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ON THE CONSUMMATE AFFAIRS OF PERFECT MATCHINGS

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Per Anam… il senzanome

Caminante, son tus huellas
el camino y nada más;
Caminante, no hay camino,
se hace camino al andar.

—ANTONIO MACHADO

ABSTRACT

Snarks and *Hamiltonicity*: two prominent areas of research in graph theory. As the title of the thesis suggests, here we study how perfect matchings behave together, more precisely, their union and intersection, in each of these two settings.

Snarks, which for us represent Class II bridgeless cubic graphs, are crucial when considering conjectures about bridgeless cubic graphs, and, if such statements are true for snarks, then they would be true for all bridgeless cubic graphs. One such conjecture which is known for its simple statement, but still indomitable after half a century, is the Berge–Fulkerson Conjecture which states that every bridgeless cubic graph G admits six perfect matchings such that every edge in G is contained in exactly two of these six perfect matchings. In this thesis we study two other related and well-known conjectures about bridgeless cubic graphs, both consequences of the Berge–Fulkerson Conjecture which are still very much open: the Fan–Raspaud Conjecture (Fan and Raspaud, 1994) and the S_4-Conjecture (Mazzuoccolo, 2013), dealing with the intersection of three perfect matchings, and the complement of the union of two perfect matchings, respectively. We give an equivalent formulation of the Fan–Raspaud Conjecture which at first glance seems stronger, and show that the S_4-Conjecture is true for bridgeless cubic graphs having oddness at most 4. We also show that the S_4-Conjecture can be stated in terms of a variant of Petersen-colourings, and discuss it in relation to bridgeless cubic multigraphs and certain cubic multigraphs having bridges.

Given the obstacles encountered when dealing with such problems, many have considered trying to bridge the gap between Class I and Class II bridgeless cubic graphs by looking at invariants that measure how far Class II bridgeless cubic graphs are from being Class I. This is done in an attempt to further refine the class of snarks, and thus, enlarging the set of cubic graphs for which such conjectures can be verified. In this spirit we consider a parameter which gives the least number of perfect matchings (not necessarily distinct) needed to be added to a bridgeless cubic graph such that the resulting multigraph is Class I. We show that the Petersen graph is, in some sense, the only obstruction for a bridgeless cubic graph to have a finite value for the parameter studied. We also relate this parameter to already well-studied concepts: the excessive index, and the length of a shortest cycle cover of a bridgeless cubic graph. In particular, we show that an infinite family of non-trivial snarks, a generalisation of treelike snarks, have a shortest cycle cover with length strictly greater than $4/3$ their size. This is done in the first part of the thesis.

In the second part, we study a concept about *Hamiltonicity* first considered in the 1970s by Las Vergnas and Häggkvist, which was generalised and recently brought to the limelight again by Fink (2007). In this part we look at Hamiltonian circuits in graphs having an even order, which is a necessary condition for a graph to admit a perfect matching. In such graphs, a Hamiltonian circuit can be seen as the union of two perfect matchings. If every perfect matching of a graph G extends to a Hamiltonian circuit, we say that G has the Perfect-Matching-Hamiltonian property (for short the PMH-property). A somewhat stronger property than the PMH-property is the following: a graph G has the Pairing-Hamiltonian property if every pairing of G (that is, a perfect matching of the complete graph having the same vertex set as G) can be extended to a Hamiltonian circuit of the underlying complete graph using only edges from G, that is, by using a perfect matching of G.

A characterisation of all the cubic graphs having the PH-property was done by Alahmadi et al. (2015), and the same authors attempt to answer a most natural question, that of characterising all 4-regular graphs having the same property. They do this by posing the following problem: for which values of p and q does the Cartesian product $C_p \square C_q$ of two circuits on p and q vertices have the PH-property? We show that this only happens when both p and q are equal to four, namely for $C_4 \square C_4$, the 4-dimensional hypercube. We continue this study of quartic graphs in relation to the above properties by proposing a class of quartic graphs on two parameters, accordion graphs, a class which we believe is a rich one in this sense. A complete characterisation of which accordion graphs are circulant is also given.

Hamiltonicity was also heavily studied with respect to line graphs by Kotzig (1964), Harary and Nash-Williams (1965), and Thomassen (1986), amongst others, and along the same lines, we give sufficient conditions for a graph in order to guarantee the PMH-property in its line graph. We do this for subcubic graphs, complete graphs, and arbitrary traceable graphs. Moreover, we also give a complete characterisation of which line graphs of complete bipartite graphs admit, not only the PMH-property, but the PH-property.

ABSTRACT (ITALIAN VERSION)

Titolo: Sulle ineccepibili relazioni tra i matching perfetti

Snarks e *Hamiltonicità* sono due rilevanti aree di ricerca nella Teoria dei Grafi. Come suggerisce il titolo, in questa tesi studieremo come interagiscono tra loro i matching perfetti di un grafo, in particolare ci occuperemo di studiare la loro unione e intersezione nei due ambiti sopra indicati.

Gli *snarks*, grafi cubici privi di ponti e di Classe II, sono oggetti cruciali quando si considerano congetture su grafi cubici senza ponti, perchè, tipicamente, se una congettura viene verificata per gli snarks allora è valida in generale. Una di queste congetture, nota per il suo enunciato particolarmente semplice ma completamente aperta dopo oltre mezzo secolo, è la Congettura di Berge–Fulkerson. Nella prima parte di questa tesi studieremo altre due ben note congetture entrambe conseguenze della congettura di Berge–Fulkerson e che sono anche esse ancora aperte: la Congettura di Fan–Raspaud (Fan & Raspaud 1994) e la Congettura S_4 (Mazzuoccolo 2013). La prima riguarda il comportamento dell'intersezione di tre matching perfetti e l'altra il complemento dell'unione di due matching perfetti. Diamo una formulazione equivalente della Congettura di Fan–Raspaud che in letteratura appariva essere una versione più forte, e mostriamo che la Congettura S_4 è vera per grafi cubici senza ponti di oddness al più 4.

A causa degli ostacoli che si incontrano nello studiare tali tipi di problemi, sono stati fatti molti tentativi di colmare il gap tra grafi cubici di Classe I e di Classe II introducendo invarianti che misurino quanto un grafo di Classe II è lontano dall'essere di Classe I. In questo spirito, proponiamo di considerare il minimo numero di matching perfetti (non necessariamente distinti) che è necessario aggiungere a un grafo cubico senza ponti per ottenere un multigrafo di Classe I. Dimostriamo che il grafo di Petersen è sostanzialmente l'unica ostruzione per questo problema, nel senso che non ammette un numero finito di matching perfetti con la proprietà richiesta. Inoltre, colleghiamo lo studio di questo problema ad altri ben noti: l'excessive index e la lunghezza della più corta copertura in cicli.

Nella seconda parte della tesi, studiamo un problema legato all'*Hamiltonicità* di un grafo, già introdotto negli anni settanta da Las Vergnas e Häggkvist, e poi generalizzato più di recente da Fink (2007). Ci riferiamo a grafi Hamiltoniani con un numero pari di vertici (condizione necessaria per avere un matching perfetto): in tali grafi, un ciclo Hamiltoniano lo si può vedere come unione di due matching

perfetti. Diremo che G ha la Perfect-Matching-Hamiltonian property (in breve la PMH-property) se ogni suo matching perfetto si può estendere a un ciclo Hamiltoniano. Una proprietà ancora più forte è la seguente: un grafo G ha la Pairing-Hamiltonian property (in breve la PH-property) se ogni pairing di G (cioè un matching perfetto del grafo completo definito sugli stessi vertici di G) può essere esteso a un ciclo Hamiltoniano del grafo completo soggiacente usando un matching perfetto di G. Una caratterizzazione dei grafi cubici con la PH-property è stata fornita da Alahmadi e al. (2015). Gli stessi autori hanno solo parzialmente tentato una caratterizzazione anche dei grafi 4-regolari con la stessa proprietà. Noi risolviamo uno dei problemi da loro proposti e mostriamo una famiglia di grafi 4-regolari, che chiameremo accordion, che riteniamo interessante in quest'ambito.

Le proprietà di Hamiltonicità sono state ampiamente studiate anche per i line-graphs da, tra gli altri, Kotzig (1964), Harary & Nash-Williams (1965) e Thomassen (1986). In questa linea di ricerca diamo condizioni sufficienti per un grafo che garantiscano la PMH-property per il suo line-graph. Otteniamo tali risultati per grafi di grado al più 3, grafi completi e grafi arbitrarily traceable. Infine otteniamo una caratterizzazione completa dei line graphs dei grafi bipartiti completi che ammettono la PH-property.

LIST OF PAPERS

This thesis is based on the following papers, referred to later on by their respective roman numeral. Each of the papers is presented in a different chapter, and at the beginning of such chapters the reader is informed accordingly. We tacitly assume that statements in such chapters having no reference, are statements taken from the respective paper.

[I] G. Mazzuoccolo and J.P. Zerafa, An equivalent formulation of the Fan–Raspaud Conjecture and related problems, *Ars Math. Contemp.* **18(1)** (2020), 87–103, [DOI].

[II] E. Máčajová, G. Mazzuoccolo, V. Mkrtchyan and J.P. Zerafa, Some snarks are worse than others, submitted 2020, arXiv:2004.14049.

[III] M. Abreu, J.B. Gauci, D. Labbate, G. Mazzuoccolo and J.P. Zerafa. Extending perfect matchings to Hamiltonian cycles in line graphs, *Electron. J. Combin.* **28(1)** (2021), #P1.7, [DOI].

[IV] M. Abreu, J.B. Gauci and J.P. Zerafa, Saved by the rook, submitted 2020.

[V] J.B. Gauci and J.P. Zerafa, A note on perfect matchings and Hamiltonicity in the Cartesian product of cycles, submitted 2020, arXiv:2005.02913.

[VI] J.B. Gauci and J.P. Zerafa, On a family of quartic graphs: Hamiltonicity, matchings and isomorphism with circulants, submitted 2020, arXiv:2011.04327.

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I would also like to thank Gloria Rinaldi for always being present and very helpful—together with Arrigo Bonisoli and Simona Bonvicini, the above four have created a very warm working environment for which I am very grateful to have been part of.

I go on to thank Edita Máčajová (and Martin Škoviera) for accepting me for a four month research visit at the Comenius University in Bratislava. As if moving from the salty Żurrieq to the foggy Emilia (which unfortunately is not new to a number of Maltese people) was not already a long shot, my stay along the Danube would not have been as barcarollesque if it were not for you. I also want to thank John Baptist Gauci for accepting me at the University of Malta for a "two/three plus" month research period—having always been at the mathematics department as a student, this was an unusual but certainly a very memorable and pleasant time (although technically still as a student). I would also like to thank him for believing in me and accompanying me in my mathematical journey since my last year of undergraduate studies. Another thanks goes to co-authors and collaborators not yet mentioned, especially for their words of encouragement: Marién Abreu, Jan Goedgebeur, Domenico Labbate and Vahan Mkrtchyan.

No man is an island, they say, and although I'm not one to boast of social powers, I must thank all my friends and colleagues from via Giuseppe Campi, but also from the *Combinatorics* group from all over Italy. Most especially, I thank Davide Mattiolo for his camaraderie during the past three years... grazie vez. I also cannot not thank the *Cheers-Capitone* gang for the good times we spent together on the southern bank of the river Po. Thank you all for accepting me as I am—my journey would have been different had it not been for you.

Lastly, I thank my friends, Nannu Skippy, and most of all, my family: Vincent, Rita, Ruth and Ġipsi for their endless patience, support and love, during both my darkest and colourful times.

To everyone who has been part of this hike: *grazzi u maħfra.*

A NOTE FOR ASPIRING / CURRENT PHD STUDENTS

I don't have much to say (Kipling's advice at the end is already more than enough), except: to try and be the best whole person you can possibly be, to stay humble, and to not let hurdles discourage you—and by hurdles I also mean coming from a working class family or being a first generation PHD student. Take care of your mental health, be kind to yourself, try to enjoy life (while you're still in the pink), and...

Let everything happen to you: beauty and terror.
Just keep going. No feeling is final.

—RAINER MARIA RILKE

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PREFACE

Matchings…in real life you would consider yourself very lucky to experience one, let alone a perfect matching! Graph theory can model such real-life experiences quite well. Take Hall's marriage theorem, for example. It is maybe due to their rarity (or is it just because it is simply true?) that some of us have created the *perfect is boring* mantra; something which will not be delved into deeper in what follows. However, perfect matchings in graph theory are very much interesting—to say the least—so much so that they have occupied my mind and a big part of my life during the last three years.

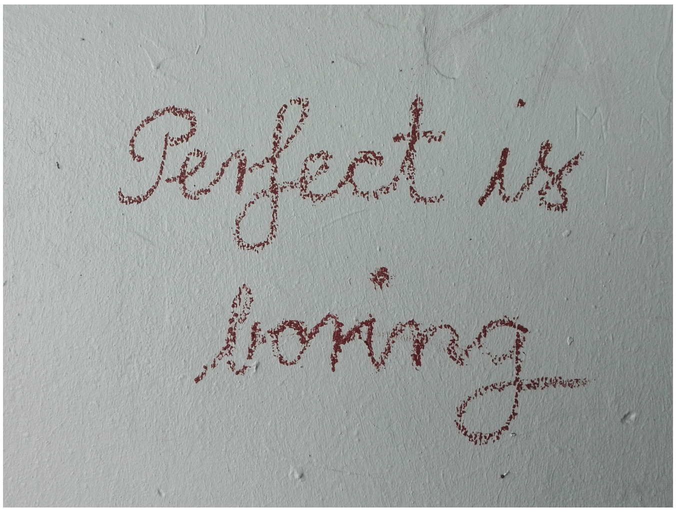

Figure 0.1: Somewhere in Koper, Slovenia, on the 3rd of Febraury, 2017

1

MOTIVATION AND BACKGROUND

This introductory chapter serves a number of purposes. We first introduce the notion of perfect matchings in graphs and the different aspects of our work regarding their behaviour. As the title suggests, this will be the recurring theme in this thesis, which is divided in two parts—representing the two different scenarios within which our work takes place:

(i) perfect matchings in bridgeless cubic graphs (more precisely, snarks) in Part i, and

(ii) perfect matchings and Hamiltonicity in Part ii.

As we shall see later on, this will involve dealing with the union and intersection of perfect matchings. Moreover, a general historical background and major results in the respective areas are given. Then, Section 1.4 will be dedicated to gather all the definitions and notation used throughout the whole thesis in one place—this is done in a way that definitions and notation not used in both the two main parts of the thesis, are put in a respective subsection such that it will be easier for the reader to filter through Section 1.4. The reader who is conversant in graph theoretical terms can skip this section, or consult it when needed, if such a need arises.

1.1 INTRODUCTORY REMARKS

What is a perfect matching of a graph G? A *perfect matching* is a pairing of the vertices of G such that each pair of vertices in the pairing is an edge in G. In other words, a perfect matching of G is a set of independent edges of G which covers the vertex set of G. Clearly, an obvious necessary condition for G to admit a pairing of the vertices is that $|V(G)|$, the order of G, is even. In what follows, we shall tacitly assume that G is of even order, unless otherwise stated.

One of the early classical results on perfect matchings dates back to 1891 and was made by Julius Petersen.

Theorem 1.1.1 (Petersen, 1891 [78]). *Every bridgeless cubic graph admits a perfect matching.*

Not only do bridgeless cubic graphs admit a perfect matching, but in 2011, one of the most prominent conjectures about perfect matchings in bridgeless cubic graphs was completely solved by Louis Esperet et al. in [24]. The conjecture, proposed by László Lovász and Michael David Plummer in the 1970s, stated that the number of perfect matchings in a bridgeless cubic graph grows exponentially with its order (see [61]). The authors in [24] in fact show that every bridgeless cubic graph has at least $2^{|V(G)|/3656}$ perfect matchings.

Many interesting problems in graph theory are in fact about the intersection and union of perfect matchings in cubic graphs as we shall see in Section 1.2 and later on in Part i. However, before continuing about this let us mention one other classical result about perfect matchings in general graphs, not necessarily bridgeless and cubic. The following theorem by William Thomas Tutte, from 1947, gives a necessary and sufficient condition for the existence of a perfect matching in a graph.

Theorem 1.1.2 (Tutte, 1947 [98]). *A graph G admits a perfect matching if and only if the number of odd components in $G - S$ is at most $|S|$, for every $S \subseteq V(G)$.*

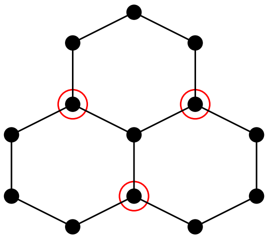

Figure 1.1: Removing the 3 red vertices leaves 4 odd components

1.2 CUBIC GRAPHS

As already mentioned above, Part i is dedicated to perfect matchings in cubic graphs. One of the most elegant statements in this area is the following conjecture from 1971, which although simple and uncomplicated, remains still unsolved after all these years.

Conjecture 1.2.1 (Fulkerson, 1971 [32]). *Every bridgeless cubic graph G admits six perfect matchings such that each edge in G is contained in exactly two of these six perfect matchings.*

In other words, a bridgeless cubic graph G admits six perfect matchings (with duplicates allowed) whose union gives the edge set of G twice. Although initially stated by Delbert Ray Fulkerson, the conjecture of Fulkerson is also attributed to Claude Berge and has been widely referred to as the Berge–Fulkerson conjecture. In hindsight, such a name was more than appropriate as Fulkerson's conjecture was actually shown to be equivalent to the following seemingly weaker conjecture made by Berge.

Conjecture 1.2.2 (Berge, unpublished). *Every bridgeless cubic graph G admits five perfect matchings such that every edge in G is contained in at least one of these five perfect matchings.*

Let G be a bridgeless cubic graph. Let us call a set of six perfect matchings of G (not necessarily distinct) having the properties mentioned in the first conjecture a *Fulkerson cover*, and a set of five perfect matchings of G (not necessarily distinct) having the properties mentioned in the second conjecture a *Berge cover*. It is clear that the first conjecture implies the second: removing one of the perfect matchings from a Fulkerson cover of G, a Berge cover is obtained. The reverse implication, that is, if Berge's conjecture is true for all bridgeless cubic graphs, then, Fulkerson's conjecture is also true for all bridgeless cubic graphs, was proved by Giuseppe Mazzuoccolo in 2010 (see [71]). Having said this, we remark that given a Berge cover of a bridgeless cubic graph, it is still not generally known how to obtain a Fulkerson cover of the same graph, if such a cover exists. Consequently, for simplicity, we shall refer simultaneously to both these conjectures as the *Berge–Fulkerson Conjecture*, with Fulkerson's version generally referred to as the classical Berge–Fulkerson Conjecture if one wants to distinguish between the two.

A bridgeless cubic graph which needs no introduction and which admits a Fulkerson cover is the well-known Petersen graph depicted in Figure 1.2. This graph crops up in various instances throughout this thesis and it is so important that there is a whole book dedicated to it (see [45]). However, one of the reasons why it distinguishes itself so much from other bridgeless cubic graphs (and most probably from any other graph) is because of the Petersen Colouring Conjecture by Francois Jaeger, from 1988.

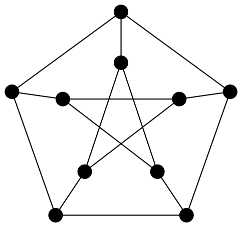

Figure 1.2: The Petersen graph

Conjecture 1.2.3 (Petersen Colouring Conjecture—Jaeger, 1988 [49])**.** *Every bridgeless cubic graph admits a Petersen colouring.*

Proving Jaeger's Conjecture would mean a lot of things: it would prove the Berge–Fulkerson Conjecture, for instance, but not only. It would also confirm the Cycle Double Cover Conjecture [105] which is a conjecture stated for general graphs and not only for cubic graphs. It is due to these huge consequences that the Petersen Colouring Conjecture is, arguably, one of the most trying and arduous conjectures in graph theory.

A possibly weaker conjecture than the Berge–Fulkerson Conjecture, proposed by Genghua Fan and André Raspaud in 1994 (see [26]), states that every bridgeless cubic graph has three perfect matchings whose intersection is empty (see Conjecture 3.1.1). In Chapter 3 we answer a question recently proposed by Vahan Mkrtchyan and Gagik N. Vardanyan in [75], by proving that a seemingly stronger version of the Fan–Raspaud Conjecture is actually equivalent to the classical formulation (Theorem 3.2.3). We also study a possibly weaker conjecture originally proposed by Mazzuoccolo [69], which states that in every bridgeless cubic graph there exist two perfect matchings such that the complement of their union is a bipartite graph (see Conjecture 3.1.3). We show that this conjecture (referred to as the S_4-Conjecture) can be equivalently stated using a variant of Petersen-colourings (Proposition 3.3.1) and we prove it for graphs having oddness at most 4 (Theorem 3.4.4). Even though all the above conjectures are a consequence of one single conjecture, Jaeger's Conjecture, they each expose different shades and particularities of the behaviour of perfect matchings in bridgeless cubic graphs: the union of six or five perfect matchings, the intersection of three (or two, as seen in Conjecture 3.1.2) perfect matchings, and the complement of the union of two perfect matchings—behaviours which are interesting in their own right, but whose significance is accentuated by their interconnections and their relation to Conjecture 1.2.3.

Although all mentioned conjectures are about simple cubic graphs without bridges, in Chapter 3 we also extend our study of the union of two perfect matchings to bridgeless cubic multigraphs and to particular cubic graphs having bridges (see Section 3.5.1 and Section 3.5.2).

Despite the challenging nature of the above conjectures, it is not very hard to show the existence of certain cubic graphs admitting, say, a Fulkerson cover. In fact, there are an infinite number of cubic graphs which trivially satisfy the properties mentioned in the conjectures stated in Section 1.2—these are the 3-edge-colourable cubic graphs known as Class I cubic graphs. In fact, the three colours of a proper edge-colouring of these graphs correspond to three perfect matchings, and by taking twice each of these perfect matchings one can easily form a Fulkerson cover, for example. In this case, a Berge cover is also easily obtained, but one might say that there are "a lot" of repeated perfect matchings in these covers. And rightly so! Indeed, the most compelling part of the above problems is analysing them in relation to Class II cubic graphs (see *snarks*), and in this spirit, a lot of research has been done in order to study invariants or parameters that measure how far Class II cubic graphs are from being 3-edge-colourable, that is, from being Class I. As a result, Class II graphs are often called *uncolourable*, and the mentioned invariants are often referred to as measures of *edge-uncolourability* (see [30]).

Two such parameters are the excessive index of a graph and the oddness of a graph. The oddness of a graph is studied in Chapter 3 whilst dealing with the S_4-Conjecture mentioned above and other related problems. The excessive index of a bridgeless cubic graph, introduced by Arrigo Bonisoli and the late David Cariolaro in [13], is the minimum number of perfect matchings needed to cover its edge set, which should be at most five by the Berge–Fulkerson Conjecture. In Chapter 4 we deal with the fact that the family of potential counterexamples to many interesting conjectures can be narrowed even further to the family $\mathcal{S}_{\geq 5}$ of bridgeless cubic graphs whose edge set cannot be covered with four perfect matchings. The Cycle Double Cover Conjecture, the Shortest Cycle Cover Conjecture [3] and the Fan–Raspaud Conjecture are examples of statements for which $\mathcal{S}_{\geq 5}$ is crucial.

In Chapter 4, we also study parameters which have the potential to further refine $\mathcal{S}_{\geq 5}$ and thus enlarge the set of cubic graphs for which the mentioned conjectures can be verified. We show that $\mathcal{S}_{\geq 5}$ can be naturally decomposed into subsets with increasing complexity, thereby producing a natural scale for proving these conjectures. More precisely, we consider the following parameters and questions: given a bridgeless cubic graph, (i) how many perfect matchings need to be added (see Section 4.2), (ii) how many copies of the same perfect matching need to be added (see Sections 4.3 and 4.4), and (iii) how many 2-factors need to be added so that the resulting regular multigraph is Class I (see Section 4.5)? We present new results for these parameters and we also establish some strong relations between these problems and some long-standing conjectures.

1.3 HAMILTONICITY

Another well-known aspect of graph theory which surely needs no introduction is Hamiltonicity. This concept had its beginnings in the mid-1850s thanks to the *Icosian Game* by Sir William Rowan Hamilton whose aim was to find (what we now refer to as) a Hamiltonian circuit of the graph of the *dodecahedron* (see Figure 1.3), which has twenty (in Greek *icos*) vertices and twelve (in Greek *dodec*) faces.

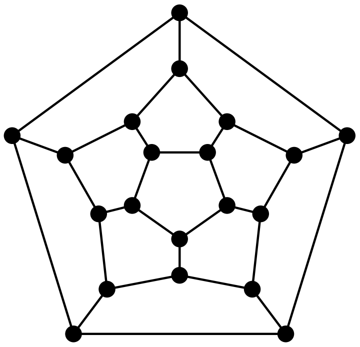

Figure 1.3: The graph of the dodecahedron

This was the year 1856. However, a year before, Reverend Thomas Penyngton Kirkman had already studied "Hamiltonian" circuits on general polyhedra, and despite preceding Hamilton, these connected 2-regular spanning subgraphs were thereafter known as Hamiltonian circuits.

One of the first momentous statements about Hamiltonian circuits came in 1880 by Peter Guthrie Tait who made the following conjecture, which if true would have implied the Four Colour Theorem, at the time still a conjecture as well (see [5, 6] and [82]).

Conjecture 1.3.1 (Tait, 1880)**.** *Every 3-connected planar cubic graph is Hamiltonian.*

This was eventually disproved by Tutte more than 60 years later in [97] by exhibiting what is known as the Tutte graph. In addition, Tutte [95] also showed that every 4-connected planar graph is Hamiltonian.

When talking about Hamiltonicity in graphs, one naturally comes to the conclusion that a graph having vertices with large degree is more likely to be Hamiltonian. However, it would take quite a long time for a general theorem about Hamiltonian graphs to appear. In 1952, Gabriel Andrew Dirac [21] proved that any graph G on more than three vertices in which every vertex has degree at least $|V(G)|/2$ is Hamiltonian. Eight years later, Oystein Ore proved a more general theorem which gives Dirac's Theorem as a consequence.

Theorem 1.3.2 (Ore, 1960 [77])**.** *Let G be a graph of order at least three. If $deg(u) + deg(v) \geq |V(G)|$ for each pair u, v of non-adjacent vertices of G, then G is Hamiltonian.*

It was a decade later that our main area of interest was studied for the first time, combining the study of perfect matchings with that of Hamiltonicity. But before continuing, one might rightfully ask: what do perfect matchings have to do with Hamiltonian circuits? Let G be a graph of even order and let M_1 and M_2 be perfect matchings of G. The subgraph of G induced by the edges in M_1 and M_2, denoted by $G[M_1 \cup M_2]$, is made up of circuits (the edges in $M_1 \triangle M_2$) and isolated edges (the edges in $M_1 \cap M_2$). Moreover, if $M_1 \cap M_2$ is empty, then $G[M_1 \cup M_2]$ is a 2-factor of G, and if this 2-factor has exactly one component, then $G[M_1 \cup M_2]$ is a Hamiltonian circuit of G. Thus, a Hamiltonian circuit of a graph G of even order can be seen as the disjoint union of two perfect matchings of G.

In this sense, we study the following property. Let G be a graph admitting a perfect matching. We shall say that G has the Perfect-Matching-Hamiltonian property, for short the PMH-property, if every perfect matching of G belongs to a Hamiltonian circuit of G. This means that for every perfect matching M of G, there exists another perfect matching N of G such that $M \cup N$ forms a Hamiltonian circuit of G. As far as we know, the first result with regards to the above mentioned property was in 1972. Michel Las Vergnas, in his doctoral thesis, proved the following Ore-type condition.

Theorem 1.3.3 (Las Vergnas, 1972 [58]). *Let G be a bipartite graph, with partite sets U and V, such that $|U| = |V| = \frac{|V(G)|}{2} \geq 2$. If for each pair of non-adjacent vertices $u \in U$ and $v \in V$ we have $deg(u) + deg(v) \geq \frac{|V(G)|}{2} + 2$, then G has the PMH-property.*

The lowerbound $\frac{|V(G)|}{2} + 2$ is best possible, as Figure 1.4 shows.

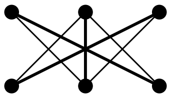

Figure 1.4: A bipartite graph G is not necessarily PMH if $deg(u) + deg(v) \geq \frac{|V(G)|}{2} + 1$ for every pair of non-adjacent vertices

Some years later, Roland Häggkvist proved another theorem having a similar flavour.

Theorem 1.3.4 (Häggkvist, 1979 [37]). *Let G be a graph, such that the order of G is even and at least four. If for each pair of non-adjacent vertices u and v we have $deg(u) + deg(v) \geq |V(G)| + 1$, then G has the PMH-property.*

However, graphs having the "PMH-property" were not initially known in these terms. Not having access to Las Vergnas' doctoral thesis, we refer to [37]. In fact, in the introduction of his paper, Häggkvist wrote that his aim was to study questions of the following type: *"Suppose that we are given a graph G and a set F of independent paths in G. What conditions should be imposed on G and F in order for F to be contained in a Hamiltonian circuit or path in G?"*

If G would eventually admit a Hamiltonian circuit or path containing F, he wrote that the graph is said to be *F-Hamiltonian* or *F-semihamiltonian*, respectively. In his paper, he also referred to a graph G being F-Hamiltonian or F-semihamiltonian for every perfect matching F of G, as a *HAC-graph* (short for *Hamiltonian alternating circuit* graph) or a *HAP-graph* (short for *Hamiltonian alternating path* graph), respectively. For completeness' sake, we remark that in [37], Häggkvist used the word cycle instead of circuit, and for instance, *HAC* was short for *Hamiltonian alternating cycle*. The difference between the usage of cycles and circuits is explained in Subsection 1.4.1.

We also remark that Häggkvist considered mostly the case when F is a perfect matching, and, in this case, the theorems were not stated in terms of HAC-graphs (or HAP-graphs), but in terms of F being a perfect matching and G eventually being F-Hamiltonian (or F-semihamiltonian), after satisfying some set of conditions.

Before continuing, we would like to remark on the choice of the word "pairing" when defining perfect matchings earlier on. In what follows, the word "pairing" shall represent a sort of "generalised" perfect matching, and we let a *pairing* of a graph G to be a pairing of the vertices of G regardless of whether the two vertices in each pair are edges in G or not. Hence, a pairing of a graph G can also be seen as a perfect matching of the complete graph on the same vertex set of G. As far as we know, the word pairing was first used as defined above in 2015, by the authors in [2].

The same authors say that a graph G (of even order) has the *Pairing-Hamiltonian property* (for short the *PH-property*) if for every pairing M of G there exists a perfect matching N of G such that $M \cup N$ is a Hamiltonian circuit H of K_G. In this sense, in order to find something which can easily contrast with "the PH-property" and also respects the definition of F-Hamiltonian (when F is a perfect matching), the authors of [III] suggest using the PMH-property, which shall be used hereafter instead of F-Hamiltonian and HAC. For simplicity and continuity, we shall refer to a graph having the PMH-property as a PMH-graph or just PMH.

In [37], a stronger conjecture than Theorem 1.3.4 was stated, and this was eventually proven by Kenneth A. Berman. The statement of the theorem is the following.

Theorem 1.3.5 (Berman, 1983 [7]). *Let G be a graph of order at least three. If $deg(u) + deg(v) \geq |V(G)| + 1$ for every pair of non-adjacent vertices u and v, then every set of independent edges of G lies in a circuit.*

In 2008, Denise Amar, Evelyne Flandrin and Grzegorz Gancarzewicz gave a degree sum condition for three independent vertices under which every matching of a graph lies in a Hamiltonian circuit (see [4]). More literature about matchings being extended to Hamiltonian circuits can be found in [48, 83, 102, 104].

In particular, in 1993, Frank Ruskey and Carla Savage [83] asked whether every matching in the n-dimensional hypercube $\mathcal{Q}_n$, for $n \geq 2$, extends to a Hamiltonian circuit of $\mathcal{Q}_n$. This was in fact shown to be true for $n = 2, 3, 4$ (see [29]) and for $n = 5$ (see [101]). Moreover, Jiří Fink [29] also showed that $\mathcal{Q}_n$ has the PH-property. This clearly implies that $\mathcal{Q}_n$ is a PMH-graph, and thus answering a conjecture made by Germain Kreweras in 1996 (see [57]). More recently, Fink also proved a weaker conjecture than that of Ruskey and Savage: he proved that every matching of $\mathcal{Q}_n$ can be extended to a 2-factor. This was shown in [28], and was initially considered by Jennifer Vandenbussche and Douglas Brent West in [99].

More results about the PH-property can be found in [2], the paper where the term *Pairing-Hamiltonian property* was first used. In particular, the authors show that the only cubic graphs having the PH-property are the complete graph K_4, the complete bipartite graph $K_{3,3}$, and the 3-dimensional cube $\mathcal{Q}_3$ (depicted in Figure 1.5).

They also study the PH-property in the Cartesian product of a complete graph and a circuit (see Theorem 6.1.3), and the Cartesian

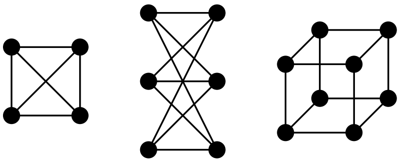

Figure 1.5: The only cubic graphs having the PH-property

product of a path and the hypercube. Other extensions of the PH-property for hypercubes are also studied.

Having a complete characterisation of which cubic graphs have the PH-property, a natural pursuit would be to characterise 4-regular graphs having the same property, as also suggested by the authors in [2]. Although Seongmin Ok and Thomas Perrett privately communicated to the authors of [2] the existence of an infinite family of quartic graphs (discussed in more detail in Chapter 8) having the PH-property, it was suggested to tackle this characterisation problem by looking at the Cartesian product of two circuits $C_p \square C_q$ (Open Problem 3 in [2]). In particular, the authors ask for which values of p and q does $C_p \square C_q$ have the PH-property.

This problem is solved in Chapter 7 where we show that $C_p \square C_q$ has the PH-property only when both p and q are equal to 4. In fact, the graph $C_4 \square C_4$ is isomorphic to the 4-dimensional hypercube $\mathcal{Q}_4$, which was already proved to have the PH-property in [29] together with all other n-dimensional hypercubes. More precisely, we show that except for $\mathcal{Q}_4$, $C_p \square C_q$ is not PMH (see Theorem 7.2.2).

Later on, in Chapter 8, we propose a class of quartic graphs on two parameters n and k which we shall call the class of *accordion graphs* $A[n,k]$, and we show that the quartic graphs having the PH-property mentioned by Adel Alahmadi et al. in [2] and discovered by Ok and Perrett, are in fact members of the class of accordion graphs (see Section 8.3.2). We also study the PMH-property in this class of accordion graphs, and although a complete characterisation of which accordion graphs admitting the above properties is still elusive, we think that this is a rich class of graphs which can contain many possible candidates admitting the PH-property or just the PMH-property. Finally, we also give a complete characterisation of which accordion graphs are circulant (see Section 8.5).

However, quartic graphs are not only interesting because of the above characterisation problem to determine which quartic graphs admit the PH-property or the PMH-property. As suggested above by the theorems of Ore, Häggkvist and Las Vergnas, the more edges a graph has, the greater the chances are for it to be Hamiltonian or PMH. Let us just remark that up till now, we do not know of a sufficient Ore-type condition for a graph to have the PH-property. In this sense, we think that studying graphs having small degree with respect to the properties mentioned above is a challenging and quite an intriguing endeavour.

For a smooth transition from cubic to quartic graphs, in Chapter 5 we study the PMH-property of the line graph of a cubic graph, which is itself quartic. Hamiltonicity of a line graph $L(G)$ is another extensively studied property: in 1964, Anton Kotzig [55] proved that the existence of a Hamiltonian circuit in a cubic graph is both a necessary and sufficient condition for a partition of $L(G)$ in two Hamiltonian circuits. Furthermore, a necessary and sufficient condition for Hamiltonicity in $L(G)$ (for G not necessarily cubic) was proved in 1965 by Frank Harary and Crispin St. John Alvah Nash-Williams [42], whilst in 1986 Carsten Thomassen [93] conjectured that every 4-connected line graph is Hamiltonian. As one can see, the class of line graphs of connected graphs is a compelling class of graphs for which a great deal is known regarding Hamiltonicity—but not only. Indeed, it is also well-known that if G is connected and has an even number of edges, then its line graph admits a perfect matching (see Section 5.2 for more details).

In this spirit, in Chapter 5 we establish some sufficient conditions for a graph G in order to guarantee that its line graph $L(G)$ has the PMH-property. In particular, we prove that this happens when G is (i) a Hamiltonian graph with maximum degree at most 3 (see Section 5.2), (ii) a complete graph (see Section 5.3.1), or (iii) an arbitrarily traceable graph (see Section 5.3.2).

The techniques used in Chapter 5 to prove that the line graph of a complete graph is PMH can be used to show that the line graph of a balanced complete bipartite graph with at least 100 vertices is PMH. However, by using different techniques we give a complete characterisation of which line graphs of complete bipartite graphs (not necessarily balanced) have, not only the PMH-property, but the PH-property. This is done in Chapter 6 in which we link this problem to a mathematical chess problem, since the line graph of a complete bipartite graph is isomorphic to the rook graph, that is, the Cartesian product of two complete graphs. We remark that the style of writing in this chapter is given in a more informal way, as the topic dealt with here can be easily understood by a wider audience. We thus took this opportunity so that the thesis contains at least one chapter which is more accessible to mathematics students and other interested people who are not very much familiar with graph theory. Finally, in Chapter 9 we study the PMH-property with respect to cubic graphs and we suggest a possible problem to be tackled, relating the PMH-property (and similar properties) to 3-edge-colourings of Class I cubic graphs (see Problem 9.1.2).

