut of G be X. Since G is 2FH, we cannot have that both G_1 and G_2 admit a 2-factor F_1 and F_2, respectively, such that $e_1 \notin E(F_1)$ and $e_2 \notin E(F_2)$, because otherwise, F_1 together with F_2 would form a 2-factor of G which does not intersect its principal 2-edge-cut, and so is not a Hamiltonian cycle. Therefore, without loss of generality, we can assume that e_1 is in every 2-factor of G_1. Let F' be a 2-factor of G_1, and let F'' be a 2-factor of G_2 which contains e_2. These 2-factors do exist since G is 2FH. Clearly, $F' \cup F'' \cup X - \{e_1, e_2\}$ is a 2-factor of G, and since G is

2FH, $F' \cup F'' \cup X - \{e_1, e_2\}$ is a Hamiltonian cycle of G. Consequently, F' and F'' are Hamiltonian cycles of G_1 and G_2, respectively.

($\Leftarrow$) For each $i \in \{1,2\}$, let $e_i = x_iy_i$. Consider a 2-factor F of G. Due to the condition on G_1, any 2-factor of G must contain the principal 2-edge-cut of G, because otherwise, this would create a 2-factor in G_1 not containing the edge e_1, a contradiction. Thus, $X \subset E(F)$. By our assumptions on G_1 and G_2, it follows that $(F \cap G_i) \cup \{e_i\}$ is a (connected) 2-factor of G_i, for each $i \in \{1,2\}$. Consequently, for each $i \in \{1,2\}$, $F \cap G_i$ is a Hamiltonian path of G_i with endvertices x_i and y_i, implying that F is a Hamiltonian cycle of G. $\square$

We first note that the graph G_2 in the above theorem is not necessarily 2FH, and can admit a 2-factor which is not a Hamiltonian cycle. In fact, there exist 2FH-graphs which are obtained by applying a 2-cut connection between graphs which are not both 2FH. Let G_1 be the cycle on four vertices, and let e_1 be one of its edges. Let G_2 be the graph obtained by applying a star product between two copies of K_4, and let e_2 be one of the edges of the principal 3-edge-cut. The graph G_1 is 2FH, whilst G_2 is not, as already stated above. However, $G_1(e_1)\#G_2(e_2)$ is still 2FH (see Figure 14).

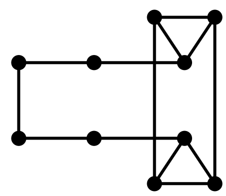

Figure 14: A 2FH-graph arising from a 2-cut connection between two graphs one of which is not 2FH.

We further remark that, in the above theorem, the graphs G_1 and G_2 cannot both be just 2FH-graphs without any further properties, because a 2-cut connection between two copies of $K_{3,3}$ is not 2FH. Moreover, the condition on G_1 (that is, every 2-factor of G_1 is a Hamiltonian cycle containing e_1) cannot be relaxed to be equivalent to the condition on G_2 (that is, every 2-factor of G_2 containing e_2 is a Hamiltonian cycle). In fact, let G_1 and G_2 be two copies of the graph obtained by applying a star product between two copies of K_4, and let e_1 and e_2 be one of the edges of the principal 3-edge-cut of G_1 and G_2, respectively. For each $i \in \{1,2\}$, every 2-factor of G_i containing e_i is a Hamiltonian cycle of G_i, however, the graph $G_1(e_1)\#G_2(e_2)$, portrayed in Figure 15, is not 2FH.

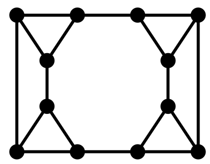

Figure 15: A graph arising from a 2-cut connection which is not 2FH.

Acknowledgements

The authors would like to thank Marién Abreu, John Baptist Gauci, Domenico Labbate and Giuseppe Mazzuoccolo for useful discussions about the topic.

References

[1] M. Abreu, J.B. Gauci, D. Labbate, G. Mazzuoccolo and J.P. Zerafa, Extending perfect matchings to Hamiltonian cycles in line graphs, *Electron. J. Combin.* **28(1)** (2021), #P1.7.

[2] M. Abreu, J.B. Gauci, D. Labbate, F. Romaniello and J.P. Zerafa, Perfect matchings, Hamiltonian cycles and edge-colourings in a class of cubic graphs, *Ars Math. Contemp.*, **23** (2023), #P3.01

[3] M. Abreu, J.B. Gauci and J.P. Zerafa, *Saved by the rook: a case of matchings and Hamiltonian cycles*, submitted, arXiv:2104.01578.

[4] A. Alahmadi, R.E.L. Aldred, A. Alkenani, R. Hijazi, P. Solé and C. Thomassen, Extending a perfect matching to a Hamiltonian cycle, *Discrete Math. Theor. Comput. Sci.*, **17(1)** (2015), 241–254.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Grad. Texts in Math. 244. Springer-Verlag London, 2008.

[6] J. Fink, Perfect matchings extend to Hamilton cycles in hypercubes, *J. Combin. Theory Ser. B*, **97** (2007), 1074–1076.

[7] J.L. Fouquet and J.M. Vanherpe, On the perfect matching index of bridgeless cubic graphs, (2009), hal-00374313.

[8] M. Funk, B. Jackson, D. Labbate and J. Sheehan, 2-Factor hamiltonian graphs, *J. Combin. Theory Ser. B* **87** (2003), 138–144.

[9] J.B. Gauci and J.P. Zerafa, Accordion graphs: Hamiltonicity, matchings and isomorphism with quartic circulants, *Discrete Appl. Math.* **321** (2022), 126–137.

[10] J.B. Gauci and J.P. Zerafa, Perfect Matchings and Hamiltonicity in the Cartesian Product of Cycles, *Ann. Comb.* **25** (2021), 789–796.

[11] R. Häggkvist, On F-Hamiltonian graphs, in: J.A. Bondy, U.S.R. Murty (eds.), *Graph Theory and Related Topics*, Academic Press, New York, 1979, 219–231.

[12] G. Kreweras, Matchings and Hamiltonian cycles on hypercubes, *Bull. Inst. Combin. Appl.* **16** (1996), 87–91.

[13] D. Labbate, Characterizing minimally 1-factorable r-regular bipartite graphs, *Discrete Math.* **248** (2002), 109–123.

[14] D. Labbate, On 3-cut reductions of minimally 1-factorable cubic bigraphs, *Discrete Math.* **231** (2001), 303–310.

[15] M. Las Vergnas, *Problèmes de couplages et problèmes hamiltoniens en théorie des graphes*, Thesis, University of Paris 6, Paris, 1972.

[16] L. Lovász, Matching structure and the matching lattice, *J. Combin. Theory Ser. B* **43** (1987), 187–222.

[17] T. Schönberger, Ein Beweis des Petersenschen Graphensatzes, *Acta Sci. Math. (Szeged)* **7** (1934), 51–57.

[18] Z. Yang, On F-Hamiltonian graphs, *Discrete Math.* **196** (1999), 281–286.