1.4 DEFINITIONS AND NOTATION

In this section, as already stated above, we give most of the definitions and notation needed for the thesis. Having said this, we shall leave out some specific definitions which are given later on when needed. This section is divided into three subsections: general definitions and notation needed throughout the whole thesis (see Subsection 1.4.1), and particular definitions and notation needed for Part i and Part ii (see Subsections 1.4.2 and 1.4.3, respectively).

1.4.1 *General definitions and notation*

In what follows, we use the letter G to denote a graph, unless otherwise stated. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Unless otherwise stated, all graphs considered are connected and *simple*, that is, without loops and parallel edges. Graphs that may contain parallel edges are referred to as *multigraphs*. The number of edges incident to a vertex u is said to be the *degree of* u, denoted by $deg(u)$. A vertex of degree 1 is called an *end-vertex*. Regular graphs in which all the vertices are of degree 3, and 4, are referred to as *cubic* and *quartic* graphs, respectively. On the other hand, graphs in which all the vertices have degree at most 3 are called *subcubic*.

A *matching* of a graph G is a subset of $E(G)$ such that any two of its edges do not share a common vertex. For any positive integer $k \geq 0$, a *k-factor* of G is a spanning subgraph of G (not necessarily connected) in which the degree of every vertex is k. In particular, a *perfect matching* is the edge-set of a 1-factor, that is, a matching which covers all the vertices.

The *complete graph* on n vertices is denoted by K_n, and a *clique* in a graph G is a complete subgraph of G. Consequently, we may sometimes refer to K_n as an *n-clique*. The *complete bipartite graph* with partite sets of order m_1 and m_2 is denoted by K_{m_1,m_2}, and is said to be *balanced* if $m_1 = m_2$.

A *walk* of length k (for some non-negative integer k) in a graph G is a sequence $v_1, \ldots, v_{k+1}$ of vertices of G with corresponding edge set $\{v_i v_{i+1} : i \in [k]\}$ (if $k > 0$). If $v_1 = v_{k+1}$, the walk is said to be *closed* and is denoted by $(v_1, \ldots, v_{k+1} = v_1)$. A *path* on t vertices, denoted by P_t, is a walk of length $t - 1$ in which all the vertices and edges are distinct. We also refer to P_t as a *t-path*. A *circuit* of length $k \geq 3$, denoted by C_k, is a closed walk of length k in which all the vertices are distinct, except for the first and last. For simplicity, we denote a circuit of length k by $(v_1, \ldots, v_k)$, instead of $(v_1, \ldots, v_{k+1} = v_1)$. We remark that in literature, C_k is most commonly referred to as the "cycle graph" instead of the "circuit graph". However, as we shall see later on, the word "cycle" shall be used for another definition, and so, to avoid confusion, we use the word "circuit" to denote C_n. The *girth* of a graph is the length of a shortest circuit contained in the graph.

Let $U \subseteq V(G)$. The graph on U whose edge set consists of those edges of G having both end-vertices in U is denoted by $G[U]$, and is referred to as the *induced* subgraph of G on U. Now, let $W \subseteq V(G)$ such that $U \cap W = \emptyset$. The set consisting of all the edges having exactly one end-vertex in U and one end-vertex in W is denoted by $[U, W]$. When W is equal to $V(G) - U$, the set $[U, W]$ is denoted by $\partial_G U$, or equivalently $\partial_G W$, and when it is obvious to which graph G we are referring we just write ∂U. When U consists of only one vertex, say u, we write ∂u, instead of $\partial\{u\}$, for simplicity.

Subgraphs can also be induced by sets of edges, and for $M \subseteq E(G)$, the *edge-induced* subgraph $G[M]$ is the subgraph of G obtained by first deleting from G the edges in $E(G) - M$ and then deleting the resulting isolated vertices.

For $X \subseteq E(G)$, the graph $G - X$ denotes the graph obtained by deleting all the edges of X from G, but leaving the vertices and the remaining edges intact. Similarly, for $W \subseteq V(G)$, the graph $G - W$ denotes the graph obtained by deleting from G all the vertices in W and all the edges incident to u, for every $u \in W$. For simplicity, when $X = \{e\}$ and $W = \{u\}$, we denote $G - X$ and $G - W$ by $G - e$ and $G - u$, respectively, instead of $G - \{e\}$ and $G - \{u\}$.

An *edge-cut*, or simply a *cut*, in a graph G is any set $X \subseteq E(G)$ such that $G - X$, denoted by $\overline{X}$, has more components than G, and no proper subset of X has this property, that is, for any $X' \subset X$, $\overline{X'}$ does not have more components than G. A graph G is said to be *k-edge-connected* if the cardinality of the smallest edge-cut of G is at least k. We refer to 2-edge-connected graphs as *bridgeless*. A cut X is said to be *odd* if there exists a subset W of $V(G)$ having odd cardinality such that $X = \partial W$.

We also make use of the following standard operations on cubic graphs known as *Y-reduction* (shrinking a triangle to a vertex), and of its inverse, *Y-extension* (expanding a vertex to a triangle), illustrated in Figure 1.6.

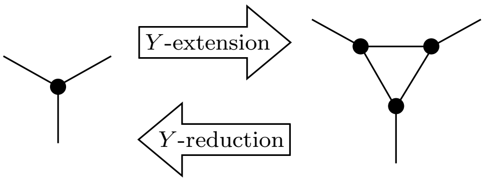

Figure 1.6: Y-operations

Finally we remark that for any integer $n \geq 1$, the set $\{1, \ldots, n\}$, shall be denoted by $[n]$.

1.4.2 *Definitions and notation used for Part i*

Let G be a bridgeless cubic graph. The least number of odd circuits in a 2-factor of G amongst all 2-factors of G, is called the *oddness* of G and is denoted by $\omega(G)$. The oddness is clearly even since a cubic graph has an even number of vertices. When M is a perfect matching of G, $\overline{M}$ is a 2-factor of G. In this case, following the terminology used for instance in [30], if $\overline{M}$ has $\omega(G)$ odd circuits, then M is said to be a *minimal perfect matching*.

A graph is *cyclically separable* if it admits an edge-cut whose removal separates two circuits. Clearly, a graph G is cyclically separable if and only if it admits two disjoint circuits, and, in this case, G is said to be *cyclically k-edge-connected* if no set of fewer than k edges separates two circuits of G. The largest integer k for which G is *cyclically k-edge-connected* is the *cyclic connectivity* of G.

Let G be a bridgeless cubic graph having a 2-edge-cut X. A 2-*edge-reduction* on X is the graph operation on G which creates two new smaller bridgeless cubic graphs by joining the degree two vertices in each component of $G - X$ by an edge. Moreover, for a bridgeless cubic graph G having a 3-edge-cut X, a 3-*edge-reduction* on X is the graph operation on G which creates two new bridgeless cubic graphs by introducing a new vertex to each of the components of $G - X$ and joining it to the degree two vertices in the respective component.

In the opposite direction, we define the following standard operation on graphs. Let G_1 and G_2 be two bridgeless cubic graphs, and let e_1 and e_2 be two edges such that $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. A 2-*cut-connection* on e_1 and e_2 is a graph operation that consists of constructing the new graph $(G_1 - e_1) \cup (G_2 - e_2) \cup \{u_1u_2, v_1v_2\}$. Clearly, the graph obtained is also bridgeless and cubic. Moreover, let $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$ such that the vertices adjacent to w_1 are x_1, y_1, z_1, and those adjacent to w_2 are x_2, y_2, z_2. A 3-*cut-connection* (sometimes also known as the star product, see for instance [33]) on w_1 and w_2 is a graph operation that consists of constructing the new graph $(G_1 - w_1) \cup (G_2 - w_2) \cup \{x_1x_2, y_1y_2, z_1z_2\}$. It is clear that the resulting graph is also bridgeless and cubic. The 3-edge-cut $\{x_1x_2, y_1y_2, z_1z_2\}$ is referred to as the *principal* 3-*edge-cut* (see for instance [31]).

Unlike 3-edge-reductions, a different labelling of the vertices in G_1 and G_2 can result in a completely different graph when applying a 3-cut-connection on w_1 and w_2. In what follows, it is not important how the adjacencies in the principal 3-edge-cut look like, and we just say that the resulting graph was obtained by a 3-cut-connection on w_1 and w_2, occasionally denoted by $G_1(w_1) * G_2(w_2)$. A similar argument can be applied to 2-edge-reductions.

A *cycle* is an even subgraph of the graph, that is, a subgraph with the degree of all its vertices being even. Observe that when a graph is cubic, any cycle of the graph is a collection of vertex-disjoint circuits.

A *k-cycle cover* is a cycle cover consisting of at most k cycles. A cycle cover $\mathcal{C}$ of a graph G is said to be a *cycle double cover* if every edge of G is contained in exactly two cycles of $\mathcal{C}$. The (total) length of a cycle cover $\mathcal{C}$ is the sum of the lengths of all the circuits making up the cycles in $\mathcal{C}$.

An *edge-colouring* of a graph G is a function that assigns a colour to each edge of G. If edges having a common end-vertex are assigned distinct colours, then the edge-colouring is called a *proper edge-colouring*. Moreover, if a proper edge-colouring uses k colours we say that graph G is *k-edge-colourable*. The minimum positive integer k such that G is k-edge-colourable is said to be the *chromatic index* of the graph G and denoted by $\chi'(G)$.

Let S be a finite set of colours containing at least two distinct colours a and b. In an edge-colouring of $E(G)$, if e is an edge assigned colour a, the (a,b)*-Kempe chain* of G containing e is the maximal connected subset of $E(G)$ which contains e and whose edges are all coloured either a or b.

A classical result by Vadim Georgievich Vizing [100] naturally divides cubic graphs in two classes according to the value of the chromatic index with respect to the maximum degree Δ. A simple graph G has $\chi'(G)$ either equal to $\Delta(G)$ or to $\Delta(G)+1$, and is said to be a *Class I* or *Class II* graph, respectively. In case of a multigraph G, we say that G is *Class I* if $\chi'(G) = \Delta(G)$, and *Class II*, otherwise.

The bridgeless cubic graphs having chromatic index 4 are referred to as *snarks*, and we denote the set of all snarks by $\mathcal{S}$. We remark that in literature one may find a stronger and more refined definition of snarks, which refers only to those graphs in $\mathcal{S}$ which are cyclically 4-edge-connected and with girth at least 5. In this thesis we shall use the broader definition of snarks and refer to the more specifically defined snarks as *non-trivial snarks*.

A *dangling edge* is an edge having exactly one end-vertex. The subgraph of G with vertex set U resulting by considering $G[U]$ together with the dangling edges arising from ∂U (and having end-vertices in U), is said to be a *k-pole*, where $k = |\partial U|$. A dangling edge with end-vertex x is said to be *joined* to a vertex y, if the dangling edge is deleted and x and y are made adjacent. In a similar way, two dangling edges are *joined* if they are both deleted and their end-vertices are made adjacent.

1.4.2.1 *Other definitions and notation used for Part i*

Finally, we give some non-graph-theoretical definitions and standard results needed for Chapter 2. Let $x_1, \ldots, x_k$ be k vectors in $\mathbb{R}^n$. A *convex combination* of $x_1, \ldots, x_k$ is a vector equal to $\sum_{i=1}^{k} \alpha_i x_i$, where the coefficients α_i are non-negative and real-valued, and $\sum_{i=1}^{k} \alpha_i = 1$. The *convex hull* of a set $X \subseteq \mathbb{R}^n$ is the set of all convex combinations of vectors in X. A *polytope* is the convex hull of a finite number of vectors in $\mathbb{R}^n$, and a vector x of a polytope $\mathcal{P}$ is said to be a *polytope vertex* if it cannot be expressed as a convex combination of elements belonging to $\mathcal{P} - \{x\}$. We remark that a polytope $\mathcal{P}$ can be represented as $\{x \in \mathbb{R}^n : Ax \leq b\}$, for some appropriate matrices A and b, whereby $Ax \leq b$ we mean that the result obtained after performing the matrix multiplication of the i^{th} row of the matrix A to the vector x is less than the i^{th} entry of b. In this sense, it is widely known that if A and b are *rational*, or in other words have rational entries, then every polytope vertex of $\mathcal{P}$ is rational. Furthermore, it is also well-known that if v is a polytope vertex of the polytope $\mathcal{P} \subset \mathbb{R}^n$, with the latter represented as $\{x \in \mathbb{R}^n : Ax \leq b\}$, then there are n rows of A such that the result obtained after performing the matrix multiplication of such rows to the vector v is exactly equal to the corresponding entry in b. In other words, there are n constraints valid for $\mathcal{P}$ which are tight at v.

1.4.3 *Definitions and notation used for Part ii*

A circuit passing through all vertices of a graph G is a *Hamiltonian circuit* and if such a circuit exists then G is said to be *Hamiltonian*. Similarly, a *Hamiltonian path* is a path passing through all the vertices of a graph.

A *tour* of G is a closed walk having no repeated edges, and an *Euler tour* is one that traverses all the edges of G. In the latter case, the graph is said to be *Eulerian*. A *dominating tour* of G is a tour in which every edge of G is incident with at least one vertex of the tour. In particular, a dominating tour which is 2-regular is referred to as a *dominating circuit*. In general, if a walk does not pass through some vertex v, we say that v is *untouched* or *uncovered*.

The *line graph* $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G, and two vertices of $L(G)$ are adjacent if the corresponding edges in G are incident to a common vertex.

For any graph G, K_G denotes the complete graph on the same vertex set $V(G)$ of G. Let G be of even order. A perfect matching of K_G is said to be a *pairing* of G. Given a pairing M of G, we say that M can be *extended* to a Hamiltonian circuit H of K_G if we can find another perfect matching N of G such that $M \cup N = E(H)$, where $E(H)$ is the set of edges of H. By using this terminology, the authors of [2] say that G has the *Pairing-Hamiltonian property* (or, for short, the *PH-property*), if every pairing M of G can be extended to a Hamiltonian circuit H of K_G, where $E(H) - M \subseteq E(G)$. For simplicity, we shall also say that a graph G is PH if it has the PH-property. As mentioned before, on a similar flavour, the authors of [III] define the *Perfect-Matching-Hamiltonian property* (or, for short, the *PMH-property*) for a graph G admitting a perfect matching, if every perfect matching of G can be extended to a Hamiltonian circuit of K_G, which in this case, would also be a Hamiltonian circuit of G itself. We remark that we only consider graphs admitting a perfect matching in order to avoid trivial cases. Henceforth, if a graph has the Perfect-Matching-Hamiltonian property, we say that it is a PMH-graph or simply that it is PMH. It can be easily seen that if a graph does not have the PMH-property, then it surely would not have the PH-property, although the converse is not true. In the sequel we shall also be referring to the following observation.

Remark 1.4.1. If a graph G contains a spanning subgraph which has the PH-property, then the graph G itself has the PH-property.

The *Cartesian product* $G\square H$ of two graphs G and H is a graph whose vertex set is the Cartesian product $V(G) \times V(H)$ of $V(G)$ and $V(H)$. Two vertices (u_i, v_j) and (u_k, v_l) are adjacent precisely if $u_i = u_k$ and $v_jv_l \in E(H)$ or $u_iu_k \in E(G)$ and $v_j = v_l$. Thus, $V(G\square H)$ is equal to

$$\{(u_r, v_s) : u_r \in V(G) \text{ and } v_s \in V(H)\},$$

and $E(G\square H)$ is equal to

$$\{(u_i, v_j)(u_k, v_l) : u_i = u_k, v_jv_l \in E(H) \text{ or } u_iu_k \in E(G), v_j = v_l\}.$$

We refer the reader to [12] for further definitions and notation not explicitly stated in the above subsections.

Part I

PERFECT MATCHINGS IN SNARKS

"It's a Snark!" was the sound that first came to their ears,
And seemed almost too good to be true.
Then followed a torrent of laughter and cheers:
Then the ominous words "It's a Boo–"

Then, silence. Some fancied they heard in the air
A weary and wandering sigh
That sounded like "–jum!" but the others declare
It was only a breeze that went by.

They hunted till darkness came on, but they found
Not a button, or feather, or mark,
By which they could tell that they stood on the ground
Where the Baker had met with the Snark.

In the midst of the word he was trying to say,
In the midst of his laughter and glee,
He had softly and suddenly vanished away—
For the Snark was *a Boojum, you see.*

—LEWIS CARROLL, The Hunting of the Snark [15]

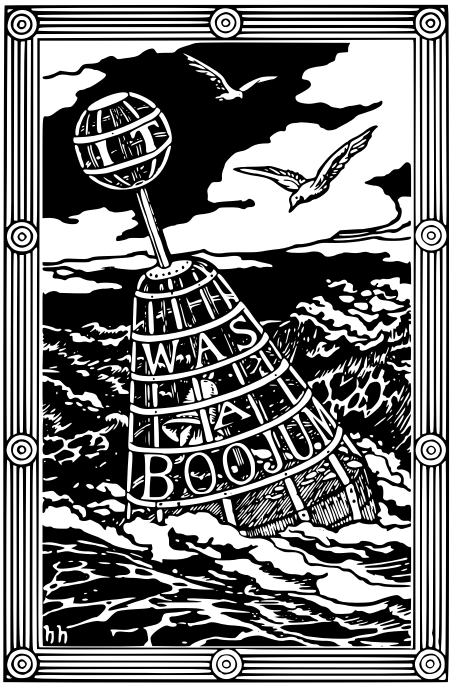

Figure 1.7: Illustration by Henry Holiday [15]

2

THE PERFECT MATCHING POLYTOPE

In some parts of Chapters 3 and 4 the notion of the perfect matching polytope and related ideas are used or cited. In this sense we believe that a short introductory chapter dedicated to this idea would be appropriate.

For a matching M of G, let χ^M be the corresponding characteristic vector (with dimension $|E(G)|$), such that for any $e \in E(G)$

$$\chi^M(e) = \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

The *matching polytope* $\mathcal{M}(G)$ of G is the convex hull of $\{\chi^M : M \text{ a matching of } G\}$. Similarly, the *perfect matching polytope* $\mathcal{P}(G)$ of G is the convex hull of $\{\chi^M : M \text{ a perfect matching of } G\}$. In 1965, Jack Robert Edmonds [22] managed to describe the matching polytope and the perfect matching polytope of a graph in terms of linear inequalities. The two statements are the following.

Theorem 2.0.1 (Edmonds, 1965 [22]). *The matching polytope of a graph G is the set of all vectors $w \in \mathbb{R}^{|E(G)|}$ that satisfy:*

(1a) $w(e) \geq 0$*, for each* $e \in E(G)$*,*

(1b) $w(\partial v) \leq 1$*, for each* $v \in V(G)$*, and*

(1c) $w(E(G[U])) \leq \lfloor \frac{1}{2}|U| \rfloor$*, for each* $U \subseteq V(G)$ *of odd cardinality.*

Theorem 2.0.2 (Edmonds, 1965 [22]). *The perfect matching polytope of a graph G is the set of all vectors $w \in \mathbb{R}^{|E(G)|}$ that satisfy:*

(2a) $w(e) \geq 0$*, for each* $e \in E(G)$*,*

(2b) $w(\partial v) = 1$*, for each* $v \in V(G)$*, and*

(2c) $w(\partial U) \geq 1$*, for each* $U \subseteq V(G)$ *of odd cardinality.*

We note that for any $E' \subseteq E(G)$, $w(E') = \sum_{e \in E'} w(e)$, where $w(e)$ is the value of w at the entry corresponding to the edge e.

Here we only give a proof of Theorem 2.0.2 as we shall not be making use of the matching polytope in what follows. The proof we give is based on a short direct proof given by Alexander Schrijver [87].

Proof of Theorem 2.0.2. Let $\mathcal{W}(G)$ be the set of all vectors $w \in \mathbb{R}^{|E(G)|}$ satisfying inequalities (2a), (2b) and (2c) in the statement of Theorem 2.0.2. We first remark that it can be easily seen that $\mathcal{W}(G)$ is a polytope. Next, suppose the theorem is not true and let G be a minimal counterexample with respect to $|V(G)| + |E(G)|$. Every perfect matching of G satisfies the three inequalities mentioned above and so $\mathcal{P}(G) \subseteq \mathcal{W}(G)$. Therefore, by our assumption, $\mathcal{W}(G) \not\subseteq \mathcal{P}(G)$. We remark that G is connected and of even order, otherwise we would have that $\mathcal{W}(G) = \emptyset = \mathcal{P}(G)$, a contradiction.

Let w be a polytope vertex of $\mathcal{W}(G)$ not belonging to $\mathcal{P}(G)$. We claim that for every edge $e \in E(G)$, $0 < w(e) < 1$, for if there exists an edge $e = uv$ such that $w(e) = 0$ or $w(e) = 1$, the graphs $G - e$ or $G - \{u, v\}$ would contradict the minimality of $|V(G)| + |E(G)|$, respectively. Consequently, the minimum degree of G is at least 2, implying that $|E(G)| \geq |V(G)|$. Moreover, if G is a circuit, it can be easily seen that $\mathcal{W}(G) = \mathcal{P}(G)$, and so $|E(G)| > |V(G)|$.

Since w is a polytope vertex, it must satisfy $|E(G)|$ linearly independent inequalities from (2a), (2b) and (2c), with equality. Furthermore, since $0 < w(e) < 1$ for every edge $e \in E(G)$, and since there are $|V(G)|$ inequalities of type (2b), w satisfies at least $|E(G)| - |V(G)|$ inequalities of type (2c) with equality. Therefore, there exists a set $U_1 \subseteq V(G)$ of odd cardinality such that $3 \leq |U_1| \leq |V(G)| - 3$ and $w(\partial U_1) = 1$. Let $U_2 = V(G) - U_1$ and, for each $i \in [2]$, let G_i be the graph obtained from G after contracting U_i into a single vertex u_i. We remark that G_1 and G_2 may contain parallel edges, and that the corresponding edges in ∂U_1 and ∂u_1 are referred to by the same name and treated as the same edge, for simplicity. The same applies for ∂U_2 and ∂u_2.

For each $i \in [2]$, let the vector w_i in $\mathbb{R}^{|E(G_i)|}$ be the restriction of w to the edges in G_i. Since $w(\partial_G U_1) = w(\partial_G U_2) = 1$, we claim that $w_i \in \mathcal{W}(G_i)$. Inequalities (2a) and (2b) are easily satisfied. So let X be a subset of $V(G_i)$ of odd cardinality. If $u_i \notin X$, then the inequality of type (2c) clearly holds for w_i. So assume $u_i \in X$ and let $Y = (X - \{u_i\}) \cup U_i$. Since Y is of odd cardinality and $w \in \mathcal{W}(G)$, $w(\partial_G Y) \geq 1$. Consequently, $w_i(\partial_{G_i} X) \geq 1$ since $w(\partial_G Y)$ is equal to $w_i(\partial_{G_i} X)$, proving our claim.

Thus, for each $i \in [2]$, the vector w_i belongs to $\mathcal{P}(G_i)$ due to the minimality of G, and so can be expressed as a convex combination of characteristic vectors of perfect matchings of G_i. Moreover, since w is rational (see Subsection 1.4.2), w_i is rational as well, and so there exist K perfect matchings $M_{i,1}, \ldots, M_{i,K}$ of G_i (not necessarily distinct) such that $w_i = \frac{1}{K} \sum_{j=1}^{K} \chi^{M_{i,j}}$.

Now, for every edge $e \in \partial_G U_i$, or equivalently $\partial_{G_i} u_i$, we have $w_1(e) = w(e) = w_2(e)$, and so the number of indices $j \in \{1, \ldots, K\}$ for which e belongs to $M_{1,j}$ (or equivalently $M_{2,j}$) is $K \cdot w(e)$. Hence, without loss of generality, we can assume that for each $j \in \{1, \ldots, K\}$,

$M_{1,j} \cap M_{2,j} \cap \partial_G U_1 \neq \emptyset$. As a consequence, for each $j \in \{1, \ldots, K\}$ we can define $M_j = M_{1,j} \cup M_{2,j}$, which is a perfect matching of G, and as a result, $w = \frac{1}{K}\sum_{j=1}^{K} \chi^{M_j}$. However this means that $w \in \mathcal{P}(G)$, a contradiction. □

We remark that when considering a bipartite graph G, the perfect matching polytope of G is equal to those vectors in $\mathbb{R}^{|E(G)|}$ obeying only the statements (2a) and (2b) from Theorem 2.0.2. In this case, the matching polytope of G is equal to $\mathcal{P}(G)$ together with those vectors w from $\mathbb{R}^{|E(G)|}$ having $w(e) \geq 0$ for every edge $e \in E(G)$, and $w(\partial v) < 1$ for every vertex $v \in V(G)$.

2.1 FRACTIONAL PERFECT MATCHINGS

In Chapter 3 we shall use the notion of fractional perfect matchings. Having the above mentioned ideas in place, it is now very easy to define what a fractional perfect matching is. The set of all fractional perfect matchings of a graph G is equal to the perfect matching polytope $\mathcal{P}(G)$, and so a *fractional perfect matching* is a vector in $\mathcal{P}(G)$. Two recent papers about fractional perfect matchings in bridgeless cubic graphs are [53] and [68], where in particular, the authors of the former paper study the maximum possible size of the union of a given number of perfect matchings in a bridgeless cubic graph. More precisely they show that by using fractional perfect matchings, any bridgeless cubic graph of size m, admits two perfect matchings that cover at least $3m/5$ edges, and three perfect matchings that cover at least $27m/35$ edges. Without going into too much detail, we remark that since any two perfect matchings of the Petersen graph intersect in exactly one edge, and any three perfect matchings of the same graph do not intersect, the value arising from the union of two perfect matchings is best possible, whilst the other value for three perfect matchings is very close to being so.

Finally, we suggest [85] for more general information about fractional notions in graph theory.

3

AN EQUIVALENT FORMULATION OF THE FAN–RASPAUD CONJECTURE AND RELATED PROBLEMS

This chapter is based on a joint work with Giuseppe Mazzuoccolo [I].

3.1 INTRODUCTION

One of the aims of this chapter is to study the behaviour of perfect matchings in cubic graphs, more specifically the union of two perfect matchings (see Section 3.3 and Section 3.4). We relate this to well-known conjectures, in particular: the Berge–Fulkerson Conjecture and the Fan–Raspaud Conjecture.

The Berge–Fulkerson Conjecture [32] (see also Conjectures 1.2.1 and 1.2.2) states that every bridgeless cubic graph G admits six perfect matchings $M_1, \dots, M_6$ such that any edge of G belongs to exactly two of them.

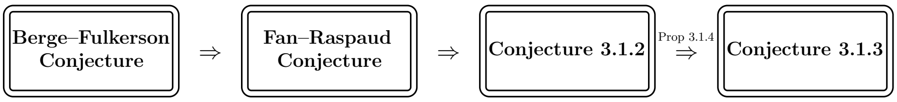

Figure 3.1: Conjectures mentioned and how they are related

Here, we also state other (possibly weaker) conjectures implied by the above conjecture.

Conjecture 3.1.1 (Fan–Raspaud Conjecture, 1994 [26]). *Every bridgeless cubic graph admits three perfect matchings M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

In the sequel we will refer to three perfect matchings satisfying Conjecture 3.1.1 as an *FR-triple*. We can see that Conjecture 3.1.1 is immediately implied by the Berge–Fulkerson Conjecture, since we can take any three perfect matchings out of the six which satisfy Berge–Fulkerson Conjecture. A still weaker statement implied by the Fan–Raspaud Conjecture is the following.

Conjecture 3.1.2 (Máčajová and Škoviera, 2005 [63]). *For each bridgeless cubic graph G, there exist two perfect matchings M_1 and M_2 such that $M_1 \cap M_2$ contains no odd-cut of G.*

We claim that any two perfect matchings out of the three in an FR-triple have no odd-cut in their intersection, in other words that Conjecture 3.1.1 implies Conjecture 3.1.2. For, suppose not. Then, without loss of generality, suppose that $M_2 \cap M_3$ contains an odd-cut

X. Hence, since every perfect matching has to intersect an odd-cut at least once, $|M_1 \cap (M_2 \cap M_3)| \geq |M_1 \cap X| \geq 1$, a contradiction, since we assumed that $M_1 \cap M_2 \cap M_3 = \emptyset$. In relation to the above, Mazzuoccolo proposed the following conjecture.

Conjecture 3.1.3 (S_4-Conjecture [69])**.** *For any bridgeless cubic graph G, there exist two perfect matchings such that the deletion of their union leaves a bipartite subgraph of G.*

For reasons which shall be obvious in Section 3.3 we let such a pair of perfect matchings be called an *S_4-pair* of G and refer to Conjecture 3.1.3 as the S_4-Conjecture. We proceed by first showing that this conjecture is implied by Conjecture 3.1.2, and so, by what we have said so far, is a consequence of the Berge–Fulkerson Conjecture. In particular, we can see the S_4-Conjecture as Conjecture 3.1.2 restricted to odd-cuts $\partial V(C)$, where C is an odd circuit of G.

Proposition 3.1.4. *Conjecture 3.1.2 implies the S_4-Conjecture.*

Proof. Let M_1 and M_2 be two perfect matchings such that their intersection does not contain any odd-cut. Consider $\overline{M_1 \cup M_2}$, and suppose that it contains an odd circuit C. Then all the edges of $\partial V(C)$ belong to $M_1 \cap M_2$. If $\overline{\partial V(C)}$ has exactly two components, then $\partial V(C)$ is an odd-cut belonging to $M_1 \cap M_2$, a contradiction. Therefore, $\overline{\partial V(C)}$ must have more than two components, say k, denoted by $C_1, C_2, \ldots, C_k$, where the first component C_1 is the circuit C. Let $[C_1, C_j]$ denote the set of edges between C_1 and C_j, for $j \in \{2, \ldots, k\}$. Since $\sum_{j=2}^{k} |[C_1, C_j]| = |\partial V(C)|$ is odd, there exists at least one j' for some $j' \in \{2, \ldots, k\}$, such that $|[C_1, C_{j'}]|$ is odd. However, $[C_1, C_{j'}]$ is an odd-cut which belongs to $M_1 \cap M_2$, a contradiction. □

3.2 STATEMENTS EQUIVALENT TO THE FAN–RASPAUD CONJECTURE

Let $M_1, \ldots, M_t$ be a list of perfect matchings of G, and let $a \in E(G)$. We denote the number of times a occurs in this list by $\nu_G[a : M_1, \ldots, M_t]$. When it is obvious to which list of perfect matchings or which graph we are referring, we simply denote this as $\nu(a)$ and refer to $\nu(a)$ as the *frequency of a*. We shall sometimes need to refer to the frequency of an ordered list of edges, say (a, b, c). This is done by saying that the frequency of (a, b, c) is (i, j, k), for some integers i, j, k. Mkrtchyan et al. [75] showed that the Fan–Raspaud Conjecture, that is, Conjecture 3.1.1, is equivalent to the following.

Conjecture 3.2.1. *[75] For each bridgeless cubic graph G, any edge $a \in E(G)$ and any $i \in \{0, 1, 2\}$, there exist three perfect matchings M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$ and $\nu_G[a : M_1, M_2, M_3] = i$.*

In other words they show that if a graph has an FR-triple then, for every i in $\{0, 1, 2\}$, there exists an FR-triple in which the frequency of a pre-chosen edge is exactly i. In the same paper, Mkrtchyan et al. state the following seemingly stronger version of the Fan–Raspaud Conjecture.

Conjecture 3.2.2. *[75] Let G be a bridgeless cubic graph, w a vertex of G and i, j, k three integers in $\{0, 1, 2\}$ such that $i + j + k = 3$. Then, G has an FR-triple in which the edges incident to w in a given order have frequencies (i, j, k).*

This means that we can prescribe the frequencies to the three edges incident to a given vertex. At the end of [75], the authors remark that it would be interesting to show whether Conjecture 3.2.2 is equivalent to the Fan–Raspaud Conjecture. Here, we prove that this is actually the case.

Theorem 3.2.3. *Conjecture 3.2.2 is equivalent to the Fan–Raspaud Conjecture.*

Proof. Since the Fan–Raspaud Conjecture is equivalent to Conjecture 3.2.1, it suffices to show the equivalence of Conjectures 3.2.1 and 3.2.2. The latter clearly implies the former, so assume Conjecture 3.2.1 is true and let a, b, c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). It is sufficient to show that there exist two FR-triples in which the frequencies of (a, b, c) are $(2, 1, 0)$ in one FR-triple (**Case 1** below) and $(1, 1, 1)$ in the other FR-triple (**Case 2** below).

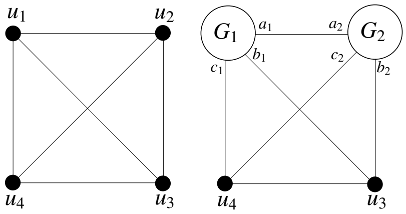

Figure 3.2: The graphs K_4 and K_4^* in **Case 1**

Case 1. Let u_1, u_2, u_3, u_4 be the vertices of the complete graph K_4 as in Figure 3.2. Consider two copies of G, and let the vertex w in the i^{th} copy of G be denoted by w_i, for each $i \in [2]$. We apply a 3-cut-connection on u_i and w_i, for each $i \in [2]$. With reference to this resulting graph, denoted by K_4^*, we refer to the copy of the graph $G - w$ at u_1 as G_1, and to the corresponding edges a, b, c as a_1, b_1, c_1, respectively. The graph G_2 and the edges a_2, b_2, c_2 are defined in a similar way, such that b_1 and b_2 are adjacent, and also c_1 and c_2, as Figure 3.2 shows. Note also that a_1 and a_2 coincide in K_4^*. By our assumption, there exists an FR-triple M_1, M_2, M_3 of K_4^* in which the edge u_3u_4 has frequency 2. Without loss of generality, let $u_3u_4 \in M_1 \cap M_2$. Then, a_1 (and so a_2) must belong to $M_1 \cap M_2$. Clearly, a_1 (and so a_2) cannot belong to M_3, and so the principal 3-edge-cuts with respect to G_1 and G_2 do not belong to M_3. If $b_1 \in M_3$, then we are done, as then M_1, M_2, M_3 restricted to G_1, together with a and b having the same frequencies as a_1 and b_1, induce an FR-triple of G such that the frequencies of (a,b,c) are $(2,1,0)$. So suppose $c_1 \in M_3$. Then, $b_2 \in M_3$, and so by a similar argument applied to G_2 and the corresponding edges, M_1, M_2, M_3 induce an FR-triple in G such that the frequencies of (a,b,c) are $(2,1,0)$.

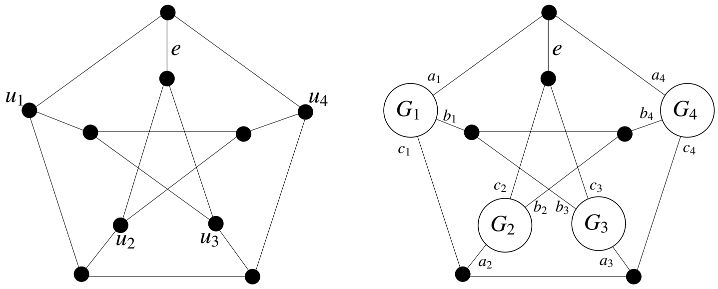

Figure 3.3: The graphs P and P^* in **Case 2**

Case 2. Let P be the Petersen graph and $\{u_1, u_2, u_3, u_4\}$ be a maximum independent set of vertices in P as in Figure 3.3. Consider four copies of G. Let the vertex w in the i^{th} copy of G be denoted by w_i, for each $i \in [4]$. Let P^* be the graph obtained by applying a 3-cut-connection on each u_i and w_i, as shown in Figure 3.3. Similar to Case 1 we refer to the copy of $G - w$ at u_i as G_i and to the corresponding edges a, b, c as a_i, b_i, c_i, respectively. Since we are assuming that Conjecture 3.2.1 is true, we can consider an FR-triple M_1, M_2, M_3 of P^* in which the edge e incident to both a_1 and a_4 has frequency 2. Without loss of generality, let the two perfect matchings containing e be M_1 and M_2. The edges a_1, c_2, c_3, a_4 are not contained in M_1 and neither M_2, since they are all incident to e, and so no principal 3-edge-cut leaving G_i belongs to M_1 or M_2. Then, M_1 and M_2 induce perfect matchings of P (clearly distinct), and since there are exactly two perfect matchings of P containing e, we can assume that M_1 contains $\{e, b_1, a_2, a_3, b_4\}$, and M_2 contains $\{e, c_1, b_2, b_3, c_4\}$.

If the third perfect matching M_3 induces a perfect matching of the Petersen graph, then the induced perfect matching cannot be one of the perfect matchings induced by M_1 and M_2 in P. Hence, since every two distinct perfect matchings of P intersect in exactly one edge of P, there exists at least one G_i such that the frequencies of (a_i, b_i, c_i) are $(1,1,1)$ and so, M_1, M_2, M_3 restricted to G_i, together with a, b, c having the same frequencies as a_i, b_i, c_i, induce an FR-triple in G with the needed property.

Therefore, suppose M_3 contains the principal 3-edge-cut of one of the G_is, say G_1 by symmetry of P^*. Thus, a_1, b_1, c_1 belong to M_3. The perfect matching M_3 can intersect the principal 3-edge-cut at G_2 either in b_2 or c_2 (not both). If $c_2 \in M_3$, we are done by the same reasoning above but now applied to G_2 and the corresponding edges. So suppose $b_2 \in M_2 \cap M_3$. Then, clearly $c_4 \in M_3$, and M_3 can only intersect the principal 3-edge-cut at G_3 in c_3, implying that the frequencies of (a_3, b_3, c_3) are $(1, 1, 1)$ in P^* and that M_1, M_2, M_3 restricted to G_3, together with a, b, c having the same frequencies as a_3, b_3, c_3, induce an FR-triple in G with the needed property. □

In [75] it is also shown that a minimal counterexample to Conjecture 3.2.2 is cyclically 4-edge-connected. It remains unknown whether a smallest counterexample to the original formulation of the Fan–Raspaud Conjecture has the same property. Indeed, we only prove that the two assertions are equivalent, but we cannot say whether a possible counterexample to Conjecture 3.2.2 is itself a counterexample to the original formulation.

3.3 STATEMENTS EQUIVALENT TO THE S_4-CONJECTURE

As already stated in Chapter 1 all conjectures presented in Section 3.1 are implied by Conjecture 1.2.3, that is, the Petersen Colouring Conjecture by Jaeger. In order to understand more what it states, we need the following definitions. Let G and H be two cubic graphs. An *H-colouring* of G is a proper edge-colouring f of G with edges of H, such that for each vertex $u \in V(G)$, there exists a vertex $v \in V(H)$ with $f(\partial_G u) = \partial_H v$. If G admits an H-colouring, then we write $H \prec G$. The importance of H-colourings is mainly due to Jaeger's Conjecture which states that for each bridgeless cubic graph G, one has $P \prec G$ (where P is again the Petersen graph). Although beyond the scope of this thesis, we remark that this conjecture has an equivalent formulation due to a result by Jaeger [50] shown in 1985, where he showed that for any cubic graph G, $P \prec G$ if and only if G admits a normal 5-edge-colouring. For recent results on Petersen-colourings, normal edge-colourings and necessary definitions, see for instance [8, 9, 27, 39, 72, 73, 74, 79, 84].

In this chapter we consider S_4-colourings of bridgeless cubic graphs, where S_4 is the multigraph shown in Figure 3.4. Since S_4 is not cubic, we remark that in an S_4-colouring of a bridgeless cubic graph G, the above definition of H-colourings applies, but for all vertices $u \in V(G)$, $\partial_G u$ is not mapped to $\partial_{S_4} z$, where z is the vertex of degree 1 in S_4 (see Figure 3.4).

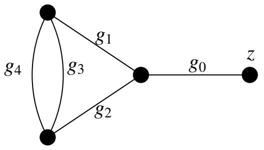

Figure 3.4: The multigraph S_4

The following proposition shows why we choose to refer to a pair of perfect matchings whose deletion leaves a bipartite subgraph as an S_4-pair.

Proposition 3.3.1. *Let G be a bridgeless cubic graph, then $S_4 \prec G$ if and only if G has an S_4-pair.*

Proof. Along the entire proof we denote the edges of S_4 by using the same labelling as in Figure 3.4. Let M_1 and M_2 be an S_4-pair of G. The graph induced by $M_1 \cup M_2$, denoted by $G[M_1 \cup M_2]$, is made up of even circuits and isolated edges, whilst the bipartite graph $\overline{M_1 \cup M_2}$ is made up of even circuits and paths. We obtain an S_4-colouring of G as follows:

- the isolated edges in $M_1 \cup M_2$ are given colour g_0;
- the edges of the even circuits in $M_1 \cup M_2$ are properly edge-coloured with g_3 and g_4; and
- the edges of the paths and even circuits in $\overline{M_1 \cup M_2}$ are properly edge-coloured with g_1 and g_2.

One can clearly see that this gives an S_4-colouring of G. Conversely, assume that $S_4 \prec G$. We are required to show that there exists an S_4-pair of G. Let M_1 be the set of edges of G coloured g_3 and g_0, and let M_2 be the set of edges of G coloured g_4 and g_0. If e and f are edges of G coloured g_3 (or g_4) and g_0, respectively, then e and f cannot be adjacent, otherwise we contradict the S_4-colouring of G. Thus, M_1 and M_2 are matchings. Next, we show that they are in fact perfect matchings. This follows since for every vertex v of G, $f(\partial_G v)$ is equal to $\{g_1, g_3, g_4\}$, or $\{g_2, g_3, g_4\}$, or $\{g_0, g_1, g_2\}$. Thus, $\overline{M_1 \cup M_2}$ is the graph induced by the edges coloured g_1 and g_2, which clearly cannot induce an odd circuit. □

Hence, by the previous proof, Conjecture 3.1.3 can be stated in terms of S_4-colourings, which clearly shows why we choose to refer to it as the S_4-Conjecture. In analogy to what we did for FR-triples, here we prove that for S_4-pairs we can prescribe the frequency of an edge and the frequencies of the edges leaving a vertex (the proof of the latter implies also that we can prescribe the frequencies of the edges of each 3-cut). Consider the following conjecture, analogous to Conjecture 3.2.1.

Conjecture 3.3.2. *For any bridgeless cubic graph G, any edge $a \in E(G)$ and any $i \in \{0, 1, 2\}$, there exists an S_4-pair, say M_1 and M_2, such that $\nu_G[a : M_1, M_2] = i$.*

In Theorem 3.3.3 we show that the latter conjecture is actually equivalent to the S_4-Conjecture. The proof given in [75] to show the equivalence of the Fan–Raspaud Conjecture and Conjecture 3.2.1 is very similar to the proof we give here for the analogous case for the S_4-Conjecture, however we need a slightly more complicated tool in our context.

Theorem 3.3.3. *Conjecture 3.3.2 is equivalent to the S_4-Conjecture.*

Proof. Clearly, Conjecture 3.3.2 implies the S_4-Conjecture so it suffices to show the converse. Assume the S_4-Conjecture to be true and let f_1, f_2, f_3 be three consecutive edges of K_4 inducing a path. Consider two copies of G. Let the edge a in the i^{th} copy of G be denoted by a_i, for each $i \in [2]$. Let K_4' be the graph obtained by applying a 2-cut-connection on f_i and a_i for each $i \in [2]$. We refer to the copy of the graph $G - a$ on f_i as G_i.

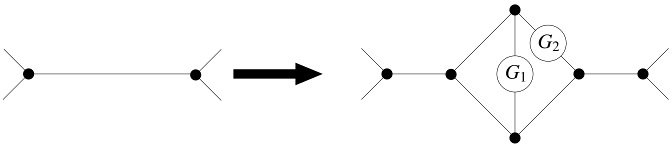

Figure 3.5: An edge in P transformed into the corresponding structure in H

Let $\{e_1, \dots, e_{15}\}$ be the edges of the Petersen graph and let $T_1, \dots, T_{15}$ be fifteen copies of K_4'. For every $i \in [15]$, apply a 2-cut-connection on e_i and the edge f_3 of T_i. Consequently, every edge e_i of the Petersen graph is transformed into the structure E_i as in Figure 3.5, and we refer to G_1 and G_2 on E_i as G_1^i and G_2^i, respectively. Let H be the resulting graph. By our assumption, there exists an S_4-pair of H, say M_1 and M_2, which induces a pair of two distinct perfect matchings in P, say N_1 and N_2, respectively. There exists an edge of P, say e_j, for some $j \in [15]$, such that $\nu_P[e_j : N_1, N_2] = 1$, since every two distinct perfect matchings of P have exactly one edge of P in common. Hence, the restriction of M_1 and M_2 to the edge set of G_1^j, together with the edge a having the same frequency as e_j, gives rise to an S_4-pair of G in which the frequency of a is 1.

Moreover, there exists an edge of P, say e_k, for some $k \in [15]$, such that $\nu_P[e_k : N_1, N_2] = 2$. Restricting M_1 and M_2 to the edge set of G_1^k, together with the edge a having the same frequency as e_k, gives rise to an S_4-pair of G, in which the frequency of a is 2. Also, the restriction of M_1 and M_2 to the edge set of G_2^k gives rise to an S_4-pair of G (G_2^k together with a), in which the frequency of a is 0, because if not, then there exists an odd circuit in G, say of length α, passing through a and having all its edges with frequency 0. However, this would mean that there is an odd circuit of length $\alpha + 4$ on E_k in $\overline{M_1 \cup M_2}$ (in H), a contradiction. □

As in Section 3.2, we state an analogous conjecture to Conjecture 3.2.2, but for S_4-pairs.

Conjecture 3.3.4. *Let G be a bridgeless cubic graph, w a vertex of G and i, j, k three integers in $\{0, 1, 2\}$ such that $i + j + k = 2$. Then, G has an S_4-pair in which the edges incident to w in a given order have frequencies (i, j, k).*

The following theorem shows that this conjecture is actually equivalent to Conjecture 3.3.2, and so to the S_4-Conjecture by Theorem 3.3.3.

Theorem 3.3.5. *Conjecture 3.3.4 is equivalent to the S_4-Conjecture.*

Proof. Since the S_4-Conjecture is equivalent to Conjecture 3.3.2, it suffices to show the equivalence of Conjectures 3.3.2 and 3.3.4. Clearly, Conjecture 3.3.4 implies Conjecture 3.3.2 and so we only need to show the converse. Let a, b, c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). We only need to prove the case when (i, j, k) is equal to $(1, 1, 0)$, as all other cases follow from Conjecture 3.3.2.

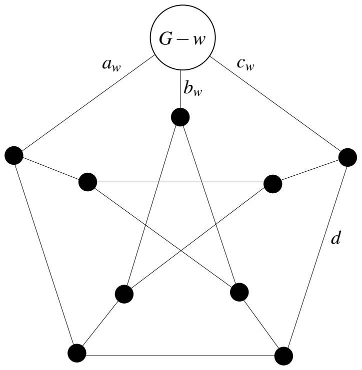

Figure 3.6: The graph $G(w) * P(v)$

Consider the graph $G(w) * P(v)$, where P is the Petersen graph and v is any vertex of P. We refer to the edges corresponding to a, b, c in $G(w) * P(v)$, as a_w, b_w, c_w. Let d be an edge originally belonging to P and adjacent to c_w in $G(w) * P(v)$. Since we are assuming Conjecture 3.3.2 to be true, there exists an S_4-pair in $G(w) * P(v)$ in which d has frequency 2. If the frequencies of (a_w, b_w, c_w) are $(1, 1, 0)$, then we are done, because the S_4-pair for $G(w) * P(v)$ restricted to the edges in $G - w$, together with a and b having the same frequencies as a_w and b_w, give an S_4-pair for G with the desired property. We claim that this must be the case. For, suppose not. Then, without loss of generality, the frequencies of (a_w, b_w, c_w) are $(2, 0, 0)$. This implies that all the edges of $G(w) * P(v)$ originally in P have either frequency 0 or 2, since the two perfect matchings in the S_4-pair induce the same perfect matching in P. However, this implies that $\overline{M_1 \cup M_2}$ is not bipartite, a contradiction. □

As in [75], a minimal counterexample to Conjecture 3.3.4 (but not necessarily to the S_4-Conjecture) is cyclically 4-edge-connected. We omit the proof of this result as it is very similar to the proof of Theorem 2 in [75].

3.4 FURTHER RESULTS ON THE S_4-CONJECTURE

Little progress has been made on the Fan–Raspaud Conjecture so far. Bridgeless cubic graphs which trivially satisfy this conjecture are those which can be edge-covered by four perfect matchings. In this case, every three perfect matchings from a cover of this type form an FR-triple since every edge has frequency one or two with respect to this cover. Therefore, a possible counterexample to the Fan–Raspaud Conjecture should be searched for in the class of bridgeless cubic graphs whose edge-set cannot be covered by four perfect matchings, see for instance [25]. In 2009, Máčajová and Škoviera [66] shed some light on the Fan–Raspaud Conjecture by proving it for bridgeless cubic graphs having oddness 2. One of the aims of this chapter is to show that even if the S_4-Conjecture is still open, some results are easier to extend than the corresponding ones for the Fan–Raspaud Conjecture. Clearly, the result by Máčajová and Škoviera in [66] implies the following result.

Theorem 3.4.1. *Let G be a bridgeless cubic graph of oddness 2. Then, G has an S_4-pair.*

We first give a proof of Theorem 3.4.1 in the same spirit of that used in [66], however much shorter since we are proving a weaker result.

Proof 1 of Theorem 3.4.1. Let M_1 be a minimal perfect matching of G, and let C_1 and C_2 be the two odd circuits in $\overline{M_1}$. Colour the even circuits in $\overline{M_1}$ using two colours, say 1 and 2. For each $i \in [2]$, let E_i be the set of edges belonging to the even circuits in $\overline{M_1}$ and having colour i. In G, there must exist a path Q whose edges alternate in M_1 and E_1 and whose end-vertices belong to C_1 and C_2, respectively, since C_1 and C_2 are odd circuits. Note that since the edges of C_1 and C_2 are not edges in $M_1 \cup E_1$, every other vertex on Q which is not an end-vertex does not belong to C_1 and C_2.

For each $i \in [2]$, let v_i be the end-vertex of Q belonging to C_i, and let M_{C_i} be the unique perfect matching of $C_i - v_i$. Let $M_2 = (M_1 \cap E(Q)) \cup (E_1 - E(Q)) \cup M_{C_1} \cup M_{C_2}$. Clearly, M_2 is a perfect matching of G which intersects C_1 and C_2, and so $\overline{M_1 \cup M_2}$ is bipartite. □

We now give a second alternative proof of the same theorem using fractional perfect matchings, a very convenient tool which we shall use for graphs having larger oddness. The following lemma is presented in [53] and is a consequence of Edmonds' characterisation of perfect matching polytopes in [22].

Lemma 3.4.2. *If w is a fractional perfect matching in a graph G, and $c \in \mathbb{R}^{|E(G)|}$, then G has a perfect matching N such that $c \cdot \chi^N \geq c \cdot w$, where $\cdot$ denotes the inner product. Moreover, there exists a perfect matching* *satisfying the above inequality and which contains exactly one edge of each odd-cut X with $w(X) = 1$.*

Remark 3.4.3. If we let $w(e) = 1/3$ for all $e \in E(G)$, for some graph G, then we know that w is a fractional perfect matching of G. Also, since the weight of every 3-cut is one, by Lemma 3.4.2 there exists a perfect matching of G containing exactly one edge of each 3-cut of G.

Proof 2 of Theorem 3.4.1. Let M_1 be a minimal perfect matching of G, and let C_1 and C_2 be the two odd circuits in $\overline{M_1}$. For each $i \in [2]$, let e_1^i and e_2^i be two adjacent edges belonging to C_i. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{2}\{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Also, as in Remark 3.4.3, we know that if we let $w(e) = 1/3$ for all $e \in E(G)$, then w is a fractional perfect matching of G. Hence, by Lemma 3.4.2, there exists a perfect matching M_2 such that $c \cdot \chi^{M_2} \geq c \cdot w$, which implies that

$$|(\cup_{i=1}^{2}\{e_1^i, e_2^i\}) \cap M_2| \geq 1/3 \times 2 \times 2 = 4/3 > 1.$$

Therefore, for each $i \in [2]$, there exists exactly one $j \in [2]$ such that $e_j^i \in M_2$. Hence, M_2 intersects C_1 and C_2 and so $\overline{M_1 \cup M_2}$ is bipartite. □

Using the same idea as in Proof 2 of Theorem 3.4.1, we also prove that the S_4-Conjecture is true for graphs having oddness 4.

Theorem 3.4.4. *Let G be a bridgeless cubic graph of oddness 4. Then, G has an S_4-pair.*

Proof. Let M_1 be a minimal perfect matching of G, and let C_1, C_2, C_3, C_4 be the four odd circuits in $\overline{M_1}$. By Remark 3.4.3, there exists a perfect matching N of G such that if G has any 3-cuts, then N intersects every 3-cut of G in one edge. Moreover, for every $i \in [4]$, there exists at least a pair of adjacent edges e_1^i and e_2^i belonging to $E(C_i) \cap E(\overline{N})$. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{4}\{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the vector $w \in \mathbb{R}^{|E(G)|}$ as follows:

$$w(e) = \begin{cases} 1/5 & \text{if } e \in N, \\ 2/5 & \text{otherwise.} \end{cases}$$

The vector w is clearly a fractional perfect matching of G because, in particular, N intersects every 3-cut in one edge and so $w(X) \geq 1$ for each odd-cut X of G. Hence, by Lemma 3.4.2, there exists a perfect matching M_2 such that $c \cdot \chi^{M_2} \geq c \cdot w$, which implies that

$$|(\cup_{i=1}^{4}\{e_1^i, e_2^i\}) \cap M_2| \geq 2/5 \times 2 \times 4 = 16/5 > 3.$$

Therefore, for each $i \in [4]$, there exists exactly one $j \in [2]$ such that $e_j^i \in M_2$. Hence, M_2 intersects C_1, C_2, C_3, C_4 and so $\overline{M_1 \cup M_2}$ is bipartite. □

As the above proofs show us, extending results with respect to the S_4-Conjecture is easier than in the case of the Fan–Raspaud Conjecture and this is why we believe that a proof of the S_4-Conjecture could be a first feasible step towards a solution of the Fan–Raspaud Conjecture. For graphs having oddness at least 6 we are not able to prove the existence of an S_4-pair and we wonder how many perfect matchings we need such that the complement of their union is bipartite. In the next proposition we use the technique used in Theorem 3.4.4 and show that given a bridgeless cubic graph G, if $\omega(G) \leq 5^{k-1} - 1$ for some positive integer k, then there exist k perfect matchings such that the complement of their union is bipartite. Note that for $k = 2$ we obtain $\omega(G) \leq 4$.

Proposition 3.4.5. *Let G be a bridgeless cubic graph and let $\mathcal{C}$ be a collection of disjoint odd circuits in G such that $|\mathcal{C}| \leq 5^{k-1} - 1$ for some positive integer k. Then, there exist $k - 1$ perfect matchings of G, say $M_1, \ldots, M_{k-1}$, such that for every $C \in \mathcal{C}$, there exists $j \in [k-1]$ for which $E(C) \cap M_j \neq \emptyset$. Moreover, if $\omega(G) \leq 5^{k-1} - 1$, then there exist k perfect matchings such that the complement of their union is bipartite.*

Proof. We proceed by induction on k. For $k = 1$, the assertion trivially holds since $\mathcal{C}$ is the empty set. Assume the result is true for some $k \geq 1$ and consider $k + 1$. Let $C_1, C_2, \ldots, C_t$, with $t \leq 5^k - 1$, be the odd circuits of G in $\mathcal{C}$. Let N be a perfect matching of G which intersects every 3-cut of G once. For every $i \in [t]$, there exists at least a pair of adjacent edges e_1^i and e_2^i belonging to $E(C_i) \cap E(\overline{N})$. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{t}\{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the vector $w \in \mathbb{R}^{|E(G)|}$ as follows:

$$w(e) = \begin{cases} 1/5 & \text{if } e \in N, \\ 2/5 & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 3.4.4, w is a fractional perfect matching of G and by Lemma 3.4.2 there exists a perfect matching M_k such that $c \cdot \chi^{M_k} \geq c \cdot w$. This implies that

$$|(\cup_{i=1}^{t}\{e_1^i, e_2^i\}) \cap M_k| \geq 2 \times 2/5 \times t.$$

Let $\mathcal{C}'$ be the subset of $\mathcal{C}$ which contains the odd circuits of $\mathcal{C}$ with no edge of M_k. Then, $|\mathcal{C}'| \leq |\mathcal{C}| - \frac{4}{5}t = t - \frac{4}{5}t = \frac{t}{5} \leq 5^{k-1} - \frac{1}{5}$, and so $|\mathcal{C}'| \leq 5^{k-1} - 1$. By induction, there exist $k-1$ perfect matchings of G, say $M_1, \ldots, M_{k-1}$, having the required property with respect to $\mathcal{C}'$. Therefore, $M_1, \ldots, M_k$ intersect all odd circuits in $\mathcal{C}$. The second part of the statement easily follows by considering as $\mathcal{C}$ the set of odd circuits in the complement of a minimal perfect matching M of G, since the union of M with the $k-1$ perfect matchings which intersect all the odd circuits in $\mathcal{C}$ has a bipartite complement. □

Remark 3.4.6. We note that with every step made in the proof of Proposition 3.4.5, one could update the weight w of the edges using the methods presented in [53, 68] which gives a slightly better upperbound for $\omega(G)$. For reasons of simplicity and brevity, we prefer the present weaker version of Proposition 3.4.5.

3.5 EXTENSION OF THE S_4-CONJECTURE TO LARGER CLASSES OF CUBIC GRAPHS

3.5.1 *Multigraphs*

In this section we discuss natural extensions of some previous conjectures to bridgeless cubic multigraphs. We note that bridgeless cubic multigraphs cannot contain any loops. We make use of the following standard operation on parallel edges, referred to as *smoothing*. Let G' be a bridgeless cubic multigraph. Let u and v be two vertices in G' such that there are exactly two parallel edges between them.

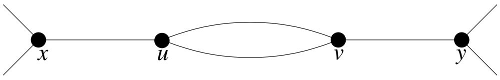

Figure 3.7: Vertices x, u, v, y in G'

Let x and y be the vertices adjacent to u and v, respectively (see Figure 3.7). We say that we *smooth* uv if we delete the vertices u and v from G' and add an edge between x and y (even if x and y are already adjacent in G'). One can easily see that the resulting multigraph, say G, after smoothing uv is again bridgeless and cubic.

In what follows, we say that a perfect matching M of G and a perfect matching M' of G' are *corresponding* perfect matchings if the following holds:

$$M = \begin{cases} M' \cup \{xy\} - \{xu, vy\} & \text{if } xu \in M', \\ M' - \{uv\} & \text{otherwise.} \end{cases}$$

The following theorem can be easily proved by using smoothing operations.

Theorem 3.5.1. *The S_4-Conjecture is true if and only if every bridgeless cubic multigraph has an S_4-pair.*

Now we show that Conjecture 3.3.4 can also be extended to multigraphs.

Theorem 3.5.2. *Let i, j, k be three integers in $\{0, 1, 2\}$ such that $i + j + k = 2$ and let w be a vertex in a bridgeless cubic multigraph G'. Then, the S_4-Conjecture is true if and only if G' has an S_4-pair in which the edges incident to w in a given order have frequencies (i, j, k).*

Proof. It suffices to assume that the S_4-Conjecture is true and only show the forward direction, by Theorem 3.5.1. Let G' be a minimal counterexample and suppose it has some parallel edges. If G' is the unique bridgeless cubic multigraph on two vertices, denoted by $C_{2,3}$, then the result clearly follows. So assume $G' \neq C_{2,3}$. Let a, b, c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). We proceed by considering two cases: when w has two parallel edges incident to it (Figure 3.8) and otherwise (Figure 3.9).

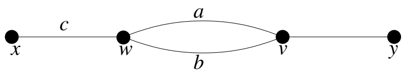

Figure 3.8: **Case 1** from the proof of Theorem 3.5.2

Case 1. Let G be the resulting multigraph after smoothing wv. By minimality of G', G has an S_4-pair (say M_1 and M_2) in which $\nu(xy) = k$. It is easy to see that a pair of corresponding perfect matchings in G' give $\nu_{G'}(c) = \nu_{G'}(vy) = k$ and can be chosen in such a way such that $\nu_{G'}(a) = i$ and $\nu_{G'}(b) = j$, a contradiction to our initial assumption. Therefore, we must have Case 2.

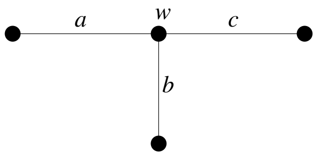

Figure 3.9: **Case 2** from the proof of Theorem 3.5.2

Case 2. Let G be the resulting multigraph after smoothing some parallel edge in G' and let a_w, b_w, c_w be the corresponding edges incident to w in G after smoothing is done. In G, there exists an S_4-pair such that the frequencies of (a_w, b_w, c_w) are equal to (i, j, k). Clearly, the corresponding perfect matchings in G' form an S_4-pair in which the frequencies of (a, b, c) are (i, j, k), a contradiction, proving Theorem 3.5.2. □

Using the same ideas as in Theorem 3.5.1 and Theorem 3.5.2 one can also state analogous results for the Fan-Raspaud Conjecture in terms of multigraphs.

3.5.2 *Graphs having bridges*

Since every perfect matching must intersect every bridge of a cubic graph, the Fan–Raspaud Conjecture cannot be extended to cubic graphs containing bridges. The situation is quite different for the S_4-Conjecture as Theorem 3.5.3 shows. By Errera's Theorem [23] we know that if all the bridges of a connected cubic graph lie on a single path, then the graph has a perfect matching. We use this idea to show that there can be graphs with bridges that can have an S_4-pair.

Theorem 3.5.3. *Let G be a connected cubic graph having k bridges, all of which lie on a single path, for some positive integer k. If the S_4-Conjecture is true, then G admits an S_4-pair.*

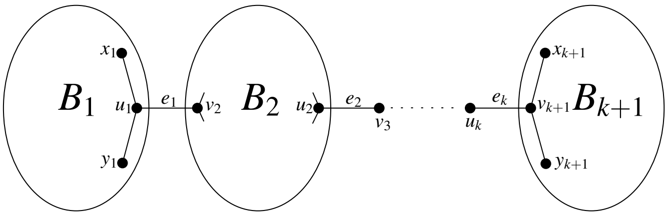

Figure 3.10: G with k bridges lying all on the same path

Proof. Let $B_1, B_2, \dots, B_{k+1}$ be the 2-connected components of G and let $e_1, \dots, e_k$ be the bridges of G such that $e_i = u_i v_{i+1}$ for each $i \in [k]$, where $u_i \in V(B_i)$ and $v_{i+1} \in V(B_{i+1})$. Let x_1 and y_1 be the two vertices adjacent to u_1 in B_1, and let x_{k+1} and y_{k+1} be the two vertices adjacent to v_{k+1} in B_{k+1}. Let $B'_1 = (B_1 - u_1) \cup \{x_1 y_1\}$ and $B'_{k+1} = (B_{k+1} - v_{k+1}) \cup \{x_{k+1} y_{k+1}\}$. Also, let $B'_i = B_i \cup v_i u_i$ for every $i \in \{2, \dots, k\}$. Clearly, $B'_1, \dots, B'_{k+1}$ are bridgeless cubic multigraphs. Since we are assuming that the S_4-Conjecture holds, by Theorem 3.5.1, for every $i \in [k+1]$, B'_i has an S_4-pair, say M^i_1 and M^i_2. Using Theorem 3.5.2, we choose the S_4-pair in:

- B'_1, such that the two edges originally incident to x_1 (not $x_1 u_1$) both have frequency 1;
- B'_i, for each $i \in \{2, \dots, k\}$, such that $\nu_{B'_i}(v_i u_i) = 2$; and
- B'_{k+1}, such that the two edges originally incident to x_{k+1} (not $x_{k+1} v_{k+1}$) both have frequency 1.

Let $M_1 = (\cup_{i=1}^{k+1} M^i_1) \cup (\cup_{j=1}^{k} \{e_j\}) - (\cup_{l=2}^{k} \{v_l u_l\})$, and let $M_2 = (\cup_{i=1}^{k+1} M^i_2) \cup (\cup_{j=1}^{k} \{e_j\}) - (\cup_{l=2}^{k} \{v_l u_l\})$. Then, M_1 and M_2 are an S_4-pair of G. □

Finally, we remark that there exist cubic graphs which admit a perfect matching however do not have an S_4-pair. For example, since the edges $u_i v_i$ in Figure 3.11 are bridges, they must be in every perfect matching. Consequently, every pair of perfect matchings do not intersect the edges of the odd circuit T. This shows that it is not possible to extend the S_4-Conjecture to the entire class of cubic graphs.

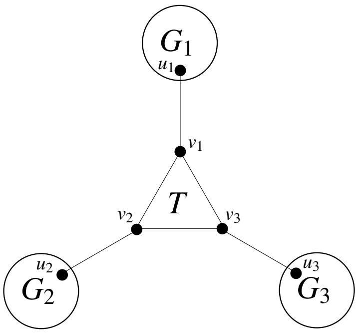

Figure 3.11: A cubic graph with bridges having no S_4-pair

3.6 FINAL REMARKS AND OPEN PROBLEMS

Many problems about the topics presented above remain unsolved: apart from asking if we can solve the Fan–Raspaud Conjecture and the S_4-Conjecture completely, or at least partially for higher oddness, we do not know which are those graphs containing bridges which admit an S_4-pair and we do not know either if the S_4-Conjecture is equivalent to Conjecture 3.1.2. Here we would like to add some other specific open problems.

For a positive integer k, we define ω_k to be the largest integer such that any graph with oddness at most ω_k, admits k perfect matchings with a bipartite complement. Clearly, for $k = 1$, we have $\omega_1 = 0$, since the existence of a perfect matching of G with a bipartite complement is equivalent to the 3-edge-colourability of G. Moreover, the S_4-Conjecture is equivalent to $\omega_k = \infty$, for $k \geq 2$, but a complete result to this is still elusive. Proposition 3.4.5 (see also Remark 3.4.6) gives a lowerbound for ω_k and it would be interesting if this lowerbound can be significantly improved. We believe that the following problem, weaker than the S_4-Conjecture, is another possible step forward.

Problem 3.6.1. Prove the existence of a constant k such that every bridgeless cubic graph admits k perfect matchings whose union has a bipartite complement.

It is also known that not every perfect matching can be extended to an FR-triple and neither to a Fulkerson cover. In fact, consider the Petersen graph P and apply a Y-extension to each of its vertices. Let the resulting graph on 30 vertices be P', and let M be the set of edges in P' corresponding to $E(P)$. The set M is a perfect matching of P', and it is not difficult to see that it cannot be extended to an FR-triple or to a Fulkerson cover. With regards to S_4-pairs, we do not see a way how to produce a similar argument. In fact, as can be seen in Figure 3.12, M (shown in blue) can be extended to an S_4-pair of P' (with the edges of the second perfect matching in red).

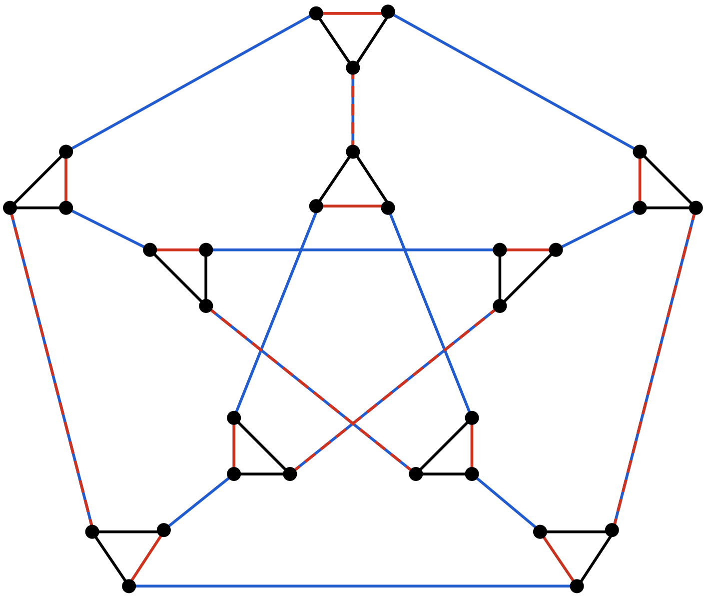

Figure 3.12: An S_4-pair of P'

We therefore suggest the following problem.

Problem 3.6.2. Establish whether any perfect matching of a bridgeless cubic graph can be extended to an S_4-pair.

It can be shown that if Problem 3.6.2 is true, then it is equivalent to saying that given any collection of disjoint odd circuits in a bridgeless cubic graph, then there exists a perfect matching which intersects all the odd circuits in this collection.

One implication is clearly obvious. Thus, assume that every perfect matching of any bridgeless cubic graph can be extended to an S_4-pair, and consider a collection of disjoint odd circuits in a bridgeless cubic graph G. Apply a Y-extension to every vertex not covered by the circuits in the collection and let the resulting bridgeless cubic graph be G'. The initial odd circuits and all the new Y-extended triangles give a 2-factor F of G', with complementary perfect matching, say M. By our assumption, there exists a perfect matching N such that $\overline{M \cup N}$ is bipartite, implying that N intersects all the odd circuits in F, including all the new Y-extended triangles. Let N_Y be the set of edges belonging simultaneously to N and the new Y-extended triangles. One can immediately see that $N - N_Y$ is a perfect matching of G intersecting all the odd circuits in the initial collection of odd circuits in G, proving the equivalence of the two statements.

Finally, together with Edita Máčajová, we also studied the problem of trying to extend S_4-pairs to an FR-triple. As already stated and seen in Section 3.4, it is somewhat easier working with S_4-pairs than with FR-triples, and it would be a profitable pursuit if we manage to tackle problems such as the Fan-Raspaud Conjecture, but work in a relatively easier environment. Due to the example given in Figure 3.12, we already know that not every S_4-pair can be extended to an FR-triple. However, the two perfect matchings considered in the S_4-pair from Figure 3.12 contain an odd-cut in their intersection. Thus, we thought that starting from a pair of perfect matchings (of a bridgeless cubic graph) having no odd-cuts in their intersection (see Conjecture 3.1.2), instead of just an S_4-pair, could be a possible way forward. However, this turned out to be a futile pursuit as the graph G in Figure 3.13 shows.

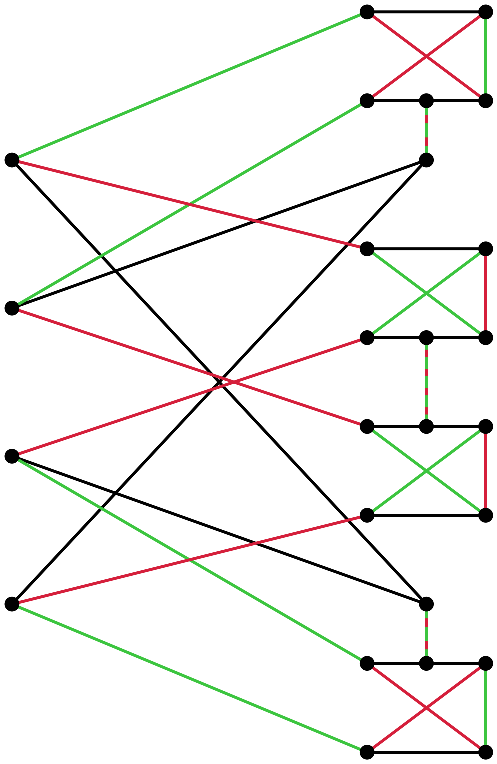

Figure 3.13: M_1 and M_2 cannot be extended to an FR-triple

In fact, let the green and red edges be the edges of our initial S_4-pair M_1 and M_2, with the additional property that $M_1 \cap M_2$ does not contain any odd-cut. If we want to find a perfect matching M_3, such that $M_1 \cap M_2 \cap M_3 = \emptyset$, that is, an FR-triple which extends the initial S_4-pair, then M_3 should be contained in $G - (M_1 \cap M_2)$. However, one can see that if we delete the vertices on the left hand side from $G - (M_1 \cap M_2)$, what remains is a collection of six odd components. Consequently, there exists a set of vertices $S \in V(G - (M_1 \cap M_2))$, such that the number of odd components in $G - (M_1 \cap M_2) - S$ is strictly greater than $|S|$, and so, by Tutte's Theorem (see Theorem 1.1.2), $G - (M_1 \cap M_2)$ does not admit a perfect matching.

Although the graph in Figure 3.13 is a legitimate counterexample to what we were searching for, we have to say that the graph is a Class I graph and, as already mentioned before, it satisfies all the above related conjectures quite easily. It would be interesting and more insightful to have a Class II counterexample.

We would like to finish this chapter by providing the theory behind the construction of the above counterexample. An S_4-pair M_1 and M_2 of a bridgeless cubic graph G can be extended to an FR-triple of G if and only if $G - (M_1 \cap M_2)$ admits a perfect matching. By using this straightforward observation together with Tutte's Theorem (Theorem 1.1.2), we manage to isolate the reason which impedes an S_4-pair from being extended to an FR-triple. This is shown in the following result.

Proposition 3.6.3. *Let G be a bridgeless cubic graph admitting an S_4-pair M_1 and M_2, such that $M_1 \cap M_2$ contains no odd-cuts. Let H be a component of $G - (M_1 \cap M_2)$. The graph G admits a perfect matching M_3 with $M_1 \cap M_2 \cap M_3 = \emptyset$ if and only if for every $S \subseteq V(H)$, the number of odd components $o(H - S)$ in $H - S$ is not equal to $|S| + 2$.*

Proof. Without loss of generality, assume that $G - (M_1 \cap M_2)$ is connected, that is, $H = G - (M_1 \cap M_2)$. One direction is clear, and so assume that $o(H - S) \neq |S| + 2$, for every $S \subseteq V(H)$. We proceed by using induction on $|S|$. We first remark that $o(H - S) \equiv |S|$ since H is of even order. We also note that every odd component in $H - S$ is connected to at least 2 vertices in S (since H is bridgeless), and every vertex in S is connected to at most 3 odd components in $H - S$, and so $o(H - S) \leq \frac{3}{2}|S|$. Consequently, one can easily see that $o(H) = 0$, and $o(H - S) \leq |S|$ when $|S| = 1$ or 2. Thus, assume $|S| > 2$. If there exists a vertex $u \in S$ which is not connected to any odd components in $H - S$, then $o(H - (S - \{u\})) = o(H - S) + 1$, and so, by induction, $o(H - S) + 1 = o(H - (S - \{u\})) \leq |S| - 1$, implying that $o(H - S) \leq |S|$, as required. On the other hand, if there exists a vertex $u \in S$ which is connected to one or two odd components in $H - S$, then we are also done, because then $o(H - (S - \{u\})) = o(H - S) - 1$, and by induction, $o(H - S) - 1 = o(H - (S - \{u\})) \leq |S| - 1$, proving our result. Therefore, assume that every vertex in S is connected to three different odd components in $H - S$. Consequently, there are no even components in $H - S$ because otherwise there would be a vertex in S which is connected to them, since H is connected. Let $v \in S$. Then, $o(H - (S - \{v\})) = o(H - S) - 3$ and so by the induction hypothesis, $o(H - S) - 3 \leq |S| - 1$, implying that $o(H - S) \leq |S| + 2$. Since $o(H - S) \equiv |S|$, we either have $o(H - S) \leq |S|$ or $o(H - S) = |S| + 2$. By our assumption, $o(H - S) \neq |S| + 2$. Result follows by Tutte's Theorem. □

In this sense, we think that the following class of subcubic graphs might be a possible way forward to address the Fan–Raspaud Conjecture.

Definition 3.6.4. Let $\mathcal{G}$ be the class of (connected) cubic Class I multigraphs admitting an even 2-factor, that is, a 2-factor made up of circuits of even length only. Let H be a graph obtained from a multigraph $G \in \mathcal{G}$ with an even 2-factor $\mathcal{C}$, by performing a series of subdivisions on edges not belonging to the circuits in $\mathcal{C}$, such that the number of degree 2 vertices in the resulting graph is even. Let $\mathcal{H}$ be the class of all such graphs H, over all possible multigraphs in $\mathcal{G}$.

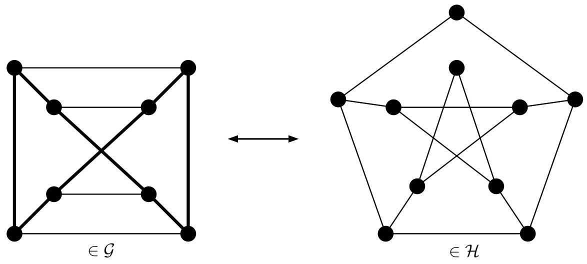

Figure 3.14: The corresponding graphs of the Petersen graph P, in $\mathcal{G}$ and $\mathcal{H}$

The graph $G - (M_1 \cap M_2)$, for a bridgeless cubic graph G with an S_4-pair M_1 and M_2 clearly belongs to the class $\mathcal{H}$ defined above. It would be intriguing to see which graphs in $\mathcal{H}$ admit a subset of vertices S, such that when removed the resulting number of odd components is equal to $|S| + 2$. An illustration of how a multigraph in $\mathcal{G}$ (with its even 2-factor shown in bold) and the corresponding graph in $\mathcal{H}$ look like is given in Figure 3.14.

4

SOME SNARKS ARE WORSE THAN OTHERS

This chapter is based on a joint work with Edita Máčajová, Giuseppe Mazzuoccolo and Vahan Mkrtchyan [II].

4.1 INTRODUCTION

It is well-known that many long standing conjectures and open problems in graph theory can be reduced to the class of cubic graphs. That is, if one can prove such a conjecture for all cubic graphs then the general statement for arbitrary graphs will immediately follow. The Cycle Double Cover Conjecture [105] falls into this category. Some other well-known conjectures are formulated directly for cubic graphs such as the Petersen Colouring Conjecture (Conjecture 1.2.3) and the Berge–Fulkerson Conjecture (Conjecture 1.2.1).

In all mentioned conjectures, only a very small subset of all cubic graphs is critical for proving them. A classical result by Vizing [100] naturally divides bridgeless cubic graphs in two classes: 3-edge-colourable cubic graphs (Class I) and snarks (which are Class II bridgeless cubic graphs). In addition, the class of snarks relevant for some old and new problems can be further restricted to a specific subset of $\mathcal{S}$, which we denote by $\mathcal{S}_{\geq 5}$ (see definition below). More precisely, $\mathcal{S}_{\geq 5}$ is shown to be critical for several, seemingly unrelated, problems. In order to define the class $\mathcal{S}_{\geq 5}$, we need the following parameter.

Definition 4.1.1. The *perfect matching index* of a graph G, denoted by $\chi'_e(G)$, is the minimum number of perfect matchings of G whose union covers the whole set $E(G)$. If such a number does not exist, $\chi'_e(G)$ is defined to be infinity. This parameter is also known in literature as the *excessive index* of a graph (see [13]).

Since bridges in cubic graphs belong to every perfect matching, the perfect matching index of a cubic graph having a bridge is infinite. Consequently, in what follows, we only consider bridgeless cubic graphs. Trivially, the chromatic index $\chi'(G)$ of a cubic graph G is 3 if and only if its perfect matching index is 3, and so, the two parameters coincide for Class I cubic graphs. The same cannot be said for Class II bridgeless cubic graphs. Indeed, there exist examples of such graphs having perfect matching index 4 and others having perfect matching index 5, such as the well-known Petersen graph. In what follows, we denote the set of snarks having perfect matching index equal to 4 by

$\mathcal{S}_4$ and the set of snarks having perfect matching index at least 5 by $\mathcal{S}_{\geq 5}$. Consequently, the following holds:

$$\mathcal{S} = \mathcal{S}_4 \cup \mathcal{S}_{\geq 5}.$$

The above situation is summarised in Table 4.1. Clearly, the Berge–Fulkerson Conjecture implies that all bridgeless cubic graphs have perfect matching index at most 5 (see also Section 1.2). If this conjecture is shown to be true, it would imply that all snarks in $\mathcal{S}_{\geq 5}$ have perfect matching index exactly equal to 5.

Cubic graph G	$\chi'(G)$	$\chi'_e(G)$	
CLASS I	3	3	
CLASS II ($\mathcal{S}$)	4	4	($\mathcal{S}_4$)
		≥ 5	($\mathcal{S}_{\geq 5}$)

Table 4.1: The relation between $\chi'(G)$ and $\chi'_e(G)$

The reason why the class $\mathcal{S}_{\geq 5}$ deserves particular attention not only in relation to the Berge–Fulkerson Conjecture but also with respect to other problems, is already very present in literature. Moreover, from amongst more than sixty million non-trivial snarks of order at most 36 (see [14]), only two belong to $\mathcal{S}_{\geq 5}$, and both of them have perfect matching index equal to 5. This suggests that the subset of snarks that is substantial for many open problems is negligible compared to its complement. On the other hand, infinite classes of non-trivial snarks belonging to $\mathcal{S}_{\geq 5}$ are constructed in [1, 25, 65].

One of the most relevant results that shows the importance of the class $\mathcal{S}_{\geq 5}$ was proven independently by Steffen [91], and by Xinmin Hou et al. [46], and states that each snark in $\mathcal{S}_4$ admits a cycle double cover. Thus, if a cubic graph is a counterexample to the Cycle Double Cover Conjecture, then it must belong to $\mathcal{S}_{\geq 5}$.

The Fan–Raspaud Conjecture (Conjecture 3.1.1) is obviously true for 3-edge-colourable cubic graphs and graphs from $\mathcal{S}_4$, making the family $\mathcal{S}_{\geq 5}$ critical once again.

Let us mention a last example: the problem of finding a shortest cycle cover of a bridgeless graph (not necessarily cubic).

Definition 4.1.2. Let G be a bridgeless graph. The minimum total length over all possible cycle covers of G is denoted by $scc(G)$, and a cycle cover having length $scc(G)$ is called a *shortest cycle cover*.

The Shortest Cycle Cover Conjecture by Noga Alon and Michael Tarsi [3] asserts that $scc(G) \leq 7/5 \cdot |E(G)|$. In [91], it is shown that if a graph G belongs to $\mathcal{S}_4$, then $scc(G) = 4/3 \cdot |E(G)|$, thus leaving, once again, the conjecture open only for graphs from $\mathcal{S}_{\geq 5}$.

All previous examples give a strong motivation to the study of the class $\mathcal{S}_{\geq 5}$. Having said this, a very recent result by Máčajová and Škoviera [64] shattered every hope that this class is also critical when dealing with the 5-flow Conjecture. A couple of years ago, it was pointed out by Abreu et al. in [1], and later on by Miguel Angel Fiol et al. in [30], that all known examples of snarks with perfect matching index equal to 5 also have circular flow number 5 (see [35] for a definition). In other words, it seemed that all snarks having the largest possible perfect matching index according to the Berge–Fulkerson Conjecture, also had the largest possible circular flow number according to the 5-Flow conjecture. Recently, however, it was showed in [64] that there exists a family of cyclically 4-edge-connected cubic graphs of girth at least 5 (non-trivial snarks) belonging to $\mathcal{S}_{\geq 5}$ and with circular flow number strictly less than 5, with the result heavily depending on the results obtained by the same two authors in [65]. We remark that there exists a large number of non-trivial snarks having circular flow number 5 and perfect matching index 4, as shown by Jan Goedgebeur et al. in [36] (see also [62]).

In this chapter, we study parameters which have a potential to further refine $\mathcal{S}_{\geq 5}$ and thus enlarge the set of cubic graphs for which the Cycle Double Cover Conjecture, the Fan–Raspaud Conjecture and other related problems can be proven. As a by-product, we also consider a parameter which identifies graphs in $\mathcal{S}_4$ that are, in a sense (explained later), closer to being 3-edge-colourable. Now we describe these parameters in more detail.

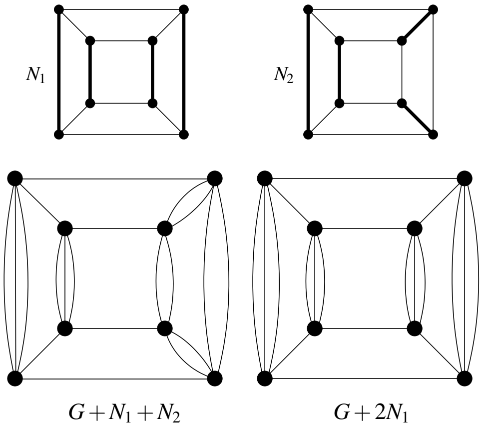

Figure 4.1: Perfect matchings N_1 and N_2 in G and the graphs $G+N_1+N_2$ and $G+2N_1$

Let G be any graph, and let $N \subseteq E(G)$. We denote by $G+N$ the multigraph obtained from G after adding a parallel edge to every edge in N. In general, let $N_1, N_2, \ldots, N_t \subseteq E(G)$. We denote by $G+N_1+\cdots+N_t$, or equivalently by $G+\sum_{i=1}^{t} N_i$, the multigraph obtained by adding to every edge of G a number of parallel edges equal to the number of times the original edge appears in $N_1, \ldots, N_t$. In the special case when we add t times the same set of edges N, the resulting multigraph is denoted by $G + tN$ (examples are given in Figure 4.1). Since in this chapter multigraphs shall be encountered more frequently than in other ones, graphs in this chapter are allowed to contain parallel edges, for simplicity.

The study of the following problem was firstly proposed to some of the authors by Gunnar Brinkmann and Eckhard Steffen during the workshop KOLKOM 2017 in Paderborn.

Problem 4.1.3 (Brinkmann and Steffen, 2017)**.** Given a bridgeless cubic graph G, when does there exist k perfect matchings $M_1, ..., M_k$ of G, for some integer $k \geq 0$, such that the graph $G + M_1 + ... + M_k$ is $(k+3)$-edge-colourable or, equivalently, is Class I?

In the sequel, a $(k+3)$-edge-colouring of the multigraph $G + M_1 + \ldots + M_k$ is sometimes considered as the proper edge-colouring of G in which every edge e is assigned $\nu(e) + 1$ colours, where $\nu(e)$ is the number of times e appears in the list $M_1, \ldots, M_k$. Moreover, if $G + M_1 + \ldots + M_k$ is $(k+3)$-edge-colourable, with the perfect matchings $F_1, \ldots, F_{k+3}$ as its colours, we write $G + M_1 + \ldots + M_k = F_1 + \ldots + F_{k+3}$.

For a graph G, we define the following parameter related to this problem.

Definition 4.1.4. Denote by $l(G)$ the minimum number of perfect matchings needed to be added to G such that the resulting graph is Class I. If such a number does not exist, then we set $l(G) = +\infty$.

Obviously, for a cubic graph G, $l(G) = 0$ if and only if G is 3-edge-colourable. Observe also that the Berge Conjecture (Conjecture 1.2.2) is true for cubic graphs G with $l(G) \leq 2$. A slight variation of the previous definition shall also be of interest later on in the chapter.

Definition 4.1.5. Let G be a graph admitting a perfect matching M. Denote by $l_M(G)$ the minimum number of copies of M which need to be added to G such that the resulting graph is Class I. If such a number does not exist for M, we set $l_M(G) = +\infty$.

Lewis Carroll has already been a great source of graph theoretical jargon, especially when dealing with snarks: with words like *"boojum"* [34, 96] and *"bandersnatch"* [90] used to represent snarks or graphs having some particular property. Below, we study what we believe is another *"unmistakable"* characteristic of snarks so much so to deserve another Carrollian word which captures this bizarre behaviour. Consequently, we say that a bridgeless cubic graph G is *frumious*† if $l_M(G) = +\infty$ for all perfect matchings M of G, and we conjecture that frumious snarks are exactly the snarks in $\mathcal{S}_{\geq 5}$.

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† Coined by Lewis Carroll and was first used in his poem *Jabberwocky*. It is the blend of fuming and furious.

In the following sections we give some results on the three parameters just defined: $l(G), l_M(G)$ and $scc(G)$, and show that the class $\mathcal{S}_{\geq 5}$ seems to be critical in the study of all of them. More precisely, in Section 4.2 we determine which bridgeless cubic graphs G admit a finite value for $l(G)$, and conclude that in some sense the Petersen graph is the only obstruction for this parameter to be finite. We also show that this parameter can be arbitrarily large (see Corollary 4.2.9). In Theorem 4.3.3 we show that there exist snarks in $\mathcal{S}_4$ which are closer to being Class I than other snarks in $\mathcal{S}_4$: we show that $l_M(G) = 1$ for any flower snark G and for any perfect matching M of G, except the Tietze graph. We remark that this was independently shown in [67]. In Section 4.4 we show that given a bridgeless cubic graph G, $scc(G)$ is equal to $4/3 \cdot |E(G)|$ if and only if there exists a perfect matching M of G for which $l_M(G)$ is finite. In particular, extending a result in [25], we prove that the graphs in an infinite family of snarks in $\mathcal{S}_5$ (a generalisation of treelike snarks) admit a shortest cycle cover whose length is strictly greater than $4/3$ their size.

4.2 THE PARAMETER $l(G)$

We recall that for a graph G, $l(G)$ denotes the minimum number of perfect matchings needed to be added to G in order to obtain a Class I graph. This section has two aims: to derive a sufficient condition for a bridgeless cubic graph G for which $l(G)$ is finite (see Lemma 4.2.6) and, in such a case, to show that $l(G)$ can be arbitrarily large (see Proposition 4.2.8 and Corollary 4.2.9). Along the entire section, let G be a bridgeless cubic graph. As already mentioned, $l(G) = 0$ if and only if G is 3-edge-colourable. Another easy observation is the following.

Proposition 4.2.1. *For every bridgeless cubic graph G, $l(G) = 1$ if and only if $\chi'_e(G) = 4$.*

Proof. If $l(G) = 1$, then G admits a perfect matching, say M, such that $G + M = F_1 + F_2 + F_3 + F_4$, where each F_i is a perfect matching of G. Clearly, $E(G) = \cup_{i=1}^{4} F_i$, implying that $\chi'_e(G) \leq 4$. Since G is not itself Class I, $\chi'_e(G) = 4$. Conversely, assume $\chi'_e(G) = 4$. Consequently, $E(G) = \cup_{i=1}^{4} M_i$, for some perfect matchings M_i of G. Each edge of G belongs to exactly one or two of these four perfect matchings. The edges belonging to exactly two of these perfect matchings induce a perfect matching which we denote by M. Since $G + M = M_1 + M_2 + M_3 + M_4$, we have $l(G) = 1$. □

By the above, we have $l(G) > 1$ if and only if $\chi'_e(G) \geq 5$. In what follows, we analyse the behaviour of $l(G)$ in the class $\mathcal{S}_{\geq 5}$. We start with the smallest bridgeless cubic graph having perfect matching index equal to 5: the Petersen graph P. In some sense, we prove that the Petersen graph is the unique obstruction for a graph G to have a finite value for $l(G)$.

We start with a simple characterisation of graphs that meet the conditions of the original problem proposed, in which the notion of the perfect matching lattice is used. A graph G is *matching covered* if any edge of G lies in a perfect matching of G. The *perfect matching lattice* $Lat(G)$ of a matching covered graph G is defined as the set of all $|E(G)|$-dimensional integral vectors over $\mathbb{Z}$ that can be represented as a sum or difference of characteristic vectors of some perfect matchings of G. In other words, for a vector $w \in \mathbb{Z}^{|E(G)|}$, we have $w \in Lat(G)$ if and only if G admits perfect matchings $J_1, ..., J_s$ and $N_1, ..., N_t$, such that

$$\vec{w} = \chi^{J_1} + ... + \chi^{J_s} - \chi^{N_1} - ... - \chi^{N_t}.$$

Let $\vec{1}$ be the $|E(G)|$-dimensional vector whose coordinates are all 1.

Proposition 4.2.2. *For a bridgeless cubic graph G, $l(G) < +\infty$ if and only if $\vec{1} \in Lat(G)$.*

Proof. Assume that $l(G) = k$. Hence $G + M_1 + \ldots + M_k$ is Class I, for some k perfect matchings $M_1, \ldots, M_k$ of G. Consequently, G admits $k + 3$ perfect matchings $F_1, ..., F_{k+3}$ which partition the edge set of $G + M_1 + ... + M_k$. One can easily see that

$$\vec{1} = \chi^{F_1} + ... + \chi^{F_{k+3}} - \chi^{M_1} - ... - \chi^{M_k},$$

as required. Conversely, assume that

$$\vec{1} = \chi^{J_1} + ... + \chi^{J_s} - \chi^{N_1} - ... - \chi^{N_t},$$

for some perfect matchings $J_1, ..., J_s$ and $N_1, ..., N_t$ of G and some integers $s, t \geq 0$. Since G is cubic, s must be equal to $t + 3$. It is not hard to see that the perfect matchings $J_1, ..., J_s$ partition the edge set of $G + N_1 + ... + N_t$. Hence $l(G) \leq t$. □

The above proposition allows us to construct an example of a bridgeless cubic graph G for which $l(G) = +\infty$. As one can expect, this is the Petersen graph, and the proof follows from [60] (see also [16, 17]).

Proposition 4.2.3. *If P is the Petersen graph, then $\vec{1} \notin Lat(P)$.*

4.2.1 *Graphs with $l(G)$ infinite*

Next we characterise bridgeless cubic graphs G for which $l(G) = +\infty$, according to their edge-connectivity. We can assume that G is connected, for if G is comprised of components $G_1, ..., G_t$, for some integer $t > 1$, then $\vec{1} \in Lat(G)$ if and only if $\vec{1} \in Lat(G_i)$, for all $i = 1, ..., t$. First we consider graphs having 2-edge-cuts.

Lemma 4.2.4. *Let G be a bridgeless cubic graph having a 2-edge-cut X. Let G_1 and G_2 be the two bridgeless cubic graphs obtained by applying a 2-edge-reduction on X. Then, $\vec{1} \in Lat(G)$ if and only if $\vec{1} \in Lat(G_1)$ and $\vec{1} \in Lat(G_2)$.*

Proof. Let $X = \{e_1, e_2\}$ and let the new edges in G_1 and G_2 be denoted by f_1 and f_2, respectively. First assume that $\vec{1} \in Lat(G)$. Any perfect matching M of G contains either both or none of the edges of X. In the former case, M gives rise to a perfect matching of G_i by simply adding f_i to $M \cap E(G_i)$, for $i = 1, 2$. Otherwise, $M \cap E(G_i)$ is a perfect matching of G_i. By using this idea and considering the new perfect matchings of G_1 and G_2 obtained from the list of perfect matchings of G whose sum and difference of their characteristic vectors give $\vec{1} \in \mathbb{Z}^{|E(G)|}$, one can easily show that $\vec{1} \in Lat(G_1)$ and $\vec{1} \in Lat(G_2)$, as required.

Conversely, assume that $\vec{1} \in Lat(G_1)$ and $\vec{1} \in Lat(G_2)$. Then, G_1 admits two sets of perfect matchings $\mathcal{J}_1 = \{J_1^{(1)}, ..., J_{s+3}^{(1)}\}$ and $\mathcal{N}_1 = \{N_1^{(1)}, ..., N_s^{(1)}\}$ such that, $\vec{1} \in \mathbb{Z}^{|E(G_1)|}$ can be represented as $\sum_{J \in \mathcal{J}_1} \chi^J - \sum_{N \in \mathcal{N}_1} \chi^N$, for some integer $s \geq 0$. Similarly, G_2 admits two sets of perfect matchings $\mathcal{J}_2 = \{J_1^{(2)}, ..., J_{t+3}^{(2)}\}$ and $\mathcal{N}_2 = \{N_1^{(2)}, ..., N_t^{(2)}\}$ such that, $\vec{1} \in \mathbb{Z}^{|E(G_2)|}$ can be represented as $\sum_{J \in \mathcal{J}_2} \chi^J - \sum_{N \in \mathcal{N}_2} \chi^N$, for some integer $t \geq 0$. The number of perfect matchings in $\mathcal{J}_1 \cup \mathcal{N}_1$ which contain f_1 is odd, and is denoted by $2s' + 1$, for some integer $s' \geq 0$. Moreover, the number of perfect matchings containing f_1 in $\mathcal{J}_1$ is one more than the number of such perfect matchings in $\mathcal{N}_1$. The same applies for G_2, and, in this case, we denote the total number of perfect matchings in $\mathcal{J}_2 \cup \mathcal{N}_2$ which contain f_2 by $2t' + 1$, for some integer $t' \geq 0$.

We can further assume that $2s' + 1 = 2t' + 1$, for, suppose that $s' < t'$, without loss of generality. By taking any perfect matching F of G_1 containing f_1 (the existence is guaranteed by [86]), it is easy to see that

$$\sum_{J \in \mathcal{J}_1} \chi^J + \sum_{i=1}^{t'-s'} \chi^F - \sum_{N \in \mathcal{N}_1} \chi^N - \sum_{i=1}^{t'-s'} \chi^F = \vec{1} \in \mathbb{Z}^{|E(G_1)|}.$$

Consequently, a new list of perfect matchings of G_1 whose characteristic vectors give $\vec{1} \in \mathbb{Z}^{|E(G_1)|}$ is obtained. Moreover, exactly $2t' + 1$ perfect matchings from this list contain the edge f_1, as required, and so we can assume that $s' = t'$. By a similar reasoning we can assume that $s = t$.

Without loss of generality, let the first $s' + 1$ perfect matchings in $\mathcal{J}_1$ (and $\mathcal{J}_2$), and the first s' perfect matchings in $\mathcal{N}_1$ (and $\mathcal{N}_2$) contain f_1 (and f_2). Let $\mathcal{J} = \{J_1, \ldots, J_{s+3}\}$, where

$$J_i = \begin{cases} (J_i^{(1)} - \{f_1\}) \cup (J_i^{(2)} - \{f_2\}) \cup \{e_1, e_2\} & \text{if } i = 1, \ldots, s' + 1, \\ J_i^{(1)} \cup J_i^{(2)} & \text{otherwise.} \end{cases}$$

Similarly, let $\mathcal{N} = \{N_1, \ldots, N_s\}$, where

$$N_i = \begin{cases} (N_i^{(1)} - \{f_1\}) \cup (N_i^{(2)} - \{f_2\}) \cup \{e_1, e_2\} & \text{if } i = 1, \ldots, s', \\ N_i^{(1)} \cup N_i^{(2)} & \text{otherwise.} \end{cases}$$

One can see that $\mathcal{J}$ and $\mathcal{N}$ are two sets consisting of perfect matchings of G, such that $\sum_{J \in \mathcal{J}} \chi^J - \sum_{N \in \mathcal{N}} \chi^N = \vec{1} \in \mathbb{Z}^{E(G)}$, as required. □

The proved statement implies that $l(G) = +\infty$ if and only if $l(G_1) = +\infty$ or $l(G_2) = +\infty$. Thus, in trying to characterise the bridgeless cubic graphs G with $l(G) = +\infty$, one can focus on 3-edge-connected graphs having 3-edge-cuts. Following [60], we say that an edge-cut in G is *tight* if any perfect matching of G intersects it in exactly one edge (not necessarily the same).

Lemma 4.2.5. *Let G be a 3-edge-connected cubic graph and let X be a non-trivial tight 3-edge-cut in G. Consider the two bridgeless cubic graphs G_1 and G_2 obtained by applying a 3-edge-reduction to X. Then, $\vec{1} \in Lat(G)$ if and only if $\vec{1} \in Lat(G_1)$ and $\vec{1} \in Lat(G_2)$.*

This statement can be derived from the results of [60]. Moreover, its proof follows a similar argument to the one used in the proof of Lemma 4.2.4. For these reasons we omit the proof here.

Before we proceed to prove the next result regarding 3-edge-connected cubic graphs which do not contain non-trivial tight 3-edge-cuts we give the definition of a brick. A *brick* is a 3-connected graph such that for any two distinct vertices u and v of G, $G - \{u, v\}$ admits a perfect matching. It is easy to see that no brick can be bipartite.

Lemma 4.2.6. *Let G be a 3-edge-connected cubic graph without non-trivial tight 3-edge-cuts. Then, $\vec{1} \in Lat(G)$ if and only if G is not the Petersen graph.*

Proof. If G is the Petersen graph, then by Proposition 4.2.3, $\vec{1} \notin Lat(G)$. So assume that G is not the Petersen graph. Since G is cubic, by [54] we have that all tight edge-cuts of G are 3-edge-cuts. Thus, by our assumptions, G contains no tight edge-cuts. Hence, by [60], G is either bipartite or a brick. Now, if G is bipartite, then it is 3-edge-colourable and so $\vec{1} \in Lat(G)$. Hence, we can assume that G is a brick. The main result of [60] implies that the only cubic brick for which $\vec{1} \notin Lat(G)$ is the Petersen graph, proving our result. □

Corollary 4.2.7. *Let G be a cyclically 4-edge-connected cubic graph different from the Petersen graph. Then, $l(G)$ is finite.*

4.2.2 *Construction of cubic graphs with $l(G)$ finite but arbitrarily large*

We have already seen that $l(G) \leq 1$ if and only if $\chi'_e(G) \leq 4$. The results obtained above suggest an algorithm to check whether $l(G) =$

$+\infty$ for a given bridgeless cubic graph G. The next question that we would like to address is to see whether there exist graphs in $\mathcal{S}_{\geq 5}$ with $1 < l(G) < \infty$. In Corollary 4.2.9, we show that there exist bridgeless cubic graphs G with $l(G)$ finite but arbitrarily large.

Let G be a bipartite graph with bipartition U and W. Let $u \in U$. We say that G is *coverable with respect to* u if for every $w \in W$ there exists a parity subgraph of G in which the vertices u and w are of degree 3 and all the other vertices are of degree 1. We remark that a *parity subgraph* of G is a spanning subgraph of G with the degrees of all the vertices having the same parity in both the subgraph and in G.

Let G be a bipartite cubic graph of order $2n$ having bipartition U and W. Assume $W = \{w_1, w_2, \ldots, w_n\}$ and let $u \in U$. Let v be a vertex of the Petersen graph P, and let $P_1, \ldots, P_n$ be n copies of the Petersen graph, with the vertex corresponding to v in each copy denoted by $v_1, \ldots, v_n$, respectively. Apply a 3-cut-connection on v_i and w_i, for each $i \in [n]$ and apply a Y-extension to u. The resulting graph is called an *extension of* G *with respect to* u.

Proposition 4.2.8. *Let G be a bipartite cubic graph of order $2n$ and let H be an extension of G with respect to u, for some $u \in V(G)$. If G is coverable with respect to u, then $l(H) = n$.*

Proof. We claim that $l(H) \geq n$. Suppose that $l(H) = k < n$, for contradiction. Then, H admits k perfect matchings $M_1, ..., M_k$, such that $H + M_1 + ... + M_k$ is Class I. Since G is bipartite, if a perfect matching M of H intersects all the three edges of $\partial(P_i - v_i)$ in H, for some $i \in [n]$, then, $|M \cap \partial(P_j - v_j)| = 1$, for all $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$. In this case, M must also intersect the three edges incident to the triangle in H. Since $k \leq n-1$, there exists some $s \in [n]$ such that $\partial(P_s - v_s)$ is not contained in any perfect matching in $M_1, ..., M_k$. Thus, these perfect matchings intersect exactly one edge from the 3-edge-cut $\partial(P_s - v_s)$. Hence, $M_1, \ldots, M_k$ induce k perfect matchings of the Petersen graph (P_s), say $M'_1, \ldots, M'_k$. Let $F_1, \ldots, F_{k+3}$ be the $k+3$ colours of $H + M_1 + \ldots + M_k$. By a simple counting argument, $|F_i \cap \partial(P_s - v_s)| = 1$, for each $i \in [k+3]$. Therefore, the F_is induce $k+3$ perfect matchings of the Petersen graph (P_s), say $F'_1, \ldots, F'_{k+3}$. However, this implies that $P_s + M'_1 + \ldots + M'_k = F'_1 + \ldots + F'_{k+3}$, a contradiction to Proposition 4.2.3.

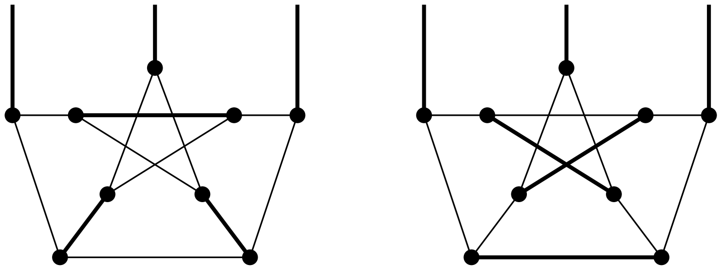

Figure 4.2: The way N_i and J_i intersect P_i

Now, we show that $l(H)$ is actually equal to n. For each $i \in [n]$, let N_i be a perfect matching of H containing $\partial(P_i - v_i)$ and intersecting $P_i - v_i$ as depicted on the left of Figure 4.2. Since G is coverable with respect to u, such a perfect matching exists. We claim that $H + N_1 + \ldots + N_n$ is Class I. For each $i \in [n]$, let J_i be the perfect matching of H equal to N_i, apart from the way it intersects the edges in $P_i - v_i$. One can see the differences in Figure 4.2.

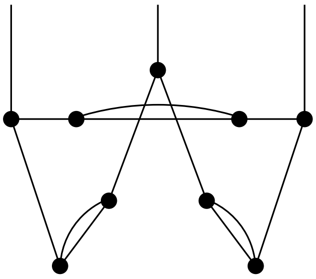

Figure 4.3: P'_i in $H + \sum_{i=1}^{n}(N_i - J_i)$

Consider the graph $H + \sum_{i=1}^{n}(N_i - J_i)$. This has the same structure as H, however, every $P_i - v_i$ is now transformed into P'_i, as shown in Figure 4.3. Since a bipartite graph is Class I and the 3-pole P'_i can be 3-edge-coloured in such a way that its three dangling edges each have a different colour, a 3-edge-colouring of G can be easily extended to a 3-edge-colouring of $H + \sum_{i=1}^{n} N_i - J_i$. Let these three colours (also perfect matchings of H) be denoted by $J_{n+1}, J_{n+2}, J_{n+3}$. Consequently, $H + N_1 + \ldots + N_n = J_1 + \ldots + J_{n+3}$, implying that $l(H) = n$, as required. $\square$

We remark that the above result holds also for bipartite cubic graphs G admitting parallel edges. Moreover, by using Proposition 4.2.8 we have the following consequence.

Corollary 4.2.9. *For each positive integer n there exists a cubic graph H with $l(H) = n$.*

Proof. Every snark having perfect matching index 4 is an example for $n = 1$. Moreover, we directly checked that the value of l for the (treelike) snark on 34 vertices, also known as *windmill* (see [1, 25]), is 2. For $n > 2$, it can be observed that if G is the circular ladder graph on $2n$ vertices (if n is even), or the Möbius ladder graph on $2n$ vertices (if n is odd), then for any vertex $u \in V(G)$, G is coverable with respect to u. Thus, the result follows from Proposition 4.2.8. $\square$

Finally, the following natural question arises.

Problem 4.2.10. Does there exist a cyclically 4-edge-connected cubic graph with arbitrarily large l?

We recall that l is always finite in the class of cyclically 4-edge-connected cubic graphs excluding the Petersen graph, as the latter is the only cyclically 4-edge-connected cubic graph for which l is infinite by Corollary 4.2.7.

4.3 THE PARAMETER $l_M(G)$

Proposition 4.2.1 states that if G belongs to $\mathcal{S}_4$, then it admits a perfect matching which when added to G the resulting graph is Class I. What happens if G belongs to $\mathcal{S}_{\geq 5}$? For sure, for any perfect matching M of G, $G + M$ is not Class I. However, what can we say about $G + tM$, for t being a positive integer strictly greater than 1?

We recall that the parameter $l_M(G)$, for a given bridgeless cubic graph G and a given perfect matching M of G, is defined as the minimum t, if such an integer exists, for which $G + tM$ is Class I. Clearly, $l_M(G) \geq l(G)$ for every perfect matching M of G, and thus, if $l(G) = +\infty$, then $l_M(G) = +\infty$ for every perfect matching M of G. We remark that, very recently, Brinkmann (private communication) found a graph G with $l(G) = 1$ and $l_M(G) = 2$, for some perfect matching M of G, disproving a hypothesis we had that there does not exist any pair (G, M) with $1 < l_M(G) < +\infty$.

Trivially, G is Class I if and only if $l_M(G) = 0$ for every perfect matching M of G. Moreover, $G \in \mathcal{S}_4$ if and only if $l(G) = 1$. The class $\mathcal{S}_4$ can be considered as the class of bridgeless cubic graphs closest to the class of 3-edge-colourable cubic graphs. Previous considerations suggest that there could be graphs inside $\mathcal{S}_4$ which are closer to being 3-edge-colourable than others: these are Class II bridgeless cubic graphs G for which $G + M$ is Class I for any one of their perfect matchings M, that is, $l_M(G) = 1$ for every M. We cannot give a complete characterisation of the graphs which have this property. However, we are able to show that an infinite family of snarks, with perfect matching index 4 (shown in [31]), have this distinctive property.

4.3.1 *Examples of cubic graphs G such that $l_M(G) = 1$ for every M*

Definition 4.3.1. The 6-pole on four vertices shown in Figure 4.4 is called a *Single-Flower 6-pole*, for short an *SF 6-pole*, whilst its vertical edge is referred to as a *spoke*.

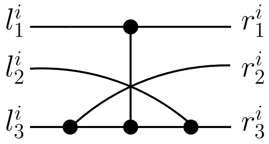

Figure 4.4: The SF 6-pole F_i

Let $n \geq 3$ be an odd integer, and let $F_1, F_2, \dots, F_n$ be n SF 6-poles. Let l_1^i, l_2^i, l_3^i and r_1^i, r_2^i, r_3^i be the *left* and *right* dangling edges of F_i, respectively, as shown in Figure 4.4. The graph obtained by joining the dangling edges r_j^i and l_j^{i+1}, for every $i \in [n]$ and for every $j \in [3]$, is called a *flower snark* and is denoted by $\mathcal{F}_n$ (see [47]). We remark that all operations in the upper indexing set are taken modulo n, with complete residue system $\{1, \ldots, n\}$. The new edge obtained after joining two dangling edges, say r_j^i and l_j^{i+1}, shall be referred to interchangeably by the same two names. To simplify the way we depict flower snarks, we look at $\mathcal{F}_n$ as a 6-pole with the left and right dangling edges being l_1^1, l_2^1, l_3^1, and r_1^n, r_2^n, r_3^n, respectively.

The 6-pole obtained by joining the right dangling edges of an SF 6-pole with the left dangling edges of another SF 6-pole in the same way as in the construction of flower snarks is called a *Double-Flower 6-pole*, for short a *DF 6-pole* (see Figure 4.5).

Figure 4.5: DF 6-poles in $\mathcal{F}_n$ with two consecutive spokes belonging to a perfect matching

Definition 4.3.2. Let X be a DF 6-pole in $\mathcal{F}_n$, with left and right dangling edges l_1^i, l_2^i, l_3^i and $r_1^{i+1}, r_2^{i+1}, r_3^{i+1}$, respectively, for some $i \in [n]$, and let M be a perfect matching of $\mathcal{F}_n$. The DF 6-pole X is said to be *good with respect* M, if there exists $j \in [3]$ such that $\partial X \cap M = \{l_j^i, r_j^{i+1}\}$.

In the sequel, we prove that given a perfect matching M of $\mathcal{F}_n$, $\mathcal{F}_n + M$ is Class I, except when $n = 3$ and M intersects exactly one spoke of $\mathcal{F}_3$. The latter case arises because the graph $\mathcal{F}_3$ is the Petersen graph P with one vertex Y-extended to a triangle ($\mathcal{F}_3$ is also known as the Tietze graph), and if $\mathcal{F}_3 + M$ is Class I for such a perfect matching M, then this would imply that $l(P) = 1$, a contradiction (see Proposition 4.2.2 and Proposition 4.2.3).

Easy direct checks show that the following remarks hold.

R.1 Let M be a perfect matching of $\mathcal{F}_3$ intersecting all three of its spokes. Then, $\mathcal{F}_3 + M$ is Class I.

R.2 Let M be a perfect matching of $\mathcal{F}_5$ intersecting exactly one spoke, say the spoke of F_3. Then, M contains one of the two matchings depicted in Figure 4.6. One can clearly see that, in any case, the colouring depicted Figure 4.6 can always be extended to a 4-edge-colouring of $\mathcal{F}_5 + M$ using the colours a, b, c, d.

R.3 As n is odd, any perfect matching of $\mathcal{F}_n$ intersects exactly one left (similarly right) dangling edge of some SF 6-pole F_i, for $i \in [n]$.

Note that R.3 follows because every perfect matching of $\mathcal{F}_n$ cannot intersect all the three left (similarly right) dangling edges of F_i. Moreover, if a perfect matching intersects exactly two left dangling

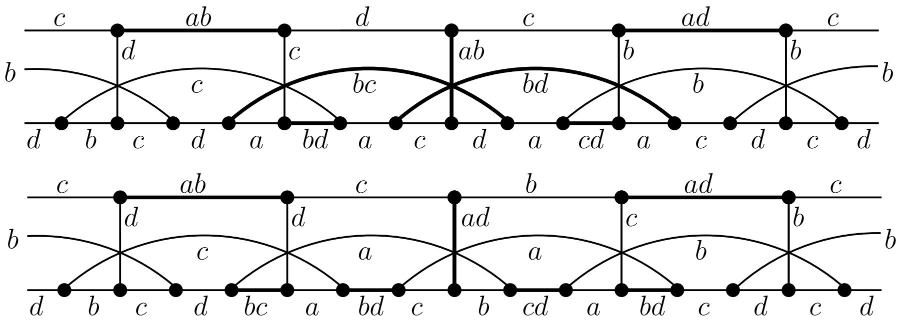

Figure 4.6: M intersecting exactly one spoke in $\mathcal{F}_5$

edges of F_i, then the right dangling edges of this 6-pole are not intersected by the perfect matching, and vice-versa. Since n is odd, this is impossible to occur.

R.4 If the two spokes of a DF 6-pole are contained in a perfect matching, then it is a good DF 6-pole with respect to that perfect matching (see Figure 4.5).

R.5 If a perfect matching M of $\mathcal{F}_n$ intersects the first and third out of three consecutive spokes, then, the second spoke must be contained in M, as well. Consequently, if a perfect matching of $\mathcal{F}_5$ intersects exactly three spokes, then they must be consecutive. Moreover, in this case, the two SF 6-poles of $\mathcal{F}_5$ whose spokes are not contained in the perfect matching form a good DF 6-pole.

R.6 As n is odd, if the spokes of three consecutive SF 6-poles, say F_1, F_2, F_3, do not belong to a perfect matching, then either F_1 and F_2, or, F_2 and F_3 form a good DF 6-pole with respect to that perfect matching.

Indeed, note that either l_1^1 and r_1^2, or, l_1^2 and r_1^3 belong to the perfect matching, and so R.6 follows by R.3.

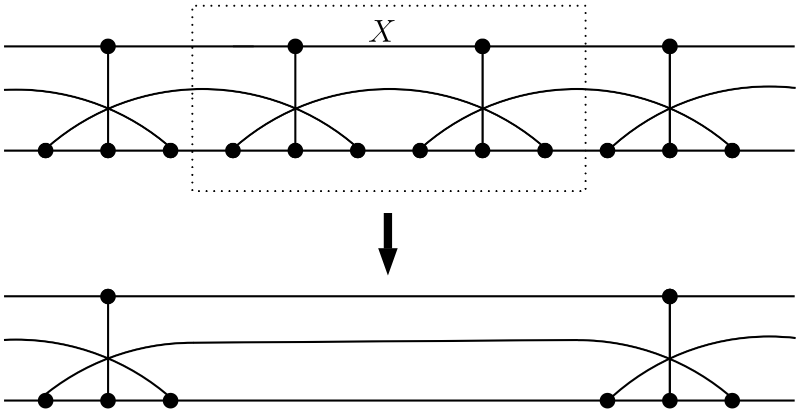

Figure 4.7: Inductive step in the proof of Theorem 4.3.3

In the next proof we make use of the following procedure: we delete a good DF 6-pole X with respect to a perfect matching M of $\mathcal{F}_n$ (shown as the dotted part in Figure 4.7) and join the remaining dangling edges accordingly together as in Figure 4.7. In this way we obtain a copy of the flower snark $\mathcal{F}_{n-2}$.

In the sequel, with a slight abuse of terminology, we refer to the three edges obtained after joining the above dangling edges as the new edges of $\mathcal{F}_{n-2}$. Moreover, since X is good, M naturally induces a perfect matching of $\mathcal{F}_{n-2}$. We denote by M_X such a perfect matching in the copy of $\mathcal{F}_{n-2}$, obtained by removing X from $\mathcal{F}_n$. Note that M_X contains exactly one of the three new edges.

Theorem 4.3.3. *Let $n \geq 5$ be an odd integer and let M be a perfect matching of $\mathcal{F}_n$. Then, $\mathcal{F}_n + M$ is Class I.*

Proof. The crucial steps of the proof of this theorem lie in the following two claims.

Claim I. Let $n \geq 5$ be an odd integer and let X be a good DF 6-pole with respect to a perfect matching M of $\mathcal{F}_n$. If $\mathcal{F}_{n-2} + M_X$ is Class I, then $\mathcal{F}_n + M$ is Class I.

Proof of Claim I. Let M_X be the perfect matching induced by M in $\mathcal{F}_{n-2}$. By assumption, $\mathcal{F}_{n-2} + M_X$ admits a 4-edge-colouring with the colours denoted by a, b, c, d. Without loss of generality, we can assume that the unique edge of $\mathcal{F}_{n-2} + M_X$ parallel to a new edge of $\mathcal{F}_{n-2}$ has colour d in the given 4-edge-colouring. Since every colour class corresponds to a perfect matching of $\mathcal{F}_{n-2}$, it follows by R.3 that each of the colours a, b, c intersects exactly one of the three new edges of $\mathcal{F}_{n-2}$. A 4-edge-colouring of $\mathcal{F}_n + M$ is constructed in the following way: if an edge does not have an end-vertex in X, then it is assigned the same colour of its corresponding edge in $\mathcal{F}_{n-2} + M_X$; all edges of $\mathcal{F}_n$ with an end-vertex in X are assigned the colours a, b, c as illustrated in Figure 4.8; and finally, all edges of M with an end-vertex in X are assigned the colour d. This gives rise to a 4-edge-colouring of $\mathcal{F}_n + M$. ▪

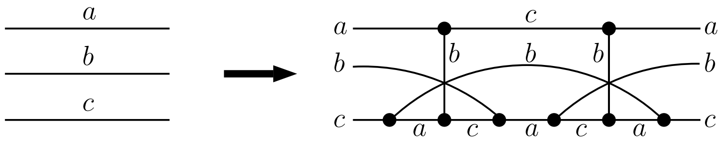

Figure 4.8: Extending the colour classes of $\mathcal{F}_{n-2}$ to $\mathcal{F}_n$

Claim II. Let $n \geq 3$ be an odd integer and let M be a perfect matching of $\mathcal{F}_n$. The graph $\mathcal{F}_n$ admits a good DF 6-pole with respect to M.

Proof of Claim II. Suppose that there is no good DF 6-pole with respect to M, for contradiction. From R.4 it follows that M cannot contain two consecutive spokes. At the same time, since n is odd, R.5 and R.6 imply that every sequence of consecutive spokes not in M has length exactly two. Hence, for every three consecutive spokes, one of them belongs to M and the other two do not. Consider three consecutive SF 6-poles in $\mathcal{F}_n$, and without loss of generality assume that M intersects only the first spoke. Since there is no good DF 6-pole with respect to M, a direct easy check shows that M can intersect these three consecutive SF 6-poles only in two possible ways, as shown in Figure 4.9.

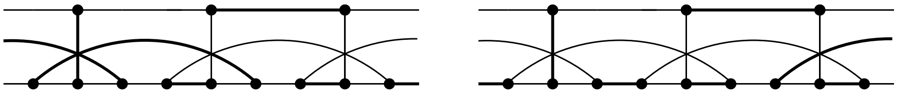

Figure 4.9: How M can intersect three consecutive SF 6-poles

The two ways M can intersect three consecutive SF 6-poles must alternate in $\mathcal{F}_n$. Hence, n is three times an even number, a contradiction, since n is assumed to be odd. ■

Now we are in a position to complete the proof of the theorem. We prove the result by induction on n. Consider first $\mathcal{F}_5$. As the spokes form an odd edge-cut, M intersects an odd number of them. By R.2 we can assume that M intersects at least three consecutive spokes of $\mathcal{F}_5$, say the spokes of F_1, F_2, F_3. Consequently, by R.4 or R.5, F_4 and F_5 form a good DF 6-pole with respect to M. Let this DF 6-pole be X. We have that M_X intersects all the three spokes of $\mathcal{F}_3$. By R.1, $\mathcal{F}_3 + M_X$ is Class I and the base case $n = 5$ follows by Claim I.

Now, assume the result holds up to $n \geq 5$, that is, $\mathcal{F}_n + M$ is Class I for every perfect matching M of $\mathcal{F}_n$. Consider $\mathcal{F}_{n+2}$ and let M be one of its perfect matchings. By Claim II, $\mathcal{F}_{n+2}$ admits a good DF 6-pole X with respect to M. By induction, $\mathcal{F}_n + M_X$ is Class I and the assertion follows by Claim I. □

The flower snark $\mathcal{F}_5$ has cyclic connectivity 5, and for every odd $n \geq 7$, $\mathcal{F}_n$ has cyclic connectivity 6. Because of Theorem 4.3.3, one may think that for every perfect matching M of a cyclically 5-edge-connected cubic graph G with perfect matching index four, $G + M$ is Class I. However, this is not true. By Theorem 1.1 in [38], there exists an infinite family of cyclically 5-edge-connected cubic graphs G having perfect matching index 4, which do not satisfy this assertion. This is true because these graphs admit a 2-factor which is not contained in any one of their cycle double covers. For, let G be such a graph, and let N be the complement of such a 2-factor C. Suppose that $G + N$ is Class I, for contradiction. Then, $G + N = \sum_{i=1}^{4} J_i$ for some perfect matchings J_i of G. Hence, $\{N \triangle J_1, \ldots, N \triangle J_4, C\}$ is a cycle double cover of G containing C. This contradicts our choice of G.

4.4 A RELATION BETWEEN $l_M(G)$ AND $scc(G)$

The main conjecture in the area of short cycle covers of bridgeless graphs is the so-called $7/5$-Conjecture (or the Shortest Cycle Cover Conjecture). It states that for any bridgeless graph G (not necessarily cubic), we have $scc(G) \leq 7/5 \cdot |E(G)|$. This conjecture is one of the many consequences of the Petersen Colouring Conjecture [81]. On the other hand, it implies the Cycle Double Cover Conjecture, see [51]. In [52] it is shown that any bridgeless cubic graph G has a cycle cover of length at most $34/21 \cdot |E(G)|$, and any bridgeless graph G of minimum degree 3 has a cycle cover of length at most $44/27 \cdot |E(G)|$.

The following conjecture can be found as Conjecture 8.11.5 in [105].

Conjecture 4.4.1. *Every bridgeless graph has a shortest* 4*-cycle cover.*

Here, we propose the following conjecture and we show that it is implied by Conjecture 4.4.1.

Conjecture 4.4.2. *For every bridgeless cubic graph* G, $\chi'_e(G) \leq 4$ *if and only if* G, $scc(G) = 4/3 \cdot |E(G)|$.

Proposition 4.4.3. *Conjecture 4.4.1 implies Conjecture 4.4.2.*

Proof. If $\chi'_e(G) \leq 4$, then by [91], $scc(G) = 4/3 \cdot |E(G)|$. So assume $scc(G) = 4/3 \cdot |E(G)|$, and let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a cycle cover of G with length $4/3 \cdot |E(G)|$. Since we are assuming Conjecture 4.4.1 to be true, we can assume $k \leq 4$. Since G is cubic and the length of $\mathcal{C}$ is $4/3 \cdot |E(G)|$, every edge of G is either covered once or twice in $\mathcal{C}$ and the edges covered twice form a perfect matching of G, say M. Let $F_i = E(C_i) \triangle M$, for every $i = 1, \ldots, k$. Since $\mathcal{C}$ is a cycle cover, the perfect matchings $F_1, \ldots, F_k$ cover the edge set of G, implying that $\chi'_e(G) \leq 4$, as required. □

A relation between $scc(G)$ and $l_M(G)$ is clearly established by the following theorem.

Theorem 4.4.4. *For every bridgeless cubic graph* G, $scc(G) > 4/3 \cdot |E(G)|$ *if and only if* G *is frumious, otherwise* $scc(G) = 4/3 \cdot |E(G)|$.

Proof. Assume that G is not frumious, that is, $G + tM$ is Class I for a perfect matching M of G and for some non-negative integer t. Let $F_1, \ldots, F_{t+3}$ be the colour classes of a $(t+3)$-edge-colouring of $G + tM$. For every $i = 1, \ldots, t+3$, let $C_i = G[M \triangle F_i]$, and let $\mathcal{C} = \{C_1, \ldots, C_{t+3}\}$. The latter is a cycle cover of G. Moreover, if $e \in M$, then e is covered exactly twice by the cycles in $\mathcal{C}$. Otherwise, if $e \notin M$, then e is covered exactly once by some cycle in $\mathcal{C}$. Since for any cubic graph G and any cycle cover $\mathcal{C}$ of G, $\mathcal{C}$ has length $4/3 \cdot |E(G)|$ if and only if the set of edges covered twice by $\mathcal{C}$ is a perfect matching of G, the result follows. Conversely, let $\mathcal{C} = \{C_1, \ldots, C_{t+3}\}$ be a shortest cycle cover of G of length $4/3 \cdot |E(G)|$, for some integer t. Let M be the set of edges covered exactly twice by $\mathcal{C}$, and let F_i be equal to $M \triangle E(C_i)$. By an argument similar to the first implication, one can see that $G + tM = F_1 + \ldots + F_{t+3}$. □

Hence, the main consequence of Conjecture 4.4.2 is that bridgeless cubic graphs having perfect matching index at least 5 would have a shortest cycle cover strictly greater than 4/3 their size. The problem seems to be very hard to solve. However, in the next section, we show that an infinite family of snarks G with perfect matching index 5 have a shortest cycle cover strictly greater than $4/3 \cdot |E(G)|$.

4.4.1 *Treelike Snarks*

We recall that a bridgeless cubic graph G is frumious if $l_M(G) = +\infty$ for all perfect matchings M of G. As already remarked, the Petersen graph is such a graph and above we conjectured (see Conjecture 4.4.2) that a bridgeless cubic graph is frumious if and only if its perfect matching index is at least 5. In order to support such a conjecture we consider an infinite family of snarks, called treelike snarks, having perfect matching index 5 and prove that they are frumious snarks. The family of treelike snarks was first introduced in [1], but here we also refer to the more general definition of treelike snarks given in [65] and prove our main result (Theorem 4.4.6) in this general setting.

In order to present such a class of snarks we need some preliminary definitions.

Definition 4.4.5. Let A be an arbitrary 4-pole. Partition its four dangling edges in ordered pairs, say (l_1, l_2), referred to as the first and second left dangling edges, and (r_1, r_2), referred to as the first and second right dangling edges. Let the end-vertices of the four dangling edges l_1, l_2, r_1, r_2 be u_1, u_2, v_1, v_2, respectively. The 4-pole A is said to be *frumious* with respect to such a partition if the graph obtained by removing the four dangling edges and adding two new vertices u and v such that u is adjacent to v, u_1, u_2, and v is adjacent to u, v_1, v_2, is a frumious snark. We refer to the latter graph as the frumious snark obtained from the 4-pole A.

Note that a 4-pole could be frumious with respect to a given partition whilst it is not with respect to another one. On the other hand, a change in the order of the left dangling edges or the right dangling edges of a frumious 4-pole produces another (possibly different) frumious 4-pole. However, although this last change may produce a different frumious 4-pole, the two frumious snarks obtained from the two 4-poles are the same. An example of a frumious 4-pole is the one obtained by removing two adjacent vertices of the Petersen graph, say u and v, with the left dangling edges corresponding to the edges originally incident to u and not v, and the right dangling edges corresponding to the edges originally incident to v and not u. In this case, the order of the dangling edges in each set of the partition is not relevant due to the symmetry of the Petersen graph.

A *Halin graph* is a plane graph consisting of a planar representation of a tree without degree 2 vertices, and a circuit on the set of its leaves (see [41]).

Let H be a cubic Halin graph consisting of the tree T and the circuit K. A *treelike snark* G is any cubic graph that can be obtained by the following procedure:

- for every vertex of degree 1 (a leaf) x of T, we add two new vertices, say x_1 and x_2, and the edges xx_1 and xx_2; and
- for every edge xy of K, with x being the predecessor of y with respect to the clockwise orientation of K, the edge xy is replaced with a frumious 4-pole, and the first and second left dangling edges of this 4-pole are joined to x_1 and x_2, respectively, whilst the first and second right dangling edges are joined to y_1 and y_2, respectively.

Let G be a treelike snark as defined above, and let the tree and the circuit defining G be T and K, respectively. Let A be a frumious 4-pole of G replacing an edge of K. We say that A is of Type ij with respect to a perfect matching M of G if M intersects the left and right dangling edges of A exactly i and j times, respectively, for some $i, j \in \{0, 1, 2\}$ with $i + j$ even. We denote this by $\text{Type}(A_M) = ij$.

In what follows we refer to the first and second left dangling edges of the 4-pole A as ${}^{-}A$ and ${}_{-}A$, respectively. The first and the second right dangling edges are similarly denoted by A^{-} and A_{-}, respectively (see Figure 4.10).

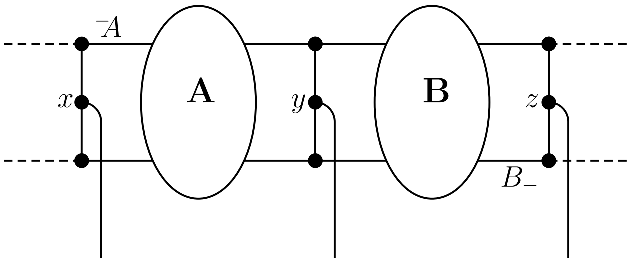

Figure 4.10: Consecutive leaves and 4-poles

Two leaves x and y of T are called *consecutive* if they are adjacent in the circuit K, and we say that the frumious 4-pole of G replacing the edge xy of K is *in between* the two leaves x and y. Moreover, two consecutive leaves are said to be *near* if they have distance two in T, that is, they have a common neighbour in T (see Figure 4.11). We remark that T always has two near leaves. Similarly, two 4-poles A and B are called *consecutive* if there exist three consecutive leaves x, y, z (that is, x and y are consecutive and y and z are consecutive) such that A is in between x and y, and B is in between y and z (see Figure 4.10). Again, we say that the leaf y is *in between* the 4-poles A and B.

Theorem 4.4.6. *Every treelike snark is frumious.*

Proof. Let G be a treelike snark. We need to prove that $l_M(G) = +\infty$ for every perfect matching M of G. Suppose, for contradiction, that G is a counterexample having the tree T defining G of minimum order. This means that $G + tM$ is Class I, for some perfect matching M of G and some positive integer t. Let the $t + 3$ colours of $G + tM$ be the perfect matchings $F_1, \dots, F_{t+3}$. It is already proved in [25] that $scc(G) > 4/3 \cdot |E(G)|$ if T has exactly one vertex of degree 3, and so, $l_M(G) = +\infty$ by Theorem 4.4.4. Therefore, we can assume that T has at least two vertices having degree 3.

Claim I. If a 4-pole of G is of Type 00 with respect to M, then there must exist exactly one perfect matching from the list of colours $F_1, \dots, F_{t+3}$ which intersects both the left (similarly right) dangling edges.

Proof of Claim I. Since the 4-pole of G is of Type 00 with respect to M, every dangling edge is contained in exactly one of the colours from the above list. Moreover, since the 4-pole is frumious, exactly one of these colours must intersect both left dangling edges and exactly one of these colours must intersect both right dangling edges (such a colour could be the same for the left and right dangling edges), otherwise one could construct a $(t + 3)$-edge-colouring of the frumious snark obtained from the 4-pole (see Definition 4.4.5), a contradiction. In this case, all the other colours from the list ($t + 1$ or $t + 2$ of them) do not intersect the four dangling edges. ∎

Claim II. If a 4-pole of G is of Type 11 with respect to M, then, there must be t perfect matchings from the list of colours $F_1, \dots, F_{t+3}$, such that each of them intersects exactly one left dangling edge and exactly one right dangling edge simultaneously. Moreover, there must also exist exactly one perfect matching from the same list which intersects both the left (similarly right) dangling edges of the 4-pole.

Proof of Claim II. Since the 4-pole is frumious, at least one colour, say F_i, must intersect both the left (right) dangling edges of the 4-pole, by the same argument used in the proof of Claim I, and once again, such colour could be the same for the left and right dangling edges. Since one of the left (right) dangling edges does not belong to M and belongs to F_i, every other colour cannot intersect this left (right) dangling edge. Hence, every perfect matching from the list of colours $F_1, \dots, F_{t+3}$ different from F_i intersects the other left (right) dangling edge at most once. More precisely, t of the colours different from F_i intersect the left and right dangling edges belonging to M exactly once. ∎

Claim III. G cannot contain two consecutive 4-poles which are respectively of Type 00 and Type 11 with respect to M.

Proof of Claim III. If two consecutive 4-poles of G are respectively of Type 00 and 11 with respect to M, then, the edge of T, say e, incident to the leaf in between these two 4-poles does not belong to M. On the other hand, by Claim I, there exists a colour F_i which contains both the right (left) dangling edges of the 4-pole of Type 00, and so it contains the edge e, as well. By Claim II, there exists a colour F_j which contains both the left (right) dangling edges of the 4-pole of Type 11, and so it contains the edge e too. We note that $j \neq i$, otherwise, F_i contains two pairs of incident edges. Consequently, the edge e belongs to two different colours and so it must belong to M, a contradiction. ∎

Claim IV. If the unique edge of T incident to a leaf x is not in M, then x is in between a 4-pole of Type 11 and a 4-pole of Type 02 or, by symmetry, a 4-pole of Type 20 and a 4-pole of Type 11, with respect to M.

Proof of Claim IV. If the unique edge of T incident to a leaf x is not in M, then one of the other two edges incident to x belongs to M. Hence, x is in between two 4-poles, one of Type 11 and the other one either of Type 00 or of Type 02 (by symmetry Type 20), with respect to M. The first possibility is already excluded by Claim III. ∎

Claim V. If two consecutive leaves are incident to edges in M not belonging to T, then the 4-pole in between them is of Type 11 with respect to M.

Proof of Claim V. Let x and y be the two consecutive leaves and let A, B, C be the three consecutive 4-poles such that x is in between A and B, and y is in between B and C. By Claim IV, either A is of Type 20 and B of Type 11, or A is of Type 11 and B of Type 02, with respect to M. The latter case is excluded by considering the pair B and C of consecutive 4-poles and Claim IV again. ∎

Claim VI. G cannot have three consecutive leaves which are incident to edges in M not belonging to T.

Proof of Claim VI. Assume there exist such three consecutive leaves, say x, y, z. By Claim V, the two 4-poles in between x and y, and in between y and z are both of Type 11 with respect to M. This implies that the edge of T incident to y is in M, a contradiction. ∎

Next, consider two near leaves of T, say x and y, as in Figure 4.11. Let e and f be the two edges of T incident to x and y, respectively. Moreover, let g be the edge of T adjacent to e and f. The perfect matching M can intersect e, f, g in two different ways.

Case 1. The edge g does not belong to M and exactly one of e and f belongs to M, say e without loss of generality.

Case 2. The edge g belongs to M.

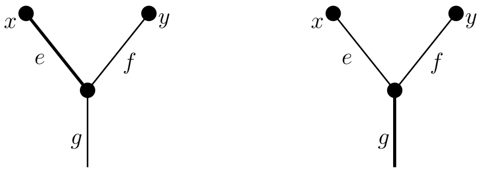

Figure 4.11: Near leaves x and y in **Case 1** and **Case 2** from the proof of Theorem 4.4.6

Consider the three consecutive 4-poles A, B, C such that x is in between A and B, and y is in between B and C, as in Figure 4.12. The list of proven claims gives some strong restrictions and information on the possible types of these 4-poles with respect to M. We briefly discuss them according to Type (B_M), ending with a summary in Table 4.2.

- Type (B_M) cannot be equal to 22 or 02, since $f \notin M$ both in Case 1 and Case 2.
- If Type $(B_M) = 00$, then Type $(C_M) = 11$ since $f \notin M$, a contradiction by Claim III.
- If Type $(B_M) = 11$, then Type $(C_M) = 02$ since $f \notin M$ and by Claim III. Moreover, if $e \in M$, then Type $(A_M) = 11$ (Case 1b)), otherwise, if $g \in M$, then Type $(A_M) = 20$ (Case 2)).
- If Type $(B_M) = 20$, then Type $(C_M) = 11$ since $f \notin M$, and Type (A_M) can be either 00 (Case 1a)) or 20 (Case 1c)).

Case	Type(A_M)	Type(B_M)	Type(C_M)
1a)	00	20	11
1b)	11	11	02
1c)	20	20	11
2)	20	11	02

Table 4.2: The types of the three consecutive 4-poles A, B, C

We prove a further last claim.

Claim VII. Let D and D' be two consecutive 4-poles of G which are respectively of Type 20 and 11 (or by symmetry 11 and 02) with respect to M. There cannot exist a colour F_j such that Type$(D_{F_j}) =$ Type$(D'_{F_j}) = 11$, and there cannot exist a colour F_l such that Type$(D_{F_l}) = 02$ or 22 (or by symmetry 20 or 22).

Proof of Claim VII. Consider two consecutive 4-poles D and D' which are respectively of Type 20 and 11 (or 11 and 02) with respect to M.

Clearly, the edge h belonging to T and incident to the leaf in between them is not in M. Since we are assuming that $G + tM$ is Class I, by Claim II there must exist a colour, say F_i, such that Type(D'_{F_i}), or from now on simply Type(D'_i), is equal to 20 or 22. Clearly, $h \in F_i$. This means that there cannot exist a colour F_j such that Type$(D_j) =$ Type$(D'_j) = 11$, as otherwise, h would be covered more than it should be. For the same reasons, there cannot exist a colour F_l such that Type$(D_l) = 02$ or 22 (by symmetry 20 or 22). ∎

Now, we use all previous claims to show that in all the four remaining cases we obtain a contradiction.

Case 1a). Type$(A_M) = 00$, Type$(B_M) = 20$, Type$(C_M) = 11$.
Since A is frumious, by Claim I there exists a colour from $F_1, \dots, F_{t+3}$, say F_1, such that Type$(A_1) = \alpha 2$, where α is either equal to 0 or 2. As the edges e and f are adjacent, the 4-poles B and C must be intersected by F_1 as in Table 4.3. In order to cover the right dangling edges of B, there must also exist two colours, say F_2 and F_3, such that Type$(B_2) =$ Type$(B_3) = 11$, since by Claim VII, Type(B_i) cannot be equal to 02, for any $i \in [t+3]$. Hence, by Claim VII, F_2 and F_3 intersect the 4-poles A, B, C as shown in Table 4.3, where $\beta, \gamma, \delta, \epsilon \in \{0, 2\}$. In any case, this means that the edge g of T is covered twice by F_2 and F_3 in $\cup_{i=1}^{t+3} F_i$, a contradiction, since $g \notin M$.

i	Type(A_i)	Type(B_i)	Type(C_i)
1	$\alpha 2$	00	11
2	$\beta 0$	11	0δ
3	$\gamma 0$	11	0ϵ

Table 4.3: **Case 1a)**

Case 1b). Type$(A_M) = 11$, Type$(B_M) = 11$, Type$(C_M) = 02$.
There must be a colour from $F_1, \dots, F_{t+3}$, here denoted by a, intersecting both the right dangling edges of A. In what follows, if Z is a set of colours, we denote by $\overline{Z}$ the set of all colours not in Z. Without loss of generality, assume $A^- \in M$. Let b and c be the colours of the two edges of G adjacent to A^-. Thus, the colours of A^- are $\overline{\{b, c\}}$, for simplicity denoted by $\overline{bc}$. Without loss of generality, let the colour of ${}^-B$ be c. This implies that c also intersects ${}_-B$, since by Claim II there must be one colour which intersects both the left dangling edges of B, and consequently the colours of ${}_-B$ are $\overline{ad}$, for some $d \in \overline{abc}$. Moreover, the edge e has colours $\overline{bd}$, as can be seen in Figure 4.12.

Once again, by Claim II, there is a colour which intersects both the right dangling edges of B. Clearly, this cannot belong to $\overline{bd}$, for otherwise, f would be coloured by a colour already used for e. Since b intersects exactly one left dangling edge of B, the right dangling

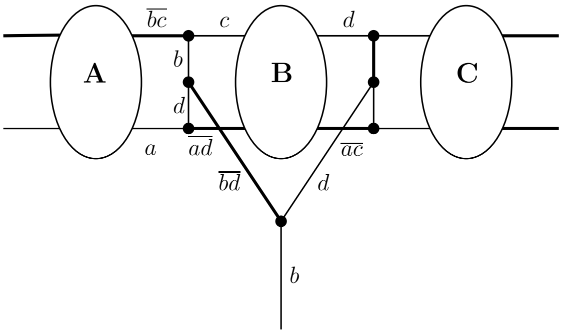

Figure 4.12: **Case 1b)**

edges of B must be intersected by d. Without loss of generality, we can assume that $B_- \in M$, and so by the above reasoning, the set of colours of B_- is $\overline{ac}$ (see Figure 4.12). At this point, we have two possible cases of how we can colour ^-C and $_-C$: we either have ^-C and $_-C$ intersected by a and c, respectively, or the other way round, as can be seen in the two figures in Figure 4.13.

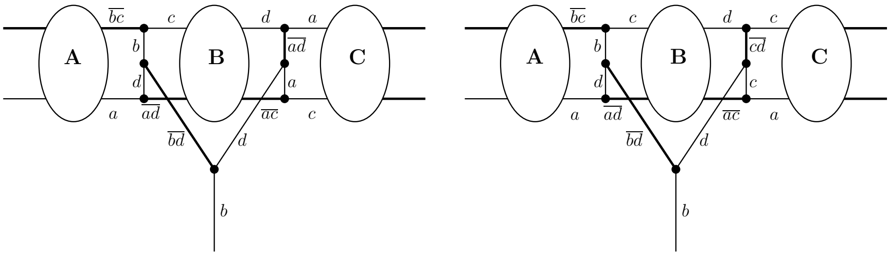

Figure 4.13: The possible colours of ^-C and $_-C$ in Case 1b)

Next, we reduce G to a smaller treelike snark following the procedure presented in Figure 4.14. Since T has at least two vertices of degree 3, the resulting graph G' is indeed a treelike snark. Let T' be the tree defining G'.

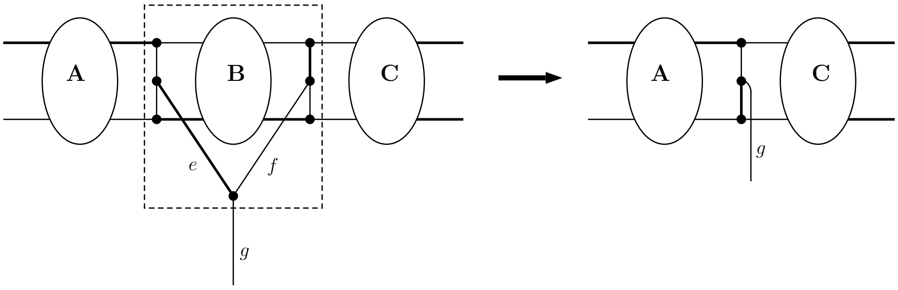

Figure 4.14: Constructing a smaller treelike snark in **Case 1b)**

Let M' be the perfect matching of G' induced by M. Without loss of generality, assume that the colours of ^-C and $_-C$ in G are a and c, respectively, and assign to the edges of G' (which correspond to edges of G) the same colours they had originally. We note that this procedure does not colour all the edges of G'. In fact, the two edges not belonging to T' which are incident to the leaf between the two 4-poles A and C in G', do not correspond to any edges of G, and so they are left uncoloured. Moreover, the edges of G' are not properly coloured, as depicted in Figure 4.15.

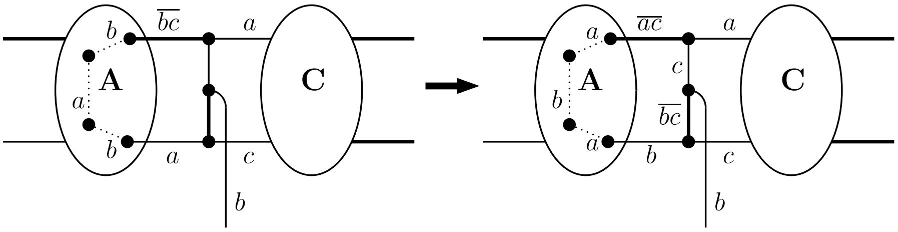

Figure 4.15: Applying a Kempe chain argument in **Case 1b)**

We claim that the (a,b)-Kempe chain in the 4-pole A starting at A_- must end at A^-. For, suppose it does not contain the latter dangling edge. Let M_A be the perfect matching induced by M in the 4-pole A. Switching the colours a and b along this chain result in a $(t+3)$-edge-colouring of the pole $A + tM_A$ in which no colour intersects the two right dangling edges of A simultaneously, contradicting Claim II. Consequently, the (a,b)-Kempe chain in G' starting from A_- must end at A^-. By switching the colours a and b along this chain and extending the colouring to a $(t+3)$-edge-colouring of $G' + tM'$ as in Figure 4.15, we obtain a contradiction due to the minimality of T.

Case 1c). Type$(A_M) = 20$, Type$(B_M) = 20$, Type$(C_M) = 11$.
This case is solved in a similar way as in Case 1b), and so this case cannot occur as well. Figure 4.16 shows the four different ways how a $(t+3)$-edge-colouring of $G + tM$ looks like in this part of G.

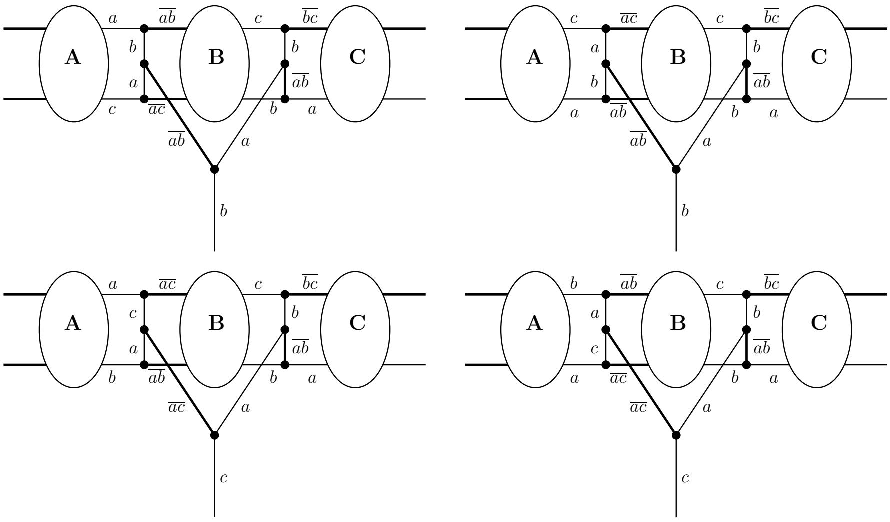

Figure 4.16: **Case 1c)**

Case 2. Type$(A_M) = 20$, Type$(B_M) = 11$, Type$(C_M) = 02$.
Since all other cases are not possible, all pairs of near leaves of G are in

between three 4-poles of these types (with respect to M). We show that in such a case, there exist three consecutive leaves all incident to edges in M not belonging to T, a contradiction by Claim VI. In fact, if T has only two vertices of degree 3, then it has exactly two pairs of near leaves, with all the four edges incident to the leaves not belonging to M. Thus, we have three consecutive leaves with the required property, contradicting Claim VI.

Therefore, T must have more than two vertices of degree 3. Remove all pairs of near leaves from T, and let the resulting tree, which still has all vertices of degree 1 and 3, be T'. In general, if x is a leaf of T' which was not a leaf in T, then the edge in T' incident to x belongs to M. Consider a pair of near leaves of T'. At least one of them was not a leaf in T, as otherwise the pair would have been deleted in the process of obtaining T'. If these two near leaves were not originally leaves in T, then the edges in T' incident to both of them belong to M, a contradiction. Hence, one leaf of the pair must also be a leaf in T, whilst the other leaf of the pair was a common neighbour to a pair of removed near leaves of T. Consequently, G contains three consecutive leaves such that the edges of T incident to them do not belong to M, contradicting Claim VI once again. Hence, $l_M(G) = +\infty$ for every perfect matching M of G. □

We complete this section with the following corollary which simply follows by Theorem 4.4.6 and Theorem 4.4.4.

Corollary 4.4.7. *For every treelike snark G, $scc(G) > 4/3 \cdot |E(G)|$.*

4.5 FINAL REMARKS AND OPEN PROBLEMS

In the following table, we summarise all the parameters discussed along the chapter and we recall one of the main conjectures proposed above. In particular, the table highlights the special role of the class $\mathcal{S}_{\geq 5}$ with regards to all the considered problems.

$\chi'(G)$	$\chi'_e(G)$	$scc(G)$	$l(G)$	$l_M(G)$
3	3	$4/3 \cdot \lvert E(G) \rvert$	0	$(\forall M)\, 0$
4	4	$4/3 \cdot \lvert E(G) \rvert$	1	$(\exists M)\, 1$
	≥ 5	$> 4/3 \cdot \lvert E(G) \rvert$ (Conj.4.4.2)	> 1	$(\forall M) = +\infty$ (Conj.4.4.2)

Let us remark that the problem of establishing the existence of a perfect matching M for which $l_M(G)$ is finite is equivalent to establishing the existence of a 2-factor (indeed the complement of M in G) which can be extendend to a cycle double cover of G. This problem was already considered for some classes of snarks (see for instance [38]). We remark that Conjecture 4.4.2 can be equivalently stated in such terms as follows.

Conjecture 4.5.1. *For every bridgeless cubic graph G, $\chi'_e(G) > 4$ if and only if every cycle double cover of G does not contain a 2-factor of G.*

In Section 4.4 we showed that the conjecture holds for a large family of snarks having perfect matching index 5. A first natural step in an attempt to understand better Conjecture 4.4.2 is trying to solve the following problem.

Problem 4.5.2. Characterise the class of bridgeless cubic graphs G for which there exists a perfect matching M, such that $G + 2M$ is Class I.

Clearly, if Conjecture 4.4.2 holds we would have a complete answer to the previous problem, and the graphs G answering Problem 4.5.2 would be those bridgeless cubic graphs having perfect matching index at most 4.

Finally, as one can notice, along the chapter we mainly focus our attention on 1-factors of G. A very similar problem for 2-factors of a snark G was communicated personally to us by Eckhard Steffen.

Problem 4.5.3 (Steffen, personal communication)**.** Let G be a bridgeless cubic graph. What is the smallest number of 2-factors that need to be added to G, such that the resulting graph is Class I?

The classical Berge–Fulkerson Conjecture is equivalent to saying that the answer for Problem 4.5.3 is at most 1, with the answer being 0 if G is already a Class I graph. Here, we would like to propose a possible approach for the study of this problem.

For a bridgeless cubic graph G, let $sp_2(G)$ be the set of all non-negative integers t such that G contains t 2-factors whose addition to G results into a Class I graph, and let $sp(G)$ be the set of all non-negative integers t such that tG is Class I, where tG represents $G + (t-1)E(G)$. These two parameters are related in the following way.

Proposition 4.5.4. *For any bridgeless cubic graph G and any integer $t \geq 0$, $t \in sp_2(G)$ if and only if $(t+1) \in sp(G)$.*

Proof. Assume that $t \in sp_2(G)$. Then, there are t 2-factors $\overline{F_1}, ..., \overline{F_t}$ of G such that $G + E(\overline{F_1}) + ... + E(\overline{F_t})$ is Class I, that is, $(2t+3)$-edge-colourable. Hence, there are $2t+3$ perfect matchings $J_1, ..., J_{2t+3}$ that partition the edge set of the graph $G + E(\overline{F_1}) + ... + E(\overline{F_t})$, and consequently

$$\vec{1} = \chi^{J_1} + ... + \chi^{J_{2t+3}} - \chi^{E(\overline{F_1})} - ... - \chi^{E(\overline{F_t})}.$$

Let F_i be the perfect matching $E(G) - E(\overline{F_i})$, for every $i \in [t]$. By noting that for every i, $\vec{1} = \chi^{F_i} + \chi^{E(\overline{F_i})}$, we have

$$\vec{1} = \chi^{J_1} + ... + \chi^{J_{2t+3}} - (\vec{1} - \chi^{F_1}) - ... - (\vec{1} - \chi^{F_t}),$$

which implies that

$$(t+1)\vec{1} = \chi^{J_1} + ... + \chi^{J_{2t+3}} + \chi^{F_1} + ... + \chi^{F_t}.$$

The latter means $(t+1)G$ is Class I, that is, $(3t+3)$-edge-colourable. The converse can be similarly proved using the same arguments. □

Part II

PERFECT MATCHINGS AND HAMILTONICITY

We now move on to study the behaviour of the union of two perfect matchings and whether such a union constitutes a Hamiltonian circuit or not. More precisely, we shall study whether a given perfect matching (or pairing) of a graph can always be extended to a Hamiltonian circuit by another perfect matching. If this holds for all perfect matchings (pairings) of a graph, we shall say that the latter has the PMH-property (or the PH-property, in case of pairings). We recall that PMH and PH stand for Perfect-Matching-Hamiltonian and Pairing-Hamiltonian, respectively.

Along the way I have made great friends and worked with a number of creative and interesting people. I have been saved from boredom, dourness, and self-absorption. One cannot ask for more.

—KAREN UHLENBECK

5

EXTENDING PERFECT MATCHINGS TO HAMILTONIAN CIRCUITS IN LINE GRAPHS

This chapter is based on a joint work with Marién Abreu, John Baptist Gauci, Domenico Labbate and Giuseppe Mazzuoccolo [III].

5.1 INTRODUCTION

In this chapter, we deal with the line graph of a graph G and search for sufficient conditions on G which result in $L(G)$ being PMH. We prove that $L(G)$ is PMH in all of the following cases:

- G is Hamiltonian with maximum degree $\Delta(G)$ at most 3 (Theorem 5.2.3);
- G is a complete graph (Theorem 5.3.2); and
- G is arbitrarily traceable from some vertex (Theorem 5.3.3).

Analogous results with regards to the line graph of complete bipartite graphs shall be discussed in Chapter 6.

Further related results and open problems regarding graphs which are hypohamiltonian, Eulerian or with large maximum degree are discussed along this chapter. We also remark that unless otherwise stated, in this chapter we let G be a graph of order n and denote its set of vertices by $\{v_1, v_2, \ldots, v_n\}$.

5.2 LINE GRAPHS OF GRAPHS WITH SMALL MAXIMUM DEGREE

For some edge $e \in E(G)$, we refer to the corresponding vertex in $L(G)$ as e, for simplicity, unless otherwise stated. A *clique partition* of a graph G is a collection of cliques of G in which each edge of G occurs exactly once. For any $v \in V(G)$, let Q_v be the set of all the edges incident to v. Clearly, Q_v induces a clique in $L(G)$ and $\mathcal{Q} = \{Q_v : v \in V(G) \text{ with degree at least } 2\}$ is a clique partition of $L(G)$. We say that $\mathcal{Q}$ is the *canonical clique partition* of $L(G)$. In the sequel, we shall refer to Q_{v_i} simply as Q_i and in order to avoid trivial cases, from now on we always assume that G is a connected graph of order larger than 2. In what follows, we shall also say that a clique $Q' \in \mathcal{Q}$ is intersected by a set of edges N of $L(G)$, and by this we mean that $E(Q') \cap N \neq \emptyset$.

For a graph F, an *F-decomposition* of G is a collection of subgraphs of G whose edges form a partition of $E(G)$ such that each subgraph in the collection is isomorphic to F. In general, it is not hard to show that every connected graph G with $|E(G)|$ even has a P_3-decomposition. This is equivalent to saying that $L(G)$ has a perfect matching (see also Corollary 3 in [92]): indeed there is a natural bijection between the paths in a P_3-decomposition of G and the edges of the corresponding perfect matching M of $L(G)$, with the two edges in a P_3 corresponding to the two end-vertices of the respective edge in M. Since we are interested in line graphs which are PMH, a necessary condition is that $L(G)$ is Hamiltonian. Harary and Nash-Williams in [42] showed that $L(G)$ is Hamiltonian if and only if G admits a dominating tour. In particular, this implies that if G is Hamiltonian or Eulerian, then, $L(G)$ is also Hamiltonian, but the converse is not necessarily true (see also [18, 42, 89]).

The following technical lemma is the main tool we use to prove Theorem 5.2.3 as well as a series of related results contained in this section. It describes a necessary and sufficient condition to extend a given perfect matching to a Hamiltonian circuit in subcubic graphs.

Lemma 5.2.1. *Let G be a connected graph such that $\Delta(G) \leq 3$. A perfect matching M of $L(G)$ can be extended to a Hamiltonian circuit if and only if there exists a dominating circuit D of G such that the vertices in G untouched by D correspond to a subset of cliques in $\mathcal{Q}$ not intersected by M, where $\mathcal{Q}$ is the canonical clique partition of $L(G)$.*

Proof. Let M be a perfect matching of $L(G)$ which can be extended to a Hamiltonian circuit H_L of $L(G)$. For some orientation of H_L, let $Q_1, Q_2, \ldots, Q_s$ be the order in which $E(H_L)$ intersects at least one edge of the cliques in $\mathcal{Q}$, where $s \in [n]$. Since $\Delta(G) \leq 3$, $\mathcal{Q}$ consists of 2-cliques and 3-cliques, implying that the sequence $Q_1, Q_2, \ldots, Q_s$ does not have repetitions. We claim that $D = (v_1, v_2, \ldots, v_s)$ is a dominating circuit of G. Clearly, D is a circuit, since consecutive cliques in the sequence $Q_1, Q_2, \ldots, Q_s$ imply the existence of an edge between the corresponding two vertices in D. We then consider two cases. If every clique in $\mathcal{Q}$ is intersected by $E(H_L)$, then $(v_1, v_2, \ldots, v_s)$ is a Hamiltonian circuit, since $s = n$. Therefore, consider the case when $\mathcal{Q}$ contains a clique, say Q, not intersected by $E(H_L)$. The edges of the other cliques in $\mathcal{Q}$ which are incident to a vertex in Q must be intersected by $E(H_L)$, as otherwise the latter is not a Hamiltonian circuit of $L(G)$. Let these cliques be denoted by $Q_{j_1}, \ldots, Q_{j_k}$, for $k = 2$ or 3 and $j_1, \ldots, j_k \in [s]$. Let the corresponding vertices of Q and $Q_{j_1}, \ldots, Q_{j_k}$, in G, be v and $v_{j_1}, \ldots, v_{j_k}$, respectively. Also, since $v \neq v_t$ for all v_t in D, and M is a perfect matching of $L(G)$, the vertices $v_{j_1}, \ldots, v_{j_k}$ are in the circuit D (not necessarily adjacent amongst themselves) and so the edges in G having v as an end-vertex have at least one end-vertex in D. Thus, since v was arbitrary, D is dominating. Moreover, every vertex in G untouched by D corresponds to a clique in $\mathcal{Q}$ not intersected by $E(H_L)$, which is a subset of the cliques in $\mathcal{Q}$ not intersected by M.

Conversely, let M be a perfect matching of $L(G)$ and let $D = (v_1, v_2, \ldots, v_s)$ be a dominating circuit in G, for some $s \leq n$, such that the untouched vertices correspond to a subset of the cliques in $\mathcal{Q}$ not intersected by M. Note that there exists a one-to-one mapping between the untouched vertices in G and the unintersected cliques in $\mathcal{Q}$, which is not necessarily onto. We traverse the cliques in $\mathcal{Q}$ as follows. Let Q be a clique in $\mathcal{Q}$, with corresponding vertex $v \in V(G)$. We consider three cases.

Case 1. $E(Q) \cap M \neq \emptyset$.
By our assumption, $v = v_i$ for some $i \in [s]$, and we traverse Q $(= Q_i)$ using the unique path joining $V(Q_{i-1}) \cap V(Q_i)$ and $V(Q_i) \cap V(Q_{i+1})$ which contains $E(Q) \cap M$.

Case 2. $E(Q) \cap M = \emptyset$ and $v \in D$.
In this case, $v = v_j$ for some $j \in [s]$, and we traverse Q $(= Q_j)$ using the edge with end-vertices $V(Q_{j-1}) \cap V(Q_j)$ and $V(Q_j) \cap V(Q_{j+1})$.

Case 3. $E(Q) \cap M = \emptyset$ and $v \notin D$.
Since M is a perfect matching, all the cliques in $\mathcal{Q}$ sharing a vertex with Q (which must be triangles in this case) are intersected by M. These 3-cliques are traversed as in Case 1, and in this way the edges of Q are not intersected.

We traverse all the cliques in $\mathcal{Q}$ in the above way and let the resulting sequence of edges be H_L. We claim that H_L induces a Hamiltonian circuit of $L(G)$ containing M. By Case 1, H_L contains M and so every vertex of $L(G)$ is covered by H_L. Also, the sequence of cliques intersected by $E(H_L)$, that is, $Q_1, Q_2, \ldots, Q_s$, corresponds to the sequence of vertices in D, and so, since D is connected and 2-regular, H_L is a connected circuit, proving our claim. □

Remark 5.2.2. Note that Lemma 5.2.1 is not true in general for $\Delta(G) > 3$. An easy example is shown in Figure 5.1: indeed, an arbitrary perfect matching of $L(G)$ can be extended to a Hamiltonian circuit, that is, $L(G)$ is PMH, but there is no dominating circuit in G.

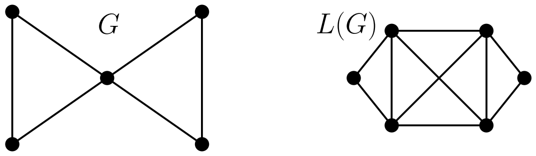

Figure 5.1: A graph with maximum degree 4 and no dominating circuit whose line graph is PMH

By using Lemma 5.2.1, we can furnish a first sufficient condition on G assuring that its line graph is PMH.

Theorem 5.2.3. *Let G be a Hamiltonian graph such that $\Delta(G) \leq 3$. Then, $L(G)$ is PMH.*

Proof. Let H be a Hamiltonian circuit of G. Given any perfect matching M of $L(G)$, since the set of vertices untouched by H in G is empty, it is trivially a subset of the cliques in $\mathcal{Q}$ not intersected by M. Consequently, by Lemma 5.2.1, M can be extended to a Hamiltonian circuit of $L(G)$. Since M was arbitrary, G is PMH. $\square$

In particular, Theorem 5.2.3 applies for all Hamiltonian cubic graphs. However, in the cubic case we can say more. As already stated before, Kotzig [55] proved that the existence of a Hamiltonian circuit in a cubic graph is both a necessary and sufficient condition for a partition of $L(G)$ in two Hamiltonian circuits. We show the following.

Corollary 5.2.4. *Let G be a Hamiltonian cubic graph and M a perfect matching of $L(G)$. Then, $L(G)$ can be partitioned in two Hamiltonian circuits, one of which contains M.*

Proof. If we extend M to a Hamiltonian circuit of $L(G)$ using the method described in Lemma 5.2.1, we obtain a Hamiltonian circuit H_1 whose edge set intersects each triangle in $\mathcal{Q}$, since G is Hamiltonian. Moreover, since $E(H_1)$ intersects $Q \in \mathcal{Q}$ in one or two edges, the edges of $L(G) - E(H_1)$ intersect Q in two edges or one, respectively. Therefore, the edges in $L(G) - E(H_1)$ induce a Hamiltonian circuit H_2 of $L(G)$ whose edges intersect the triangles in $\mathcal{Q}$ in the same order as the edges in H_1. $\square$

When considering Theorem 5.2.3, one could wonder if the two conditions on the maximum degree and the Hamiltonicity of G could be improved in some way. First of all, we remark that our result is best possible in terms of the maximum degree of G: indeed, if G is a Hamiltonian graph such that $\Delta(G) = 4$, then, $L(G)$ is not necessarily PMH. For instance, consider the Hamiltonian graph in Figure 5.2 having maximum degree 4, and let M be the perfect matching of $L(G)$ shown in the figure.

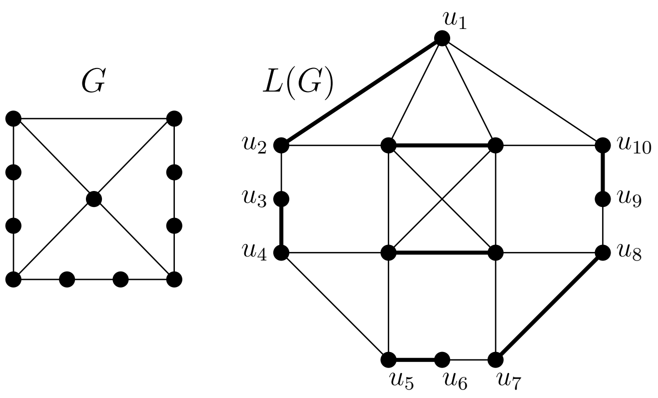

Figure 5.2: A Hamiltonian graph with maximum degree 4 whose line graph is not PMH

Suppose M can be extended to a Hamiltonian circuit. Then, it should include all edges incident to its vertices of degree 2, and so it should contain the paths $u_1, u_2, \dots, u_4$ and $u_5, u_6, \dots, u_{10}$. However, these two paths cannot be extended to a Hamiltonian circuit of $L(G)$ containing M, contradicting our assumption.

On the other hand, Hamiltonicity of G in Theorem 5.2.3 is not a necessary condition, since there exist non-Hamiltonian cubic graphs whose line graph is PMH. In particular, in Proposition 5.2.5 we prove that hypohamiltonian cubic graphs are examples of such graphs. Let us recall that a graph G is *hypohamiltonian* if G is not Hamiltonian, but for every $v \in V(G)$, $G - v$ has a Hamiltonian circuit.

Proposition 5.2.5. *Let G be a hypohamiltonian graph such that $\Delta(G) \leq 3$. Then, $L(G)$ is PMH.*

Proof. Let M be a perfect matching of $L(G)$. Since $|\mathcal{Q}| = |V(G)|$ is strictly larger than $|M| = \frac{|V(L(G))|}{2} \leq \frac{3/2 \cdot |V(G)|}{2}$, there surely exists some clique $Q \in \mathcal{Q}$ which is not intersected by M. Let v be the corresponding vertex in G. Since G is hypohamiltonian, there exists a dominating circuit in G which passes through all the vertices of G except v, and so by Lemma 5.2.1, $L(G)$ is PMH, since M was arbitrary. □

Finally, another possible improvement of Theorem 5.2.3 could be a weaker assumption on the length of the longest circuit of G: the circumference of G, denoted by $circ(G)$. However, in Proposition 5.2.8 we exhibit cubic graphs having circumference just one less than the order of G whose line graphs are not PMH.

For the proof of Proposition 5.2.8, we also need to show that each edge of $L(G)$, where G is cubic and Hamiltonian, belongs to a perfect matching. This kind of property is extensively studied in many papers and a graph G is said to be *1-extendable* if every edge in G belongs to a perfect matching of G. Theorem 2.1 in [80] states that every claw-free 3-connected graph is 1-extendable. By recalling that every line graph is a claw-free graph, we have, in particular, that $L(G)$ is 1-extendable if G is cubic and 3-edge-connected. The generalisation to an arbitrary Hamiltonian cubic graph G is not hard to achieve by using such a result, but here we prefer to present a direct short proof which is valid for any bridgeless cubic graph and which makes use of the following tool from the proof of Proposition 2 in [70].

Remark 5.2.6. [70] Let G_1 be a cubic graph of even size and M a perfect matching of $L(G_1)$, with canonical clique partition $\mathcal{Q}$. The graph G_2 obtained by removing all the edges in M from $L(G_1)$ and then applying Y-reductions to all the triangles in $\mathcal{Q}$ not intersected by M, is isomorphic to G_1.

Remark 5.2.6 follows by considering the natural bijection ϕ between $V(G_1)$ and $\mathcal{Q}$, and the function ψ_M between $\mathcal{Q}$ and $V(G_2)$, where $\psi_M(Q)$, for $Q \in \mathcal{Q}$, is defined as follows. If $E(Q) \cap M = \emptyset$, Q is mapped to the vertex in G_2 obtained after applying a Y-reduction to Q. Otherwise, if $E(Q) \cap M \neq \emptyset$, Q is mapped to the vertex in G_2 corresponding to the vertex in Q unmatched by $E(Q) \cap M$. It is not hard to prove that $\psi_M \circ \phi$ is an isomorphism between G_1 and G_2.

Lemma 5.2.7. *Let G be a bridgeless cubic graph of even size. Then, every edge of $L(G)$ belongs to a perfect matching.*

Proof. Let $e \in E(L(G))$ and let M be a perfect matching of $L(G)$. Assume $e \notin M$, otherwise the statement holds. The graph $L(G) - M$ is cubic, and by Remark 5.2.6 can be obtained by applying suitable Y-extensions to G. Since G is bridgeless, and the resulting graph after applying Y-extensions to a bridgeless graph is again bridgeless, we have that $L(G) - M$ is bridgeless as well. Moreover, in [86], Schönberger proved that every bridgeless cubic graph is 1-extendable: hence, there exists a perfect matching of $L(G) - M$ which contains e. Such a perfect matching is trivially also a perfect matching of $L(G)$ containing e. □

The following proposition shows that the Hamiltonicity condition in Theorem 5.2.3 cannot be relaxed to any other condition regarding the length of the longest circuit in G. Indeed, starting from an appropriate cubic graph and performing suitable Y-extensions, we obtain a graph of circumference one less than its order whose line graph is not PMH.

Proposition 5.2.8. *Let G be a hypohamiltonian cubic graph of odd size. Let G' be a graph obtained by performing a Y-extension to all vertices of G except one. Then, $circ(G') = |V(G')| - 1$ and $L(G')$ is not PMH.*

Proof. Let v be the vertex of G to which we do not apply a Y-extension, and let the resulting graph be G', with the vertex of G' corresponding to v denoted by v'. Since G is hypohamiltonian, G admits a circuit C of length $|V(G)| - 1$ which passes through all the vertices of G except v. Consequently, G' admits a circuit C' which passes through all the vertices of G' except v' and whose edges intersect the Y-extended triangles in the same order that C passes through all the corresponding vertices in G, resulting in the three vertices of each Y-extended triangle being consecutive in C'. Since G' is not Hamiltonian, $circ(G') = |V(G')| - 1$. We proceed by supposing that $L(G')$ is PMH, for contradiction. Denote by $Q_{v'}$ the triangle in the canonical clique partition of $L(G')$ which corresponds to the vertex v'. By construction of G', we have $|E(G')| = |E(G)| + 3(|V(G)| - 1)$. Since both $|V(G)| - 1$ and $|E(G)|$ are odd, $|E(G')|$ is even, that is, $L(G')$ has even order. Moreover, since G is hypohamiltonian, G is bridgeless. Consequently, G' is bridgeless as well, since it is obtained by applying Y-extensions to G, and so, by Lemma 5.2.7, there exists a perfect matching M of $L(G')$ which intersects a chosen edge of $Q_{v'}$. Lemma 5.2.1 assures that there exists a dominating circuit D in G' such that the set of its uncovered vertices does not contain v'. Furthermore, the edge set of every dominating circuit of G', in particular $E(D)$, intersects at least one edge of all the

Y-extended triangles. Consequently, the dominating circuit D induces a circuit in G which passes through v and also through every other vertex of G, making G Hamiltonian, a contradiction. □

As already remarked, the graph in Figure 5.2 is Hamiltonian, but not every perfect matching in its line graph can be extended to a Hamiltonian circuit. Such an example is not regular, and we are not able to find a regular one. A most natural question to ask is whether the Hamiltonicity and regularity of a graph are together sufficient conditions to guarantee the PMH-property of its line graph. Thus, we suggest the following problem.

Problem 5.2.9. Let G be an r-regular Hamiltonian graph of even size, for $r \geq 4$. Does $L(G)$ have the PMH-property?

To conclude this section, let us note that not all 4-regular (and so not all Eulerian) graphs of even size have a PMH line graph. A non-Hamiltonian example is given in Figure 5.3. It is not hard to check that every perfect matching of $L(G)$ which contains the edges e_1e_2 and e_3e_4 cannot be extended to a Hamiltonian circuit of $L(G)$.

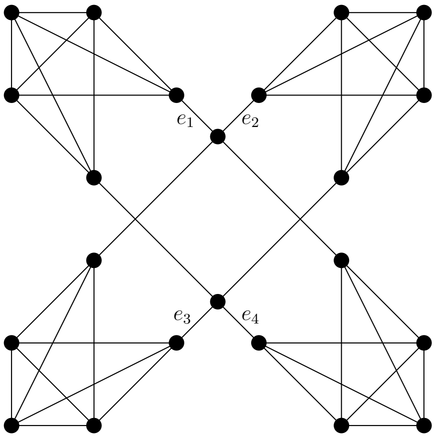

Figure 5.3: A non-Hamiltonian 4-regular graph whose line graph does not have the PMH-property

Since the graphs in Figure 5.2 and Figure 5.3 are both not simultaneously Eulerian and Hamiltonian, we pose a further problem.

Problem 5.2.10. Let G be a graph of even size which is both Eulerian and Hamiltonian. Does $L(G)$ have the PMH-property?

5.3 OTHER CLASSES OF GRAPHS WHOSE LINE GRAPHS ARE PMH

The complete graph K_n, for even n, and the complete bipartite graph $K_{m,m}$, for $m \geq 2$, are clearly PMH. To stay in line with the contents of this chapter, we now see whether their line graphs are also PMH. To this purpose, given an edge-colouring (not necessarily proper) of a Hamiltonian graph, a Hamiltonian circuit in which no two consecutive edges have the same colour is referred to as a *properly coloured Hamiltonian circuit*.

5.3.1 *Complete graphs*

First of all, we note that the line graph of a complete graph K_n has a perfect matching if and only if the number of edges in K_n is even. Hence, in the sequel we consider only complete graphs with $n \equiv 0, 1 \mod 4$.

We denote the vertices of K_n by $\{v_i : i \in [n]\}$ and the edges of K_n by $\{e_{i,j} = v_i v_j : i \neq j\}$. Moreover, $V(L(K_n))$ is denoted by $\{v_{i,j} : i \neq j\}$ where the vertex $v_{i,j}$ corresponds to the edge $e_{i,j}$ of K_n. Finally, we denote the edges of $L(K_n)$ by $\{e^i_{j,k} = v_{i,j}v_{i,k} : i \neq j \neq k \neq i\}$. Note that the upper index in the notation $e^i_{j,k}$ immediately indicates that the considered edge belongs to the clique Q_i in the canonical clique partition of $L(K_n)$, whilst the order of lower indices is irrelevant.

The proof of our main theorem in this section, Theorem 5.3.2, makes use of a special case of a result by David E. Daykin [20] from 1976 which asserts the existence of a properly coloured Hamiltonian circuit if the edges of K_n are coloured according to the following constraints.

Theorem 5.3.1. *[20] If the edges of the complete graph K_n, for $n \geq 6$, are coloured in such a way that no three edges of the same colour are incident to any given vertex, then there exists a properly coloured Hamiltonian circuit.*

In the following proof, the process of traversing one path after another is called *concatenation of paths*. If two paths P^1 and P^2 have end-vertices x, y, and y, z, respectively, we write P^1P^2 to denote the path starting at x and ending at z obtained by traversing P^1 and then P^2.

Theorem 5.3.2. *For $n \equiv 0, 1 \mod 4$, $L(K_n)$ is PMH.*

Proof. Since K_4 is Hamiltonian and cubic, by Theorem 5.2.3, the result holds for $n = 4$. Therefore, we can assume $n > 4$.

Let M be a perfect matching of $L(K_n)$. We colour the $\frac{1}{4}n(n-1)$ edges of M with $\frac{1}{4}n(n-1)$ different colours. For all $e^i_{j,k} \in M$, we colour the edges $e_{i,j}$ and $e_{i,k}$ in K_n with the same colour given to the edge $e^i_{j,k}$ in $L(K_n)$. This gives a P_3-decomposition of K_n in which each P_3 is monochromatic and the colours of all the 3-paths are pairwise distinct.

If $n = 5$, the total number of Hamiltonian circuits in K_5 is $\frac{4!}{2} = 12$. Each of the five monochromatic 3-paths in K_5 is on exactly two distinct Hamiltonian circuits. Therefore, the number of Hamiltonian circuits containing a monochromatic P_3 is at most 10, hence K_5 contains at least two (complementary) properly coloured Hamiltonian circuits. Without loss of generality, let one of them be H, say $H = (v_1, v_2, \ldots, v_5)$.

For $n \geq 8$, by Theorem 5.3.1, there exists a properly coloured Hamiltonian circuit H in K_n and again, without loss of generality, we can assume $H = (v_1, v_2, \dots, v_n)$.

Now, for all $n \geq 5$ and $n \equiv 0,1 \mod 4$, we use the properly coloured Hamiltonian circuit H in K_n to obtain a Hamiltonian circuit H_L in $L(K_n)$ containing the perfect matching M. We construct the Hamiltonian circuit H_L in such a way that it enters and exits each clique in the canonical clique partition $\mathcal{Q}$ of $L(K_n)$ exactly once. More precisely, we construct a suitable path P^i in each clique Q_i and we obtain H_L as a concatenation of such paths following the order determined by H. Consider the $(n-1)$-clique Q_i and its two vertices $v_{i-1,i}$ and $v_{i,i+1}$. The corresponding edges $e_{i-1,i}$ and $e_{i,i+1}$, in K_n, are not of the same colour since they are consecutive in H, and so the edge $e^i_{i-1,i+1} \notin M$. We assign a linear order $<_i$ to the set of edges $M \cap E(Q_i)$, with $(M \cap E(Q_i), <_i) = \mu_i$, such that:

(i) if $M \cap E(Q_i)$ contains an edge incident to $v_{i-1,i}$, such an edge is the first edge of μ_i; and

(ii) if $M \cap E(Q_i)$ contains an edge incident to $v_{i,i+1}$, such an edge is the last edge of μ_i.

Note that $<_i$ exists since $e^i_{i-1,i+1} \notin M$. Next, we construct an M-alternating path in Q_i, which we denote by P^i, starting at $v_{i-1,i}$ and ending at $v_{i,i+1}$ as follows: P^i alternates between an edge of μ_i and an edge which is simultaneously adjacent to two consecutive edges in μ_i, except possibly the first and/or last edge in P^i. Note that the choice of edges not belonging to $M \cap E(Q_i)$ as given above is always possible since Q_i is a clique. Consequently, $M \cap E(Q_i) \subset E(P_i)$.

Now we define H_L to be $P^1 P^2 \dots P^n$. Note that H_L is a circuit since the paths P^i are all internally and pairwise disjoint, and the beginning of P^1 coincides with the end of P^n. Moreover, H_L is Hamiltonian because $M \subset E(H_L)$, and so each vertex of the line graph belongs to H_L. □

5.3.2 *Arbitrarily traceable graphs*

A graph G is said to be *arbitrarily traceable* (or equivalently *randomly Eulerian*) from a vertex $v \in V(G)$ if every walk starting from v and not containing any repeated edges can be completed to an Eulerian tour. This notion was firstly introduced by Ore in [76], who proved that an Eulerian graph G is arbitrarily traceable from v if and only if every circuit in G touches v. Here we show that every perfect matching M of the line graph of an arbitrarily traceable graph can be extended to a Hamiltonian circuit.

Note that the technique used in this proof is in some way different from what was used in the case of complete graphs in Section 5.3.1. Again, a perfect matching M of $L(G)$ corresponds to a

P_3-decomposition of G, but this time we construct an Euler tour of the original graph (instead of a Hamiltonian circuit) such that two edges in the same 3-path are consecutive in the Euler tour (as opposed to what was done in Section 5.3.1 where we forbade two edges in the same 3-path to be consecutive in the Hamiltonian circuit considered in K_n).

Theorem 5.3.3. *Let G be a graph of even size. If G is arbitrarily traceable from some vertex, then its line graph is PMH.*

Proof. Let M be a perfect matching of $L(G)$. Consider the P_3-decomposition of G induced by M. Since G is arbitrarily traceable from some vertex, there exists an Euler tour in which every pair of edges in the same 3-path are consecutive. The sequence of edges in this Euler tour corresponds to a sequence of vertices in $L(G)$ which gives a Hamiltonian circuit H of $L(G)$, and since the two edges of each 3-path in the P_3-decomposition are consecutive in the Euler tour, H contains all the edges of M, as required. □

5.4 FINAL REMARKS

Along the chapter, we have proposed several sufficient conditions of different types for a graph in order to guarantee the PMH-property in its line graph. The wide variety of such conditions, ranging between sparse and dense graphs, do not allow us to easily identify non-trivial necessary conditions to this problem. This could be seemingly hard, but we still consider it an intriguing problem to be addressed in the future.

6

SAVED BY THE ROOK

This chapter is based on a joint work with Marién Abreu and John Baptist Gauci [IV].

In 1976, Chen and Daykin considered an analogous version of Theorem 5.3.1 for the complete bipartite graph $K_{m,m}$ (see [19]). A particular instance of this (Theorem $1'$ in [19]) can be stated as follows.

Theorem 6.0.1. *[19] Consider an edge-colouring of the complete bipartite graph $K_{m,m}$ such that no vertex is incident to more than k edges of the same colour. If $m \geq 25k$, then there exists a properly coloured Hamiltonian circuit.*

By considering the case $k = 2$ in the previous theorem, that is, $m \geq 50$, and by using an argument very similar to the one used for complete graphs in Section 5.3.1, one could obtain that $L(K_{m,m})$ is PMH for every even $m \geq 50$. However, in this chapter, a more complete result is given and extended by using a different and more technical approach. More precisely we prove that $L(K_{m_1,m_2})$ does not have the PH-property if and only if $m_1 = 2$ and m_2 is odd, where m_1 is an even integer and $m_2 \geq 1$.

6.1 INTRODUCTION

In the very early versions of chess, the *rook* (from the Persian and Indian words *rukh* and *ratha*, respectively) represented the *chariot*, and it is said that it is only when the game got to Europe did the word *rukh* got confused with the Italian word *rocca*, which means *fortress*. The rook chess piece is allowed to move in a horizontal and vertical manner only—no diagonal moves are permissible—and the rook graph represents all the possible moves of a rook on a chessboard. The vertices and the edges of the rook graph correspond to the cells of the chessboard, and the legal moves of the rook from one cell to the other, respectively.

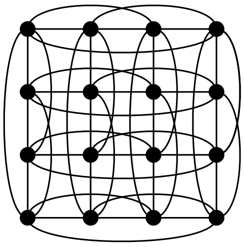

Figure 6.1: The 4×4 rook graph isomorphic to $K_4 \square K_4$

All the legal moves of a rook on a $m_1 \times m_2$ chessboard give rise to a $m_1 \times m_2$ *rook graph* which is isomorphic to the Cartesian product of the complete graphs K_{m_1} and K_{m_2}, denoted by $K_{m_1} \square K_{m_2}$. Such correlation was already considered, for example, in [59]. We also remark that the $m_1 \times m_2$ rook graph is also isomorphic to the line graph of the complete bipartite graph K_{m_1,m_2}. In what follows we consider the following problem.

Problem 6.1.1. Let G be a $m_1 \times m_2$ chessboard and let M be a set containing pairs of distinct cells of G such that each cell of G belongs to exactly one pair in M. Determine the values of m_1 and m_2 for which it is possible to construct a closed tour H visiting all the cells of the chessboard G exactly once, such that:

(i) consecutive cells in H are either a pair of cells in M, or two cells in G which can be joined by a legal rook move; and

(ii) H contains all pairs of cells in M.

In other words, given any possible choice of a set M as defined above, is a rook good enough to let one visit, exactly once, all the cells on a chessboard and finish at the starting cell, in such a way that each pair of cells in M is allowed to and must be used once? We remark that M can contain pairs of cells which are not joined by a legal rook move.

As many other mathematical chess problems (for a detailed exposition, we suggest the reader to [88]), the above problem can be restated in graph theoretical terms, as follows.

Problem 6.1.2 (restated)**.** Let G be the $m_1 \times m_2$ rook graph, or equivalently $K_{m_1} \square K_{m_2}$. Determine for which values of m_1 and m_2 does G have the PH-property.

Clearly, in order for $K_{m_1} \square K_{m_2}$ to admit a pairing, at least one of m_1 and m_2 must be even, and without loss of generality, in the sequel we shall tacitly assume that m_1 is even.

In what follows we shall consider Hamiltonian circuits of K_G (for some graph G of even order) composed of a pairing of G and a perfect matching of G. In order to distinguish between pairings of G, which may possibly contain edges not in G, and perfect matchings of G, we shall depict pairing edges as green, bold and dashed, and edges of a perfect matching of G as black and bold.

To emphasise that pairings can contain edges in G, we shall depict such edges with a black thin line underneath the green, bold and dashed edge described above. This can be clearly seen in Figure 6.2.

Finally, we also state the following theorem from [2] that shall be used in the next section to prove our main result.

Theorem 6.1.3. *[2] The Cartesian product of a complete graph K_m (m even and $m \geq 6$) and a path P_q ($q \geq 1$) has the PH-property.*

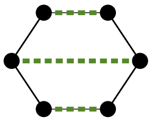

Figure 6.2: A pairing in C_6 which cannot be extended to a Hamiltonian circuit

6.2 MAIN RESULT

In this section we give a complete solution to our problem, summarised in the following theorem.

Theorem 6.2.1. *Let m_1 be an even integer and let $m_2 \geq 1$. The $m_1 \times m_2$ rook graph does not have the PH-property if and only if $m_1 = 2$ and m_2 is odd.*

Proof. When $m_2 = 1$, $K_{m_1} \square K_1$ is K_{m_1} and the result clearly follows. Consequently, we shall assume that $m_2 > 1$. By Remark 1.4.1 and Theorem 6.1.3, $K_{m_1} \square K_{m_2}$ is PH when $m_1 \geq 6$, since $K_{m_1} \square K_{m_2}$ contains $K_{m_1} \square P_{m_2}$ as a spanning subgraph.

So consider the cases when $m_1 = 2$ or 4. If $m_1 = 2$, $K_{m_1} \square K_{m_2}$ is PH if and only if m_2 is even. In fact, if m_2 is odd, the pairing consisting of the m_2-edge-cut between the two copies of K_{m_2} cannot be extended to a Hamiltonian circuit, as can be seen in Figure 6.3.

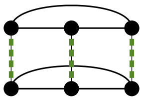

Figure 6.3: A pairing in $K_2 \square K_3$ which cannot be extended to a Hamiltonian circuit

If m_2 is even, the result follows once again by Theorem 6.1.3 when $m_2 \geq 6$. If $m_2 = 2$, the result easily follows, and when $m_2 = 4$, $K_2 \square K_4$ is PH because the 3-dimensional cube Q_3 is a subgraph of $K_2 \square K_4$ and has the PH-property by Fink's result in [29] (also referred to previously).

What remains to be considered is the case when $m_1 = 4$ and $m_2 \geq 3$. The graph $K_4 \square K_4$ contains $C_4 \square C_4$, the 4-dimensional hypercube Q_4, which is PH ([29]), and for $m_2 \geq 6$ and m_2 even, the result follows once again by Theorem 6.1.3. Therefore, what remains to be shown is the case when $m_2 \geq 3$ and m_2 is odd, which is settled in Lemma 6.2.2. ☐

Lemma 6.2.2. *For every odd $m \geq 3$, the $4 \times m$ rook graph has the PH-property.*

Proof. Let the $4 \times m$ rook graph $K_4 \square K_m$ be denoted by G. We let the vertex set of G be $\{a_i, b_i, c_i, d_i : i \in [m]\}$, such that for each i, the vertices a_i, b_i, c_i, d_i induce a complete graph on four vertices, denoted by K_4^i, and the vertices represented by the same letter induce a K_m. Let

M be a pairing of G. We consider two cases.

Case 1. M does not induce a perfect matching in each K_4^i.

Case 2. M induces a perfect matching in each K_4^i.

We start by considering **Case 1**, and without loss of generality assume that $|M \cap E(K_4^1)| < 2$. If we delete from G all the edges having exactly one end-vertex in K_4^1, we obtain two components G_1 and G_2 isomorphic to K_4^1 and $K_4 \square K_{m-1}$, respectively. Since G_1 is of even order and $M \cap E(G_1)$ is not a perfect matching of this graph, G_1 has an even number (two or four) of vertices which are unmatched by $M \cap E(G_1)$.

We pair these unmatched vertices such that $M \cap E(G_1)$ is extended to a perfect matching M_1 of G_1. By a similar reasoning, $M \cap E(G_2)$ does not induce a pairing of G_2 and the number of vertices in G_2 which are unmatched by $M \cap E(G_2)$ is again two or four. Without loss of generality, let a_1, b_1 be two vertices in G_1 unmatched by $M \cap E(G_1)$ such that $a_1b_1 \in M_1$, and let x, y be the two vertices in G_2 such that a_1x and b_1y are both edges in the pairing M of G. We extend $M \cap E(G_2)$ to a pairing M_2 of G_2 by adding the edge xy to $M \cap E(G_2)$, and we repeat this procedure until all vertices in G_2 are matched. Since $m-1$ is even, G_2 has the PH-property and so M_2 can be extended to a Hamiltonian circuit H_2 of K_{G_2}. We extend H_2 to a Hamiltonian circuit of G containing M as follows. If $c_1d_1 \in M \cap E(G_1)$, we replace the edge xy in H_2 by the edges $xa_1, a_1d_1, d_1c_1, c_1b_1, b_1y$, as in Figure 6.4.

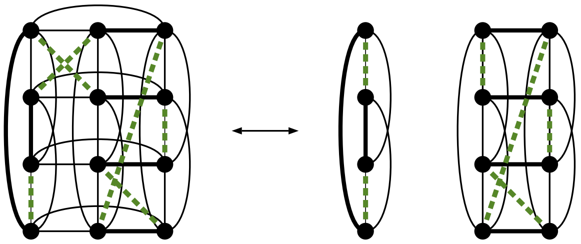

Figure 6.4: An illustration of the inductive step in **Case 1** when $m_2 = 3$

Otherwise, $c_1d_1 \in M_1 - (M \cap E(G_1))$, and so there exist two vertices u, v in G_2 such that c_1u and d_1v belong to belong to the initial pairing M, and uv belongs to M_2. In this case, we replace the edges xy and uv in H_2 by the edges xa_1, a_1b_1, b_1y, and uc_1, c_1d_1, d_1v, respectively. In either case, H_2 is extended to a Hamiltonian circuit of G containing the pairing M, as required.

Next, we move on to **Case 2**, that is, when M induces a perfect matching in each K_4^i. This case is true by Proposition 1 in [2], however, here we adopt a constructive and more detailed approach. There are three different ways how M can intersect the edges of K_4^i, namely, $M \cap E(K_4^i)$ can either be equal to $\{a_ib_i, c_id_i\}$, $\{a_ic_i, b_id_i\}$, or $\{a_id_i, b_ic_i\}$. The number of 4-cliques intersected by M in $\{a_ib_i, c_id_i\}$ is denoted by

v^{ab}_{cd}, and we shall define v^{ac}_{bd} and v^{ad}_{bc} in a similar way. Without loss of generality, we shall assume that $v^{ab}_{cd} \geq v^{ac}_{bd} \geq v^{ad}_{bc}$. We shall also assume that the first v^{ab}_{cd} 4-cliques in $\{K^i_4 : i \in [m]\}$ are the ones intersected by M in $\{a_ib_i, c_id_i\}$, and, if $v^{ad}_{bc} \neq 0$, the last v^{ad}_{bc} 4-cliques are the ones intersected by M in $\{a_id_i, b_ic_i\}$. This can be seen in Figure 6.5, in which "unnecessary" curved edges of G are not drawn so as to render the figure more clear.

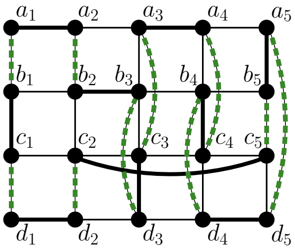

Figure 6.5: G when $v^{ab}_{cd} = 2$, $v^{ac}_{bd} = 2$ and $v^{ad}_{bc} = 1$

When $v^{ab}_{cd} = 1$, we have that $v^{ac}_{bd} = v^{ad}_{bc} = 1$, and in this case it is easy to see that M can be extended to a Hamiltonian circuit of K_G, for example, $(a_1, b_1, c_1, d_1, d_3, a_3, c_3, b_3, b_2, d_2, c_2, a_2)$. We remark that this is the only time when all the 4-cliques are intersected differently by M. Therefore, assume $v^{ab}_{cd} \geq 2$. First, let $v^{ab}_{cd} = 2$. If $v^{ad}_{bc} = 0$, then, $v^{ac}_{bd} = 1$ and it is easy to see that M can be extended to a Hamiltonian circuit of K_G, for example, $(a_1, b_1, b_2, a_2, a_3, c_3, b_3, d_3, d_2, c_2, c_1, d_1)$. The only other possibility is to have $v^{ac}_{bd} = 2$ and $v^{ad}_{bc} = 1$, and once again M can be extended to a Hamiltonian circuit of K_G, as Figure 6.5 shows.

Thus, we can assume that $v^{ab}_{cd} \geq 3$. Let $r = v^{ab}_{cd} + v^{ac}_{bd}$ and let r' be the largest even integer less than or equal to r. Moreover, let G_1 be the subgraph of G induced by the vertices $\{b_i, c_i : i \in [m]\}$ (isomorphic to $K_2 \Box K_m$) and let $M_1 = \{b_1b_2, \ldots, b_{r'-1}b_{r'}, c_1c_2, \ldots, c_{r'-1}c_{r'}, b_{r'+1}c_{r'+1}, \ldots, b_mc_m\}$. Clearly, M_1 is a pairing of G_1 which contains $M \cap E(G_1)$, and can be extended to a Hamiltonian circuit H_1 of K_{G_1} as follows: $(b_1, b_2, \ldots, b_{r'}, b_{r'+1}, c_{r'+1}, c_{r'+2}, b_{r'+2}, \ldots, b_mc_m, c_{r'}, c_{r'-1}, \ldots, c_1)$.

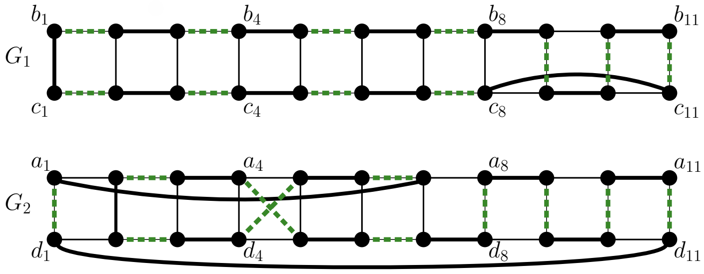

Figure 6.6: G_1 and G_2 when $v^{ab}_{cd} = 4$, $r = r' = 8$, and $m = 11$ in **Case 2**

This is depicted in Figure 6.6. We note that if $r' = m - 1$, we do not consider the index $r' + 2$ in the last sequence of vertices forming H_1. Deleting the edges belonging to $M_1 - M$ from H_1 gives a collection of r disjoint paths $\mathcal{P} = \{P^i : i \in [r]\}$. We note that the union of all the end-vertices of the paths in $\mathcal{P}$ give $\{b_i, c_i : i \in [r]\}$. If we look at the example given in Figure 6.6, the only path in $\mathcal{P}$ on more than two vertices is the path $b_8, b_9, c_9, c_{10}, b_{10}, b_{11}, c_{11}, c_8$.

Next, let G_2 be the subgraph of G induced by the vertices $\{a_i, d_i : i \in [m]\}$, which is isomorphic to $K_2 \square K_m$ as G_1. For every $i \in [r]$, we let u_i and v_i be the two end-vertices of the path P^i, and we let x_i and y_i be the two vertices in G_2 such that $u_i x_i$ and $v_i y_i$ both belong to M. We remark that $\{a_i, d_i : i \in [r]\} = \{x_i, y_i : i \in [r]\}$. Let $M_2 = \{x_1y_1, \ldots, x_ry_r\} \cup (M \cap E(G_2))$. If $r = m$, then $M \cap E(G_2)$ is empty, otherwise it consists of $\{a_{r+1}d_{r+1}, \ldots, a_md_m\}$. If ν^{ab}_{cd} is even (as in Figure 6.6), M_2 contains

$$\{a_1d_1, a_2a_3, \ldots, a_{\nu^{ab}_{cd}-2}a_{\nu^{ab}_{cd}-1}, a_{\nu^{ab}_{cd}}d_{\nu^{ab}_{cd}+1}, d_2d_3, \ldots, d_{\nu^{ab}_{cd}-2}d_{\nu^{ab}_{cd}-1}, d_{\nu^{ab}_{cd}}a_{\nu^{ab}_{cd}+1}\}.$$

Otherwise, M_2 contains $\{a_1d_1, a_2a_3, \ldots, a_{\nu^{ab}_{cd}-1}a_{\nu^{ab}_{cd}}, d_2d_3, \ldots, d_{\nu^{ab}_{cd}-1}d_{\nu^{ab}_{cd}}\}$. Moreover, if r is even, then $a_rd_r \in M_2$. In either case, M_2 can be extended to a Hamiltonian circuit H_2 of K_{G_2}, as can be seen in Figure 6.6, which shows the case when ν^{ab}_{cd} and r are both even. We remark that the green, bold and dashed edges in the figure are the ones in M_1 and M_2. If for each $i \in [r]$, we replace the edges x_iy_i in H_2 by x_iu_i, the path P^i, and v_iy_i (as in Figure 6.7), a Hamiltonian circuit of K_G containing M is obtained, proving our theorem. $\square$

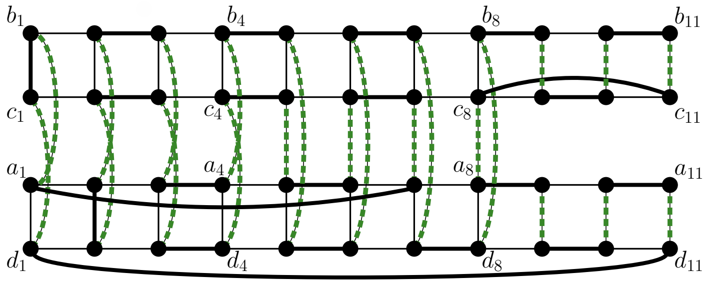

Figure 6.7: Extending H_1 and H_2 from Figure 6.6 to a Hamiltonian circuit of K_G containing M

6.3 COMPLETE BIPARTITE GRAPHS

As already mentioned, the $m_1 \times m_2$ *rook graph* can also be seen as the line graph of the complete bipartite graph K_{m_1,m_2}. In this last section we show that, unsurprisingly, the complete bipartite graph having equal partite sets (otherwise it does not admit a perfect matching) is PH, as well.

Theorem 6.3.1. *For every $m \geq 2$, the complete bipartite graph $K_{m,m}$ has the PH-property.*

Proof. Let $\{u_1, \ldots, u_m\}$ and $\{w_1, \ldots, w_m\}$ be the partite sets of $K_{m,m}$. We proceed by induction on m. When $m = 2$, result holds since

$K_{2,2} \simeq K_2 \square K_2$. So assume $m > 2$ and let M be a pairing of $K_{m,m}$. If $M = \{u_i w_i : i \in [m]\}$, then M easily extends to a Hamiltonian circuit of the underlying complete graph on $2m$ vertices. Thus, assume there exists $j \in [m]$ such that $u_j w_j \notin M$. Without loss of generality, let j be equal to m. Then, M contains the edges xu_m and yw_m, for some x and y belonging to the set $Z = \{u_i, w_i : i \in [m-1]\}$. We note that Z induces the complete bipartite graph $K_{m-1,m-1}$ with partite sets $\{u_1, \ldots, u_{m-1}\}$ and $\{w_1, \ldots, w_{m-1}\}$, which we denote by G'. The set of edges $M' = M \cup \{xy\} - \{xu_m, yw_m\}$ is a pairing of G', and so, by induction on m, M' can be extended to a Hamiltonian circuit H' of $K_{G'}$. This Hamiltonian circuit can be extended to a Hamiltonian circuit H of the underlying complete graph of $K_{m,m}$ by replacing the edge xy in H', by the edges $xu_m, u_m w_m, w_m y$. The resulting Hamiltonian circuit H clearly contains M, proving our theorem. □

Although the statement and proof of Theorem 6.3.1 are quite easy, they may lead to another intriguing problem. From Theorem 6.2.1 we know that the rook is not good enough to solve our problem on a $2 \times m_2$ chessboard when m_2 is odd. However, the above result shows that if the rook was somehow allowed to do only vertical and diagonal moves (instead of vertical and horizontal moves only), then it would always be possible to perform a closed tour on a $2 \times m_2$ chessboard in such a way that each pair of cells in M is allowed to and must be used once, no matter the choice of M.

We shall call this new hybrid chess piece the *bishop-on-a-rook*, and, as already stated, it is only allowed to move in a vertical and diagonal manner—no horizontal moves are permissible. As in the case of the rook, all the legal moves of a bishop-on-a-rook on a $m_1 \times m_2$ chessboard give rise to a $m_1 \times m_2$ *bishop-on-a-rook graph*, with m_1 corresponding to the vertical axis.

As before, for the $m_1 \times m_2$ bishop-on-a-rook graph to be PH, at least one of m_1 or m_2 must be even. Moreover, we remark that when $m_2 \leq m_1$, the $m_1 \times m_2$ bishop-on-a-rook graph contains $K_{m_1} \square K_{m_2}$ as a subgraph. Finally, we also observe that the $m_1 \times m_2$ bishop-on-a-rook graph is isomorphic to the co-normal product of K_{m_1} and $\overline{K}_{m_2}$, where the latter is the empty graph on m_2 vertices. The *co-normal product* $G * H$ of two graphs G and H is a graph whose vertex set is the Cartesian product $V(G) \times V(H)$ of $V(G)$ and $V(H)$, and two vertices (u_i, v_j) and (u_k, v_l) are adjacent precisely if $u_i u_k \in E(G)$ or $v_j v_l \in E(H)$. Thus,

$$V(G * H) = \{(u_r, v_s) : u_r \in V(G) \text{ and } v_s \in V(H)\}, \text{ and}$$
$$E(G * H) = \{(u_i, v_j)(u_k, v_l) : u_i u_k \in E(G) \text{ or } v_j v_l \in E(H)\}.$$

We wonder for which values m_1 and m_2 is the $m_1 \times m_2$ bishop-on-a-rook graph PH.

7

ON A PROBLEM BY ALAHMADI ET AL.

This chapter is based on a joint work with John Baptist Gauci [V].

7.1 INTRODUCTION

After characterising all the cubic graphs having the PH-property, Alahmadi et al. [2] attempt to characterise all 4-regular graphs having the same property by posing the following problem (Open Problem 3 in [2]): for which values of p and q does the Cartesian product $C_p \square C_q$ of two circuits on p and q vertices have the PH-property? Here, we give a complete answer and show that this only happens when both p and q are equal to four, namely for $C_4 \square C_4$, the 4-dimensional hypercube. For all other values of p and q, we show that $C_p \square C_q$ does not even admit the PMH-property.

7.2 MAIN RESULT

In this chapter we restrict our attention to the Cartesian product of a circuit graph and a path graph and to that of two circuit graphs, noting that the latter is also referred to in literature as a torus grid graph. In the sequel we tacitly assume that operations (including addition and subtraction) in the indices of the vertices of a circuit C_n are taken modulo n, with complete residue system $\{1, \ldots, n\}$. We first prove the following result.

Lemma 7.2.1. *The graph $C_p \square P_q$ is not PMH, for every $p, q \geq 3$.*

Proof. Label the vertices of C_p and P_q consecutively as $u_1, u_2, \ldots, u_p$, and $v_1, v_2, \ldots, v_q$, respectively, such that v_1 and v_q are the two end-vertices of P_q. For simplicity, we refer to the vertex (u_r, v_s) as $\omega_{r,s}$. If p is odd (and so q is even, otherwise $C_p \square P_q$ does not have a perfect matching), then there exists a perfect matching of $C_p \square P_q$ containing an odd-cut, say $\{\omega_{1,q-1}\omega_{1,q}, \ldots, \omega_{p,q-1}\omega_{p,q}\}$. Clearly, this perfect matching cannot be extended to a Hamiltonian circuit. Thus, we can assume that p is even. Let M be a perfect matching of $C_p \square P_q$ containing $\omega_{i,q-1}\omega_{i+1,q-1}$ and $\omega_{i-1,q}\omega_{i,q}$, for every odd $i \in [p]$. For contradiction, suppose that N is a perfect matching of $C_p \square P_q$ such that $M \cup N$ is a Hamiltonian circuit. Then, for every odd $i \in [p]$, N contains either $\omega_{i,q}\omega_{i+1,q}$, or the two edges $\omega_{i,q-1}\omega_{i,q}$ and $\omega_{i+1,q-1}\omega_{i+1,q}$. Therefore, $M \cup N$ contains a circuit with vertices belonging to $\{\omega_{1,q-1}, \ldots, \omega_{p,q-1}, \omega_{1,q}, \ldots, \omega_{p,q}\}$. Since $q > 2$, $M \cup N$ is not a Hamiltonian circuit, a contradiction. Consequently, $C_p \square P_q$ is not PMH. □

Now, we prove our main result.

Theorem 7.2.2. *Let $p, q \geq 3$. The graph $C_p \square C_q$ is PMH only when $p = 4$ and $q = 4$.*

Proof. The 4-dimensional hypercube $\mathcal{Q}_4 = C_4 \square C_4$ has the PH-property by Fink's result in [29]. Moreover, the authors in [2] showed that $C_4 \square C_q$ is not PMH when $q \neq 4$. Thus, in what follows we shall assume that p is even and at least 6 and that q is not equal to 4. Let the consecutive vertices of C_p and C_q be labelled $u_1, u_2, \ldots, u_p$, and $v_1, v_2, \ldots, v_q$, respectively, and as before, we refer to the vertex (u_r, v_s) as $\omega_{r,s}$.

We first consider the case when $q = 3$. For simplicity, let the vertices $\omega_{i,1}, \omega_{i,2}, \omega_{i,3}$ be referred to as a_i, b_i, c_i, for each $i \in [p]$, and let M be a perfect of $C_p \square C_3$ containing the following nine edges: $a_1a_2, b_1b_2, c_1c_2, a_3c_3, b_3b_4, a_4a_5, c_4c_5, b_5b_6, a_6c_6$, as shown in Figure 7.1. Since p is even, such a perfect matching M clearly exists.

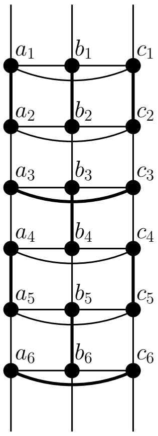

Figure 7.1: Edges belonging to the perfect matching M in $C_p \square C_3$

We claim that M cannot be extended to a Hamiltonian circuit. For, suppose not, and let N be a perfect matching of $C_p \square C_3$ such that $M \cup N$ is a Hamiltonian circuit. Each of the two sets $X_1 = \{a_3a_4, c_3c_4\}$ and $X_2 = \{a_5a_6, c_5c_6\}$ is a 2-edge-cut of the cubic graph $C_p \square C_3 - M$, and so $|X_i \cap N|$ is even for each $i \in [2]$. Moreover, the edge b_4b_5 is a bridge of the graph $C_p \square C_3 - M$, and consequently, $M \cup N$ contains a circuit of length 4, 6 or 8 with vertices belonging to $\{a_3, a_4, a_5, a_6, c_3, c_4, c_5, c_6\}$, a contradiction. Therefore, $q \geq 5$.

Similar to above, for each $i \in [p]$, let the vertices $\omega_{i,1}, \omega_{i,2}, \ldots, \omega_{i,6}$ be referred to as $a_i, b_i, \ldots, f_i$ as in Figure 7.2, with f_i being equal to a_i if $q = 5$. For each $i \in [p]$, let $L_i = b_ic_i$ and $R_i = d_ie_i$, with the sets of edges $\{L_i : i \in [p]\}$ and $\{R_i : i \in [p]\}$ denoted by $\mathcal{L}$ and $\mathcal{R}$, respectively. Let M be a perfect matching of $C_p \square C_q$ containing the following edges:

(i) a_ia_{i+1} and f_if_{i+1}, for every even $i \in [p]$,

(ii) b_ib_{i+1} and e_ie_{i+1}, for every odd $i \in [p]$, and

(iii) c_id_i, for every $i \in [p]$.

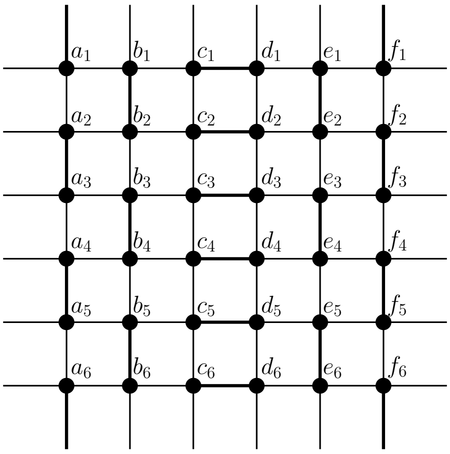

Figure 7.2: Edges belonging to the perfect matching M in $C_p\Box C_q$ when $q \geq 5$

Once again, since p is even, such a perfect matching M exists. For contradiction, suppose that N is a perfect matching of $C_p\Box C_q$ such that $M \cup N$ is a Hamiltonian circuit H of $C_p\Box C_q$. The set of edges $\mathcal{L}$ (and similarly $\mathcal{R}$) is an even cut of order p in the cubic graph $C_p\Box C_q - M$. Consequently, both $|\mathcal{L} \cap N|$ and $|\mathcal{R} \cap N|$ are even. We claim that both sets $\mathcal{L}$ and $\mathcal{R}$ must be intersected by N. For, suppose that $\mathcal{R} \cap N$ is empty, without loss of generality. In this case, $M \cup N$ forms a Hamiltonian circuit of $C_p\Box C_q - \mathcal{R}$, which is isomorphic to $C_p\Box P_q$. By a similar reasoning to that used in the proof of Lemma 7.2.1, this leads to a contradiction, and so M cannot be extended to a Hamiltonian circuit. Therefore, both $\mathcal{L} \cap N$ and $\mathcal{R} \cap N$ are non-empty.

Next, we claim that a maximal sequence of consecutive edges belonging to $\mathcal{L} - N$ (or $\mathcal{R} - N$) is of even length, whereby "consecutive edges" we mean that the indices of these edges are taken modulo p, with complete residue system $\{1, \ldots, p\}$. For, suppose there exists such a sequence made up of an odd number of edges. Without loss of generality, let L_s and L_{s+2t} be the first and last edges of this sequence, for some $s \in [p]$ and $0 \leq t < p/2$. Thus, L_{s-1} and L_{s+2t+1} are in N. In order for N to cover all the vertices of the graph it must induce a perfect matching of the path $c_s, c_{s+1}, \ldots, c_{s+2t}$, which has an odd number of vertices. This is not possible, and so our claim holds. Consequently, there exists $L_\gamma \in N$, for some odd $\gamma \in [p]$. We pair the edge L_γ with the edge $L_{\gamma'}$, where γ' is the least integer greater than γ (taken modulo p) such that $L_{\gamma'} \in N$. More formally,

$$\gamma' = \begin{cases} \min\{j \in \{\gamma+1, \ldots, p\} : L_j \in N\} & \text{if such a minimum exists,} \\ \min\{j \in \{1, \ldots, \gamma-1\} : L_j \in N\} & \text{otherwise.} \end{cases}$$

By the last claim we note that γ' is even and that the next integer $\beta > \gamma'$ (taken modulo p), if any, for which L_β is in N must be odd. Repeating this procedure on all the edges in $\mathcal{L} \cap N$ we get a partition of $\mathcal{L} \cap N$ into pairs of edges $\{L_\gamma, L_{\gamma'}\}$ where γ is odd and γ' is even. The edges in $\mathcal{R} \cap N$ are partitioned into pairs in a similar way.

We remark that if we start tracing the Hamiltonian circuit H from c_γ going towards b_γ, then H contains a path with edges alternating in N and M, starting from c_γ and ending at $c_{\gamma'}$. More precisely, if $\gamma' = \gamma + 1$, then H contains the path $c_\gamma, b_\gamma, b_{\gamma'}, c_{\gamma'}$. Otherwise, if $\gamma' \neq \gamma + 1$, then, for every even $j \in \{\gamma + 1, \ldots, \gamma' - 2\}$, N contains either $b_j b_{j+1}$ or the two edges $a_j b_j$ and $a_{j+1} b_{j+1}$. Consequently, the internal vertices on this path belong to the set $\{b_\gamma, a_{\gamma+1}, b_{\gamma+1}, \ldots, a_{\gamma'-1}, b_{\gamma'-1}, b_{\gamma'}\}$. In each of these two cases we refer to such a path between c_γ and $c_{\gamma'}$ as an *$L_\gamma L_{\gamma'}$-bracket*, or just a *left-bracket*, with L_γ and $L_{\gamma'}$ being the *upper* and *lower* edges of the bracket, respectively.

Having arrived at $c_{\gamma'}$, and noting that $c_{\gamma'} d_{\gamma'} \in M$, H also traverses this edge to arrive at vertex $d_{\gamma'}$. At this point we can potentially take one of three directions, depending on whether $R_{\gamma'}$ is in N or not. If $R_{\gamma'} \in N$, then there exists an $R_\alpha R_{\gamma'}$-bracket for some odd $\alpha \in [p]$, where α is the greatest integer smaller than γ' (taken modulo p) such that $R_\alpha \in N$. As above, this bracket consists of a path with edges alternating in N and M, starting from $d_{\gamma'}$ and ending at d_α, such that the other vertices of this path belong to:

$$\begin{array}{cl} \{e_{\gamma'}, f_{\gamma'-1}, e_{\gamma'-1}, \ldots, f_{\alpha+1}, e_{\alpha+1}, e_\alpha\} & \text{if } \alpha \neq \gamma' - 1, \\ \{e_{\gamma'}, e_\alpha\} & \text{if } \alpha = \gamma' - 1. \end{array}$$

Otherwise, if $R_{\gamma'} \notin N$, we either have $d_{\gamma'-1} d_{\gamma'} \in N$ or $d_{\gamma'} d_{\gamma'+1} \in N$. Continuing this process, the Hamiltonian circuit H must eventually reach the vertex c_γ. Thus, H contains only vertices in the set $\{a_i, b_i, c_i, d_i, e_i, f_i : i \in [p]\}$, giving a contradiction if $q \geq 7$. Henceforth, we can assume that $5 \leq q \leq 6$. Notwithstanding whether or not $R_{\gamma'}$ is in N, if $q = 6$, then there is no instance in the above procedure which leads to H passing through the vertices a_γ and $a_{\gamma'}$, a contradiction. Hence, we can further assume that $q = 5$.

We now note that for the vertices in the set $\{a_i, b_i, e_i : i \in [p]\}$ to be in H, they must belong either to a left-bracket or to a right-bracket. Thus, if $R_i \in N$ is a lower edge of a right-bracket, for some even $i \in [p]$, then, R_{i+1} must be an upper edge of another right-bracket (that is, $R_{i+1} \in N$), otherwise, the vertex e_{i+1} is not contained in any bracket. This observation, together with the fact that a maximal sequence of consecutive edges belonging to $\mathcal{R} - N$ is of even length, implies that if $R_i \notin N$, for some even $i \in [p]$, then $d_i d_{i+1} \in N$.

We revert back to the last remaining case, that is, when $q = 5$. The only way how the Hamiltonian circuit H can contain the vertices a_γ and $a_{\gamma'}$ is when both R_γ and $R_{\gamma'}$ do not belong to N, in which case

a_γ and $a_{\gamma'}$ can be reached by some right-bracket (or right-brackets). Therefore, suppose that R_γ and $R_{\gamma'}$ do not belong to N.

Consequently, tracing H starting from c_γ and going in the direction of b_γ, after traversing the $L_\gamma L_{\gamma'}$-bracket, H must then contain the path $c_{\gamma'}, d_{\gamma'}, d_{\gamma'+1}, c_{\gamma'+1}$. First assume that $\gamma' + 1 \neq \gamma$. By the same reasoning used for the edges in $\mathcal{R} \cap N$, the lower edge $L_{\gamma'}$ must be followed by an upper edge, and thus $L_{\gamma'+1} \in N$. We trace the Hamiltonian circuit through an $L_{\gamma'+1} L_{\gamma''}$-bracket, noting in particular that for $a_{\gamma''}$ to be in H, $R_{\gamma''}$ does not belong to N, and hence $d_{\gamma''} d_{\gamma''+1} \in N$, since γ'' is even. Continuing this procedure, H must eventually reach again the vertex c_γ, without having traversed any right-bracket. The same conclusion can be obtained if $\gamma' + 1 = \gamma$. In either case, the vertices a_γ and $a_{\gamma'}$, together with several other vertices of $C_p \square C_q$, are untouched by H, a contradiction. As a result M cannot be extended to a Hamiltonian circuit, proving our theorem. $\square$

8

ON A FAMILY OF QUARTIC GRAPHS: ACCORDIONS

This chapter is based on a joint work with John Baptist Gauci [VI].

8.1 INTRODUCTION

As already stated in Chapter 7, a complete characterisation of the cubic graphs having the PH-property was given in [2], and thus the most obvious next step would be to characterise 4-regular graphs which have the PH-property. This endeavour proved to be more elusive, and thus far a complete characterisation of such quartic graphs remains unknown. In an attempt to advance in this direction, in Section 8.2, we define a class of graphs on two parameters, n and k, which we call the class of *accordion graphs* $A[n,k]$. This class presents a natural generalisation of the well-known antiprism graphs, and of a class of graphs which is known to have the PH-property, corresponding to $A[n,1]$ and $A[n,2]$, respectively. In this section we discuss some fundamental properties and characteristics of accordion graphs and, in particular, we see that accordion graphs can be drawn in a grid-like manner, which resembles a drawing of the Cartesian product of two circuits $C_{n_1}\square C_{n_2}$, for appropriate circuit lengths n_1 and n_2. In 2015, Bogdanowicz [11] gave all possible values of n_1 and n_2 for which the graph $C_{n_1}\square C_{n_2}$ is circulant, namely when $\gcd(n_1,n_2)=1$. Due to the similarity between the two classes of graphs, in Section 8.5, we give a complete characterisation of which accordion graphs are circulant graphs. In Section 8.3 we prove that all antiprism graphs have the PMH-property but only four of them also have the PH-property. In the same section, we provide a proof that $A[n,2]$ has the PH-property, which result, although known to be communicated to the authors of [2], has no published proof. These encouraging outcomes motivate our proposal of the class of accordion graphs as a possible candidate for graphs having the PMH-property and/or the PH-property. Empirical evidence suggests that, apart from the above mentioned, there are (possibly an infinite number of) other accordion graphs which have the PMH-property, and possibly some of them even have the PH-property, but a proof for this is currently unavailable. In Section 8.4, by extending an argument introduced in [V], we show that we can exclude some graphs $A[n,k]$ from this search for graphs having the PMH- and/or the PH-property. In fact, we prove that the graphs $A[n,k]$ for which the greatest common divisor of n and k is at least 5 do not have the PMH-property. The technique used does not seem to lend itself when coming to show whether the remaining accordion graphs have, or do not have, the PH-property or the PMH-property. We thus pose some related questions and open problems in Section 8.6.

8.2 ACCORDION GRAPHS

Definition 8.2.1. Let n and k be integers such that $n \geq 3$ and $0 < k \leq n/2$. The *accordion graph* $A[n,k]$ is the quartic graph with vertices $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ such that the edge set consists of the edges

$$\{u_i u_{i+1}, v_i v_{i+1}, u_i v_i, u_i v_{i+k} : i \in [n]\}.$$

The edges $u_i u_{i+1}$ and $v_i v_{i+1}$ are called the *outer-circuit edges* and the *inner-circuit edges*, respectively, or simply the *circuit edges*, collectively; and the edges $u_i v_i$ and $u_i v_{i+k}$ are called the *vertical spokes* and the *diagonal spokes*, respectively, or simply the *spokes*, collectively. For simplicity, we sometimes refer to the accordion graph $A[n,k]$ as the accordion $A[n,k]$.

We remark that, henceforth, operations (including addition and subtraction) in the indices of the vertices u_i and v_i are taken modulo n, with complete residue system $\{1, \ldots, n\}$.

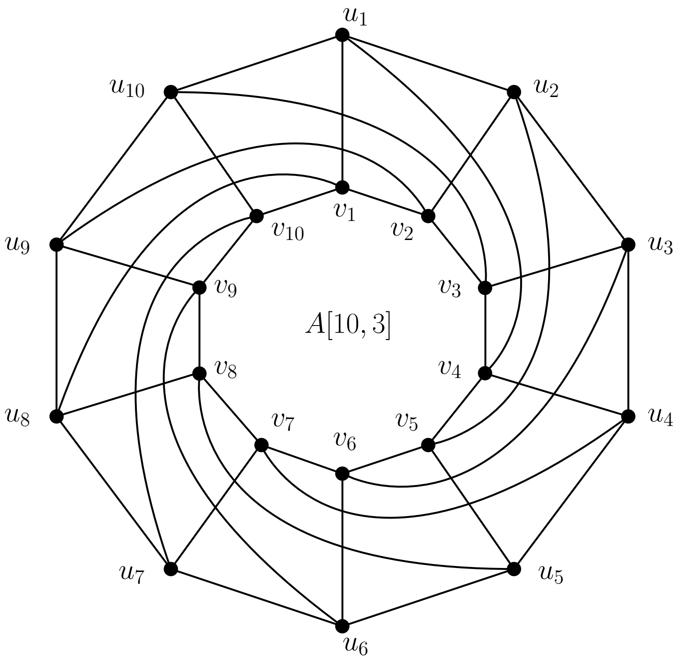

Figure 8.1: The accordion graph $A[10,3]$

An observation that will prove to be useful in the sequel revolves around the greatest common divisor of n and k, denoted by $\gcd(n,k)$.

Remark 8.2.2. The graph obtained from $A[n,k]$ after deleting the edges

$$\{u_{tq} u_{tq+1}, v_{tq} v_{tq+1} : q = \gcd(n,k) \text{ and } t \in \{1, \ldots, \tfrac{n}{q}\}\}$$

is isomorphic to the Cartesian product $C_{\frac{2n}{q}} \square P_q$. This can be easily deduced by an appropriate drawing of $A[n,k]$, as shown in Figure 8.2 for the case when $k = 5$ and $\gcd(n,k) = 5$. Thus, any perfect matching of $C_{\frac{2n}{q}} \square P_q$ is also a perfect matching of $A[n,k]$, although the converse is trivially not true.

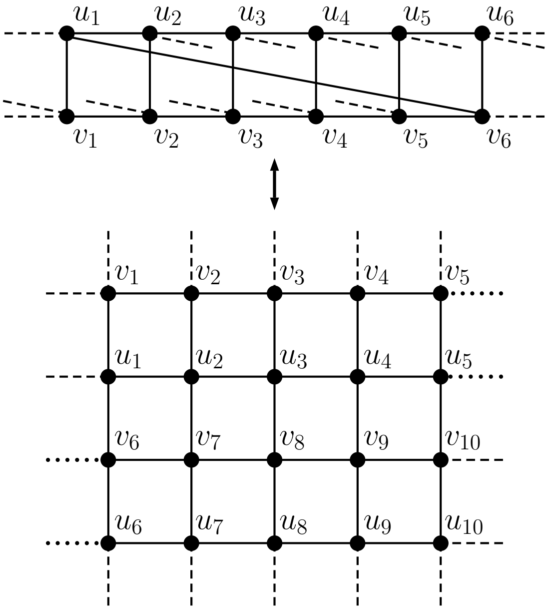

Figure 8.2: Two different drawings of $A[n,k]$ when $k = 5$ and $\gcd(n,k) = 5$

8.3 THE ACCORDION GRAPH $A[n,k]$ WHEN $k \leq 2$

8.3.1 $A[n,1]$

As already mentioned above, the accordion graph $A[n,1]$ is isomorphic to the widely known antiprism graph A_n on $2n$ vertices. Let M be a perfect matching of A_n. We note that, if M contains at least one vertical spoke, then no diagonal spoke can be contained in M, and if any inner-circuit edges are in M, then the outer-circuit edges having the same indices must also belong to M. A similar argument can be made if M contains diagonal spokes. Thus, for every $i, j \in [n]$, $u_iv_i \in M$ or $\{u_iu_{i+1}, v_iv_{i+1}\} \subset M$ if and only if $u_jv_{j+1} \notin M$ or $\{u_ju_{j+1}, v_{j+1}v_{j+2}\} \not\subset M$. This can be summarised in the following remark.

Remark 8.3.1. Let M be a perfect matching of A_n. Then, M is either a perfect matching of $A_n - \{u_iv_i : i \in [n]\}$ or of $A_n - \{u_iv_{i+1} : i \in [n]\}$.

Consequently, in what follows, without loss of generality, we only consider perfect matchings of A_n containing spokes of the type u_iv_i, that is, vertical spokes, and if a perfect matching contains the edge u_ju_{j+1}, then it must also contain the edge v_jv_{j+1}.

Theorem 8.3.2. *The antiprism A_n is PMH.*

Proof. Let M be a perfect matching of A_n. Consider first the case when $M = \{u_i v_i : i \in [n]\}$. It is easy to see that $(v_1, u_1, v_2, u_2, \ldots, v_{n-1}, u_{n-1}, v_n, u_n)$ is a Hamiltonian circuit of A_n containing M. So assume that M does not consist of only vertical spokes. Without loss of generality, we can assume that M contains the edges $u_n u_{n+1}$ and $v_n v_{n+1}$, by Remark 8.3.1. We proceed by induction on n. The antiprism A_3 was already shown to be PMH in [III], since A_3 is the line graph of the complete graph K_4. So assume result is true up to $n \geq 3$, and consider A_{n+1} and a perfect matching M of A_{n+1}. Let M' be $M \cup \{u_n v_n\} - \{u_n u_{n+1}, v_n v_{n+1}\}$. Then, M' is a perfect matching of A_n, and so, by induction, there exists a Hamiltonian circuit H' of A_n which contains M'. We next show that H' can be extended to a Hamiltonian circuit H of A_{n+1} containing M by considering each of the following possible induced paths in H' and replacing them as indicated hereunder:

(i) $u_{n-1}, u_n, v_n, v_{n-1}$ is replaced by $u_{n-1}, u_n, u_{n+1}, v_{n+1}, v_n, v_{n-1}$ (similarly u_1, u_n, v_n, v_1 is replaced by $u_1, u_{n+1}, u_n, v_n, v_{n+1}, v_1$);

(ii) u_{n-1}, v_n, u_n, u_1 is replaced by $u_{n-1}, v_n, v_{n+1}, u_n, u_{n+1}, u_1$ (similarly v_{n-1}, v_n, u_n, v_1 is replaced by $v_{n-1}, v_n, v_{n+1}, u_n, u_{n+1}, v_1$); and

(iii) u_{n-1}, u_n, v_n, v_1 or u_{n-1}, v_n, u_n, v_1 are replaced by $u_{n-1}, v_n, v_{n+1}, u_n, u_{n+1}, v_1$ (similarly v_{n-1}, v_n, u_n, u_1 is replaced by $v_{n-1}, v_n, v_{n+1}, u_n, u_{n+1}, u_1$).

Consequently, A_{n+1} is PMH, proving our theorem. □

Theorem 8.3.3. *The only antiprisms having the PH-property are* A_3, A_4, A_5 *and* A_6.

Proof. The graph A_3 is PMH as already explained in Theorem 8.3.2, so what is left to show is that every pairing M of A_3 containing some edge belonging to $E(K_{A_3}) - E(A_3)$ (referred to as a non-edge) can be extended to a Hamiltonian circuit of K_{A_3}. The pairing M can only contain one or three non-edges. When M consists of three non-edges, then $M = \{u_1 v_3, u_2 v_1, u_3 v_2\}$, and this can be extended to a Hamiltonian circuit of K_{A_3} as follows $(u_1, v_3, u_3, v_2, u_2, v_1)$. Otherwise, assume that the only non-edge in M is $u_1 v_3$, without loss of generality. Then, M is either equal to $\{u_1 v_3, u_2 v_2, u_3 v_1\}$ or $\{u_1 v_3, u_2 u_3, v_1 v_2\}$, which can be extended to $(u_1, v_3, u_3, v_1, v_2, u_2)$ or $(u_1, v_3, u_3, u_2, v_2, v_1)$, respectively.

By Remark 1.4.1, the graph A_4 has the PH-property because it contains the cube $\mathcal{Q}_3$ as a spanning subgraph, and by the main theorem in [29], all hypercubes have the PH-property .

An exhaustive computer check was conducted through Wolfram Mathematica [103] to verify that all pairings of the antiprism graphs A_5 and A_6 can be extended to a Hamiltonian circuit of the same graphs, thus proving (by brute-force) that they both have the PH-property (see Appendix A).

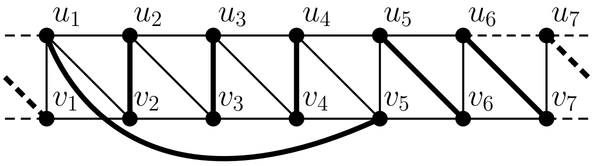

Figure 8.3: A pairing in A_n, $n \geq 7$, which is not extendable to a Hamiltonian circuit of K_{A_n}

The antiprism A_7 does not have the PH-property because the pairing $M = \{u_1v_5, u_2v_2, u_3v_3, u_4v_4, u_5v_6, u_6v_7, u_7v_1\}$, depicted in Figure 8.3, cannot be extended to a Hamiltonian circuit of K_{A_7}. For, suppose not, and let N be a perfect matching of A_7 such that $M \cup N$ gives a Hamiltonian circuit of K_{A_7}. Then, $|N \cap \{u_2u_3, u_2v_3, v_2v_3\}|$, and $|N \cap \{u_3u_4, u_3v_4, v_3v_4\}|$ must both be equal to 1. Consequently, $M \cup N$ induces a M-alternating path containing the edges $\{u_2v_2, u_3v_3, u_4v_4\}$ such that its end-vertices are $x \in \{u_2, v_2\}$ and $y \in \{u_4, v_4\}$. If $xu_1 \in N$, then $N \cap \{u_7u_1, u_1v_1, v_1v_2\}$ is empty, implying that $M \cup N$ induces the 4-circuit (u_6, u_7, v_1, v_7), a contradiction. Therefore, N contains the edge v_2v_1 implying that $x = v_2$. Consequently, N contains the edge u_7u_1 as well. However, this implies that $N \cap \{u_6u_7, u_7v_7, v_7v_1\}$ is empty, implying that $M \cup N$ induces the 4-circuit (u_5, u_6, v_7, v_6), a contradiction once again. Thus, A_7 does not have the PH-property.

The pairing M considered above can be easily extended to a pairing of A_n, for any $n > 7$, and by the same argument for A_7, we conclude that A_n does not have the PH-property for every $n \geq 7$. □

8.3.2 $A[n,2]$

In [2], it is mentioned that Ok and Perrett informed the authors that they have obtained an infinite class of 4-regular graphs having the PH-property: such a graph in this family is obtained from a circuit of length at least three, by replacing each vertex by two isolated vertices and replacing each edge by the four edges joining the corresponding pairs of vertices. More formally, the resulting graph starting from a n-circuit, for $n \geq 3$, has vertex set $\{s_i, t_i : i \in [n]\}$ such that, for every $i \in [n]$, s_i and t_i are both adjacent to $s_{i+1 \pmod n}$ and $t_{i+1 \pmod n}$. As far as we know, no proof of this can be found in literature, and in what follows we give a proof of this result. Before we proceed, we remark that the function that maps s_i to u_i, and t_i to $v_{i+1 \pmod n}$, for every $i \in [n]$, is an isomorphism between the above graph and the accordion graph $A[n,2]$.

Theorem 8.3.4. *The accordion graph $A[n,2]$ has the PH-property, for every $n \geq 3$.*

Proof. Let $A'[n,2]$ be the graph depicted in Figure 8.4, obtained from $A[n,2]$ after deleting the following set of edges: $\{u_1u_n, v_1v_n, u_{n-1}v_1,$

$u_n v_2\}$. We use induction on n to show that $A'[n,2]$ has the PH-property, for every $n \geq 3$. The result then follows by Remark 1.4.1, since $A'[n,2]$ is a spanning subgraph of $A[n,2]$.

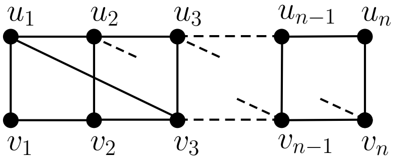

Figure 8.4: The graph $A'[n,2]$

When $n = 3$, one can show by a case-by-case analysis (or using an exhaustive computer search) that the graph $A'[3,2]$, shown in Figure 8.5, has the PH-property.

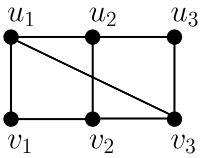

Figure 8.5: The graph $A'[3,2]$

So we assume that $n > 3$ and let M be a pairing of $A'[n,2]$, hereafter denoted by G. If M consists of only vertical spokes, that is, $M = \{u_i v_i : i \in [n]\}$, then

(i) $(v_1, u_1, v_3, u_3, \ldots, v_{n-1}, u_{n-1}, u_n, v_n, u_{n-2}, v_{n-2}, \ldots, u_2, v_2)$, when n is even, or

(ii) $(v_1, u_1, v_3, u_3, \ldots, v_n, u_n, u_{n-1}, v_{n-1}, u_{n-3} \ldots, u_2, v_2)$, when n is odd,

is a Hamiltonian circuit of K_G containing the pairing M. So assume that $M \neq \{u_i v_i : i \in [n]\}$. Consequently, there exists $i \in [n]$ such that $u_i v_i \notin M$. Let $\alpha = \max\{i \in [n] : u_i v_i \notin M\}$. We note that by a parity argument, $\alpha > 1$. Consider the two subgraphs of G induced by $\{u_1, v_1, \ldots, u_{\alpha-1}, v_{\alpha-1}\}$, denoted by G_1, and $\{u_\alpha, v_\alpha, \ldots, u_n, v_n\}$, denoted by G_2. We remark that G_1 and G_2 are the two components obtained after deleting from G the set of edges X, where $X = \{u_{\alpha-1}u_\alpha, v_{\alpha-1}v_\alpha, u_{\alpha-2}v_\alpha\}$ if $\alpha = n$, and $X = \{u_{\alpha-1}u_\alpha, v_{\alpha-1}v_\alpha, u_{\alpha-2}v_\alpha, u_{\alpha-1}v_{\alpha+1}\}$ otherwise. We also remark that depending on the value of α, we have that G_1 is isomorphic to either K_2, C_4 or $A'[\alpha-1, 2]$, and that G_2 is isomorphic to K_2, C_4 or $A'[n-(\alpha-1), 2]$. Without loss of generality, we assume that $|V(G_1)| \geq |V(G_2)|$, implying that $3 \leq \alpha \leq n$.

Next, consider the two edges yu_α and zv_α in M. Since for $i > \alpha$, $u_i v_i \in M$, then y and z both belong to $\{u_1, v_1, \ldots, u_{\alpha-1}, v_{\alpha-1}\}$. Let $M_1 = (M \cap E(K_{G_1})) \cup \{yz\}$. One can see that M_1 is a pairing of G_1, and so, by induction, M_1 is contained in a Hamiltonian circuit H_1 of K_{G_1}. Consequently, H_1 contains a Hamiltonian path of K_{G_1} with end-vertices y and z. We denote this path by H_1'.

When $\alpha = n$, we obtain a Hamiltonian circuit of K_G containing M, by adding the edges $yu_\alpha, u_\alpha v_\alpha, v_\alpha z$ to $E(H_1')$. For $\alpha \leq n-1$, we proceed as follows. Let $M_2 = (M \cap E(G_2)) \cup \{u_\alpha v_\alpha\}$. This is clearly a pairing of G_2, and so, by induction, there exists a Hamiltonian circuit H_2 of K_{G_2} containing M_2. Let H_2' be the Hamiltonian path of G_2 obtained by deleting the edge $u_\alpha v_\alpha$ from $E(H_2)$. Consequently, combining H_1' and H_2' together with the edges yu_α and zv_α, we form a Hamiltonian circuit of K_G containing M, as required. □

8.4 THE ACCORDION GRAPH $A[n,k]$ WHEN $\gcd(n,k) \geq 5$

The method adopted in this section follows a similar line of thought as that used in [V]. Let $q = \gcd(n,k) \geq 5$, let $p = \frac{2n}{\gcd(n,k)}$ and let $p' = \frac{p}{2}$. Consider a grid-like drawing of the accordion graph $A[n,k]$ as in Remark 8.2.2. For simplicity, we let the vertices $v_1, u_1, v_{1+k}, u_{1+k}, \ldots,$ $v_{1+(p'-1)k}, u_{1+(p'-1)k}$, be referred to as $a_1, a_2, \ldots, a_p$. We define the vertices $\{b_i, c_i, d_i, e_i : i \in [p]\}$ in a similar way, where, in particular, the vertices $b_1, \ldots, e_1$, and $b_2, \ldots, e_2$, represent $v_2, \ldots, v_5$, and $u_2, \ldots, u_5$, respectively. If $\gcd(n,k) = 6$, we refer to $v_6, u_6, v_{6+k}, u_{6+k}, \ldots, v_{6+(p'-1)k}$, $u_{6+(p'-1)k}$, as $f_1, f_2 \ldots, f_p$, and if $\gcd(n,k) > 6$, we simply do not label all the other vertices since we are only interested in the subgraph of $A[n,k]$ generated by the vertices $\{a_i, \ldots, f_i : i \in [p]\}$. This can be seen better in Figure 8.6.

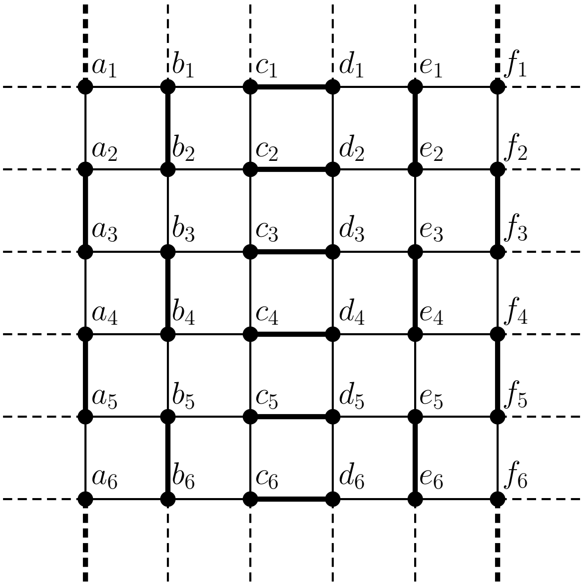

Figure 8.6: Edges belonging to S in $A[n,k]$ when $\gcd(n,k) \geq 6$

For each $i \in [p]$, let L_i and R_i represent the edges b_ic_i and d_ie_i, respectively, whilst $\mathcal{L} = \{L_i : i \in [p]\}$ and $\mathcal{R} = \{R_i : i \in [p]\}$. Let S denote the following set of edges:

(i) a_ia_{i+1}, for every even $i \in [p]$,

(ii) $b_i b_{i+1}$ and $e_i e_{i+1}$, for every odd $i \in [p]$,

(iii) $c_i d_i$, for every $i \in [p]$, and

(iv) in the case when $q \geq 6$, $f_i f_{i+1}$, for every even $i \in [p]$.

Since p is even, $A[n,k]$ has a perfect matching M that contains S.

In [V], it was shown that $C_p \square C_q$ is not PMH except when $p = q = 4$. In the case when $q \geq 6$, the proof utilises exclusively the set S of edges described above (and adapted to $C_p \square C_q$) to show that a perfect matching M containing this set cannot be extended to a Hamiltonian circuit of $C_p \square C_q$. Since the same set S of edges can also be chosen in a perfect matching of $A[n,k]$, the proof extends naturally and hence we have the following result.

Lemma 8.4.1. *$A[n,k]$ is not PMH if $\gcd(n,k) \geq 6$.*

We shall now show that $A[n,k]$ is not PMH in the case when $\gcd(n,k) = 5$. We remark that, in this case, the proof in [V] cannot be extended to $A[n,k]$ because it makes use of the edges $e_i a_i$ of $C_p \square C_q$, which are missing in $A[n,k]$. For the remaining part of this section, we shall need some results extracted from the proof of the main theorem in [V], and adapted for accordion graphs. We note that the arguments in [V] are quite elaborate and lengthy, but when adapted to our case, they remain essentially the same. Hence our decision not to reproduce them in detail here but to only give the main points in the following lemma.

Lemma 8.4.2. *[V] Let $\gcd(n,k) \geq 5$. If there exists a perfect matching M of $A[n,k]$ containing S and another perfect matching N of $A[n,k]$ such that $M \cup N$ is a Hamiltonian circuit H of $A[n,k]$, then the following statements hold.*

(i) $|\mathcal{L} \cap N|$ and $|\mathcal{R} \cap N|$ are both even and non-zero.

(ii) A maximal sequence of consecutive edges belonging to $\mathcal{L} - N$ (or $\mathcal{R} - N$) is of even length (consecutive edges are edges having indices which are consecutive integers taken modulo p, with complete residue system $\{1, \ldots, p\}$).

*(iii) The edges of $\mathcal{L} \cap N$ are partitioned into pairs of edges $\{L_\gamma, L_{\gamma'}\}$, where γ is odd and γ' is the least integer greater than γ (taken modulo p) such that $L_{\gamma'} \in N$ (and similarly for $\mathcal{R} \cap N$). In this case, if we start tracing the Hamiltonian circuit H from c_γ going towards b_γ, then H contains a path with edges alternating in N and M, starting from c_γ and ending at $c_{\gamma'}$, with the internal vertices on this path being $\{b_\gamma, b_{\gamma'}\}$, if $\gamma' = \gamma + 1$, or belonging to the set $\{b_\gamma, a_{\gamma+1}, b_{\gamma+1}, \ldots, a_{\gamma'-1}, b_{\gamma'-1}, b_{\gamma'}\}$, if $\gamma' \neq \gamma + 1$. In each of these two cases we refer to such a path between c_γ and $c_{\gamma'}$ as an $L_\gamma L_{\gamma'}$-*bracket*, or just a* left-bracket*, with L_γ and $L_{\gamma'}$ being the* upper *and* lower *edges of the bracket, respectively.* Right-brackets *are defined similarly.*

(iv) *If $L_{\gamma'}$ is a lower edge of a left bracket, then $L_{\gamma'+1}$ belongs to N, and so is an upper edge of a possibly different left bracket (and similarly for right-brackets).*

(v) *If $L_i \notin N$, for some even $i \in [p]$, then $c_i c_{i+1} \in N$ (and similarly for edges belonging to $\mathcal{R} - N$).*

Lemma 8.4.3. *$A[n,k]$ is not PMH if* $\gcd(n,k) = 5$.

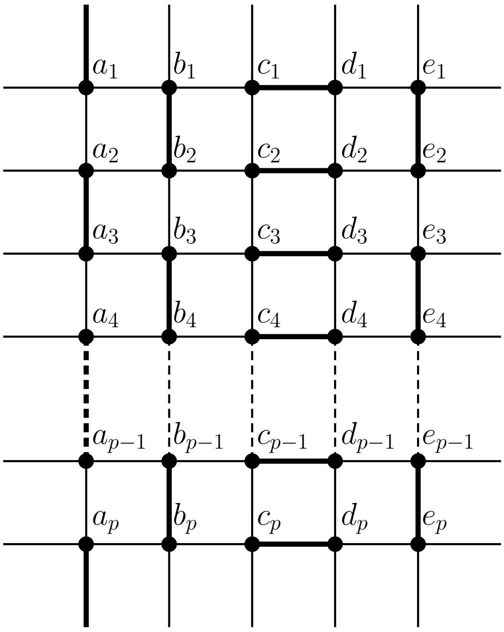

Figure 8.7: $A[n,k]$ when $\gcd(n,k) = 5$

Proof. Let M be a perfect matching of $A[n,k]$ which contains the set S of edges as shown in Figure 8.7, and let $p = 2n/5$ (as defined earlier on in this section). Suppose that there exists a perfect matching N of $A[n,k]$ such that $M \cup N$ is a Hamiltonian circuit H of $A[n,k]$.

Since $\mathcal{R} \cap N \neq \emptyset$, there exists some odd $\beta \in [p]$ such that $R_\beta \in N$. We note that the vertex e_β is adjacent to a unique a_θ, for some odd $\theta \neq \beta$. The only way how a_θ can be reached by the Hamiltonian circuit H is if it is reached by a left bracket, and so only if $a_\theta b_\theta \in N$. Consequently, $L_\theta = b_\theta c_\theta$ is not in N, implying that $c_\theta c_{\theta-1} \in N$ by Lemma 8.4.2. By a similar reasoning, since H is a Hamiltonian circuit, $d_\theta d_{\theta-1}$ cannot be contained in N, and so, in particular, R_θ and $R_{\theta-1}$ are respectively upper and lower edges belonging to N. Repeating the same procedure over again, first for R_θ and eventually for all edges in $\mathcal{R}$ having an odd index, one can deduce that $\mathcal{R} \cap N = \mathcal{R}$. Since $\mathcal{L} \cap N$ is non-empty, there exists some ν such that $L_\nu \in N$. The vertex a_ν must be reached by some right bracket in order to belong to H, however, this is impossible since $\mathcal{R} \cap N = \mathcal{R}$, contradicting the Hamiltonicity of H. □

By combining Lemma 8.4.1 and Lemma 8.4.3 we obtain the main result of this section.

Theorem 8.4.4. *The accordion graph* $A[n,k]$ *is not PMH if* $\gcd(n,k) \geq 5$.

8.5 ACCORDIONS AND CIRCULANT GRAPHS

For distinct integers a and b, $\mathrm{Ci}[2n, \{a,b\}]$ denotes the *quartic circulant graph* on the vertices $\{x_i : i \in [2n]\}$, such that x_i is adjacent to the vertices in the set $\{x_{i+a}, x_{i-a}, x_{i+b}, x_{i-b}\}$. We say that the edges arising from these adjacencies have length a and b, accordingly, and we also remark that operations in the indices of the vertices x_i are taken modulo $2n$, with complete residue system $\{1, \ldots, 2n\}$. In [11], the necessary and sufficient conditions for a circulant graph to be isomorphic to a Cartesian product of two circuits were given. Motivated by this, we show that there is a non-empty intersection between the class of accordions $A[n,k]$ and the class of circulant graphs $\mathrm{Ci}[2n, \{a,b\}]$, although neither one is contained in the other. In particular, we here show that the only accordion graphs $A[n,k]$ which are not circulant are those with both n and k even, such that $k \geq 4$. Results of a similar flavour about 4-regular circulant graphs, perfect matchings and Hamiltonicity can be found in [43].

We shall be using the following two results about circulant graphs (not necessarily quartic). Let the circulant graph on n' vertices and with r different edge lengths be denoted by $\mathrm{Ci}[n', \{a_1, \ldots, a_r\}]$. The first result, implied by a classical result in number theory, says that $\mathrm{Ci}[n', \{a_1, \ldots, a_r\}]$ has $\gcd(n', a_1, \ldots, a_r)$ isomorphic connected components (see [10]). Consequently, in our case we have that the circulant graph $\mathrm{Ci}[2n, \{a,b\}]$ is connected if and only if $\gcd(2n, a, b) = 1$. Secondly, let $\gcd(n', a_1, \ldots, a_r) = 1$. Heuberger [44] showed that $\mathrm{Ci}[n', \{a_1, \ldots, a_r\}]$ is bipartite if and only if $a_1, \ldots, a_r$ are odd and n' is even. Restated for our purposes we have that $\mathrm{Ci}[2n, \{a,b\}]$ is bipartite if and only if a and b are both odd.

Remark 8.5.1. When n and k are even, the sets $\{u_1, v_2, u_3, v_4, \ldots, u_{n-1}, v_n\}$ and $\{v_1, u_2, v_3, u_4, \ldots, v_{n-1}, u_n\}$ are two independent sets of vertices of $A[n,k]$, implying that for these values of n and k, $A[n,k]$ is bipartite. On the other hand, it is easy to see that when at least one of n and k is odd, $A[n,k]$ is not bipartite. Consequently, $A[n,k]$ is bipartite if and only if n and k are both even.

Before proceeding, we recall that operations in the indices of the vertices of $A[n,k]$ and $\mathrm{Ci}[2n, \{a,b\}]$ are taken modulo n and modulo $2n$, respectively.

Lemma 8.5.2. *The accordion graph* $A[n,2]$ *is circulant for any even integer* $n \geq 4$.

Proof. For every even integer $n \geq 4$, we claim that the following function $\phi : V(A[n,2]) \to V(\text{Ci}[2n, \{1, n-1\}])$ defined by:

- $\phi : u_i \mapsto x_{1+(n-1)(i-1) \pmod{2n}}$, for all $i \in [n]$,
- $\phi : v_1 \mapsto x_{2+n \pmod{2n}}$, and
- $\phi : v_i \mapsto x_{2+(n-1)(i-1) \pmod{2n}}$, for $2 \leq i \leq n$,

is an isomorphism. We first show that the function ϕ is bijective. Since $n-1 \geq 3$ is odd and $\gcd(n, n-1) = 1$, we have that $\gcd(2n, n-1) = 1$, and so the vertices in $\{\phi(u_i) : 1 \leq i \leq n\}$ are mutually distinct. For the same reason, since $2+n \equiv 1+(n-1)(2n-1) \pmod{2n}$, the vertex $\phi(v_1)$ is distinct from any vertex $\phi(u_i)$. Moreover, since $1 \neq n-1$ and the vertices $\phi(u_2), \ldots, \phi(u_n)$ are mutually distinct, the vertices in $\{\phi(v_j) : 2 \leq j \leq n\}$ are mutually distinct as well, and are not equal to some vertex $\phi(u_i)$. Finally, since $\gcd(n, n-1) = 1$, we have that $2+(n-1)(i-1) \not\equiv 2+n \pmod{2n}$, for any $2 \leq i \leq n$. Consequently, $\phi(v_1)$ is distinct from any vertex $\phi(v_j)$, proving that ϕ is, in fact, bijective. We next show that ϕ is an isomorphism. Since $\text{Ci}[2n, \{1, n-1\}]$ has the same number of edges as $A[n,2]$, it suffices to show that an edge in $A[n,2]$ is mapped to an edge in $\text{Ci}[2n, \{1, n-1\}]$.

We first take an edge u_iu_j from the outer-circuit of $A[n,2]$, for some $i \in [n]$ and $j \equiv i+1 \pmod{n}$, without loss of generality. Consider $\phi(u_i)\phi(u_j)$. The length of $\phi(u_i)\phi(u_j)$ can be calculated using $1+(n-1)(j-1)-(1+(n-1)(i-1)) \pmod{2n}$. This is equal to $n-1$, when $j \neq 1$, and to $-n^2+2n-1 \equiv -1 \pmod{2n}$, otherwise. Since in both cases the length of the edge $\phi(u_i)\phi(u_j)$ belongs to $\{\pm 1, \pm(n-1)\}$, $\phi(u_i)\phi(u_j) \in E(\text{Ci}[2n, \{1, n-1\}])$.

Next, we consider the vertical spoke u_iv_i, for some $i \in [n]$. The length of the edge $\phi(u_i)\phi(v_i)$ is $2+n-(1+(n-1)(i-1)) \equiv -(n-1) \pmod{2n}$, when $i = 1$, and $2+(n-1)(i-1)-(1+(n-1)(i-1)) = 1$, otherwise, implying that the length of the edge $\phi(u_i)\phi(v_i)$ belongs to $\{\pm 1, \pm(n-1)\}$ in both cases. Consequently, $\phi(u_i)\phi(v_i) \in E(\text{Ci}[2n, \{1, n-1\}])$.

We now take the diagonal spoke u_iv_j, for some $i \in [n]$, and for $j \equiv i+2 \pmod{n}$. The length of the edge $\phi(u_i)\phi(v_j)$ is:

- $2+(n-1)(j-1)-(1+(n-1)(i-1)) = 2n-1 \equiv -1 \pmod{2n}$, when $1 \leq i \leq n-2$,
- $2+n-(1+(n-1)(i-1)) = -n^2+4n-1 \equiv -1 \pmod{2n}$, when $i = n-1$, and
- $2+(n-1)(j-1)-(1+(n-1)(i-1)) = -n^2+3n-1 \equiv n-1 \pmod{2n}$, when $i = n$.

In each of these three cases, the length of $\phi(u_i)\phi(v_j)$ belongs to $\{\pm 1, \pm(n-1)\}$, implying that $\phi(u_i)\phi(v_j) \in E(\text{Ci}[2n, \{1, n-1\}])$.

Finally, we take an edge v_iv_j from the inner-circuit of $A[n, 2]$, for some $i \in [n]$ and $j \equiv i + 1 \pmod{n}$, without loss of generality. The length of the edge $\phi(v_i)\phi(v_j)$ is:

- $2 + n - (2 + (n-1)(i-1)) = -n^2 + 3n - 1 \equiv n - 1 \pmod{2n}$, when $j = 1$,
- $2 + (n-1)(j-1) - (2 + n) = -1$, when $j = 2$, and
- $2 + (n-1)(j-1) - (2 + (n-1)(i-1)) = n - 1$, when $3 \leq j \leq n$.

This implies that the length of the edge $\phi(v_i)\phi(v_j)$ belongs to $\{\pm 1, \pm(n-1)\}$ in both cases, and so $\phi(v_i)\phi(v_j) \in E(\text{Ci}[2n, \{1, n-1\}])$.

There are no more cases to consider, proving our result. □

We next show that the only circulant accordions with both n and k even are the ones having $k = 2$. By Remark 8.5.1, this means that the only circulant bipartite accordions are the ones with n even and $k = 2$.

Lemma 8.5.3. *For n and k even, the accordion graph $A[n, k]$ is circulant if and only if $k = 2$.*

Proof. Let n and k be even. By Lemma 8.5.2, it suffices to show that accordion graphs admitting $k \geq 4$ are not circulant. We can further assume that $n \geq 8$, since the only accordions with $n = 4$ or 6, and k even are $A[4, 2]$ and $A[6, 2]$. Suppose, for contradiction, that for $n \geq 8$, there exists an even integer $k \geq 4$ such that $A[n, k]$ is circulant. Then, by Remark 8.5.1 and Heuberger's result [44], $A[n, k] \simeq \text{Ci}[2n, \{a, b\}]$ for some distinct odd integers a and b. For simplicity, we refer to $A[n, k]$, or equivalently $\text{Ci}[2n, \{a, b\}]$, by G. By the definition of quartic circulant graphs we can assume that $1 \leq a < b \leq n - 1$.

Claim. $\gcd(2n, a) = \gcd(2n, b) = 1$.

Proof of Claim. Suppose that $\gcd(2n, a) \neq 1$, for contradiction. Then, the least common multiple of a and $2n$ is $2na'$, for some $a' < a$. Consequently, there exists an even integer p such that $ap = 2na'$. Moreover, since $a \neq a'$ and a is odd, $\frac{a}{a'}$ (or equivalently $\frac{2n}{p}$) is odd and at least 3, and so $p < n$. By considering the edges in G having length a, there exists a partition $\mathcal{P}$ of the $2n$ vertices of G into $\frac{2n}{p}$ sets, each inducing a p-circuit. This follows since $\gcd(2n, a) = \frac{2n}{p} \neq 1$. Furthermore, $\mathcal{P}$ has an odd number of components, namely $\gcd(2n, a)$, or equivalently $\frac{2n}{p}$.

Since G is a connected quartic graph and $\frac{2n}{p} > 1$, two vertices on a particular p-circuit in $\mathcal{P}$ are adjacent in G if and only if there is an edge of length a between them, in other words, the subgraph induced by the vertices on a p-circuit in $\mathcal{P}$ is the p-circuit itself. Therefore, the graph contains two adjacent vertices x_i and x_j belonging to two different p-circuits from $\mathcal{P}$. Consequently, since $i \equiv j \pm b \pmod{2n}$, the vertices of these two p-circuits induce $C_p \square P_2$, where P_2 is the path on two vertices. By a similar argument to that used on x_i and x_j, we deduce that G contains a spanning subgraph G_0 isomorphic to $C_p \square P_{\frac{2n}{p}}$.

We now denote the set $\{u_1, u_2, \ldots, u_n\}$ of vertices on the outer-circuit of $A[n,k]$ by $\mathcal{U}$, and the set $\{v_1, v_2, \ldots, v_n\}$ of vertices on the inner-circuit of $A[n,k]$ by $\mathcal{V}$, and claim that:

(i) given two adjacent vertices from some p-circuit in $\mathcal{P}$, say x_i and x_{i+a}, if x_i is a vertex in $\mathcal{U}$, then x_{i+a} is a vertex in $\mathcal{V}$, or vice-versa; and

(ii) given two adjacent vertices from two different p-circuits in $\mathcal{P}$, say x_i and x_{i+b}, we have that either both belong to $\mathcal{U}$ or both belong to $\mathcal{V}$.

First of all, we note that the vertices inducing a p-circuit from $\mathcal{P}$, cannot all belong to $\mathcal{U}$, since the latter set of vertices induces a n-circuit, and $p < n$. Similarly, the vertices inducing a p-circuit from $\mathcal{P}$, cannot all belong to $\mathcal{V}$. Secondly, let $i \in [2n]$, such that x_i is of degree 3 in G_0, and $x_i x_{i+b} \in E(G_0)$. Consider the 4-circuit $(x_i, x_{i+a}, x_{i+a+b}, x_{i+b})$. Since $n > 4$, these four vertices cannot all belong to $\mathcal{U}$ (or $\mathcal{V}$). Also, we cannot have three of them which belong to $\mathcal{U}$ (or $\mathcal{V}$), because otherwise we would have $k = 2$, or, $k \equiv -2 \pmod{n}$, that is, $k = n - 2$. Since we are assuming that $k \geq 4$, we must have that $k = n - 2$, but by Definition 8.2.1, k is at most $\frac{n}{2}$, and so, $n - 2 \leq \frac{n}{2}$, a contradiction, since $n \geq 8$. This means that exactly two vertices from $(x_i, x_{i+a}, x_{i+a+b}, x_{i+b})$ belong to $\mathcal{U}$, and the other two belong to $\mathcal{V}$. Without loss of generality, assume that x_i belongs to $\mathcal{U}$.

Suppose that $x_{i+b} \notin \mathcal{U}$, for contradiction. Then, $\mathcal{U}$ must contain exactly one of x_{i+a} and x_{i+a+b}. Suppose we have $x_{i+a+b} \in \mathcal{U}$. Consequently, x_{i+a} and x_{i+b} belong to $\mathcal{V}$, and so since $\frac{2n}{p} \geq 3$, x_{i+2a+b} and x_{i+a+2b} belong to $\mathcal{U}$, giving rise to a 4-circuit in G_0 with three of its vertices belonging to $\mathcal{U}$, a contradiction. Therefore, we have $x_{i+a} \in \mathcal{U}$. This means that x_{i+b} and x_{i+a+b} both belong to $\mathcal{V}$. Suppose further that $x_{i+2a} \in \mathcal{V}$. Since we cannot have three vertices in a 4-circuit belonging to $\mathcal{V}$, $x_{i+2a+b} \in \mathcal{U}$. However, this once again gives rise to a 4-circuit in G_0 with three of its vertices belonging to $\mathcal{U}$, a contradiction. Therefore, x_{i+2a} must belong to $\mathcal{U}$. By repeating the same argument we get that all the vertices in the p-circuit, from $\mathcal{P}$, containing x_i belong to $\mathcal{U}$, a contradiction. Hence, $x_{i+a} \notin \mathcal{U}$. Thus, neither one of x_{i+a} and x_{i+a+b} is in $\mathcal{U}$, contradicting our initial assumption. This implies that $x_{i+b} \in \mathcal{U}$, and that $x_{i\pm a}$ and $x_{i+b\pm a}$ belong to $\mathcal{V}$. This forces all the vertices not considered so far to satisfy the two conditions in the above claim.

Thus, by Remark 8.2.2, $\frac{2n}{p} = \gcd(n,k)$. This is a contradiction, since $\frac{2n}{p}$ is odd and the greatest common divisor of two even numbers is even. Hence, $\gcd(2n, a) = 1$, and by a similar reasoning, $\gcd(2n, b) = 1$

as well. ∎

This implies that a does not divide n, b does not divide n, and that the edges of G can be partitioned in two Hamiltonian circuits induced by the edges having length a and b, respectively.

In particular, since a does not divide $2n$, there exists an edge of length a with both end-vertices belonging to $\{u_i : i \in [n]\} \subset V(A[n,k])$. Without loss of generality, assume that u_1u_2 has length a, and consider the 4-circuit $C = (u_1, u_2, v_2, v_1)$. Since the edges having length a (and similarly the edges having length b) induce a Hamiltonian circuit, and $n > 4$, the lengths of the edges in C cannot all be the same. Hence, the lengths of the edges $(u_1u_2, u_2v_2, v_2v_1, v_1u_1)$ of C can be of Type A1 $:= (a,b,b,b)$, Type A2 $:= (a,a,b,a)$, Type A3 $:= (a,a,a,b)$, Type A4 $:= (a,b,a,a)$, Type B1 $:= (a,a,b,b)$, Type B2 $:= (a,b,b,a)$, or Type B3 $:= (a,b,a,b)$. Some of these types are depicted in Figure 8.8.

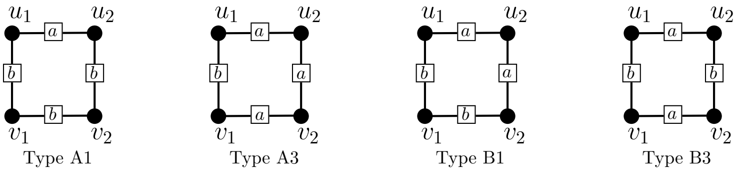

Figure 8.8: Different lengths of the edges in C

If C is of Type A1, then $a \equiv \pm 3b \pmod{2n}$. This implies that the two end-vertices of an edge of length a are also end-vertices of a 3-path whose edges are all of length b. Also, the two end-vertices of a 3-path whose edges are all of length b, must be adjacent. Consider the edge v_2v_3. Since v_1v_2 and u_2v_2 have length b, the edge v_2v_3 has length a, and so it must belong to some 4-circuit with the other three edges of the circuit having length b. We denote this 4-circuit by $C_4(v_2v_3)$. First, assume that u_2u_3 is of length a. If $u_2v_2 \in E(C_4(v_2v_3))$, then, $C_4(v_2v_3)$ contains $u_2v_{2+k \pmod{n}}$, and consequently $v_{2+k \pmod{n}}v_3$, which is impossible, since $k \geq 4$. Therefore, $C_4(v_2v_3)$ contains v_1v_2, and so $C_4(v_2v_3) = (v_3, v_2, v_1, u_1)$, implying once again that $k = 2$, a contradiction. Consequently, u_2u_3 must be of length b, implying that $C_4(v_2v_3) = (v_3, v_2, u_2, u_3)$. By using the same arguments we can deduce that the outer- and inner-circuit edges, and the vertical spokes in G have lengths as shown in Figure 8.9.

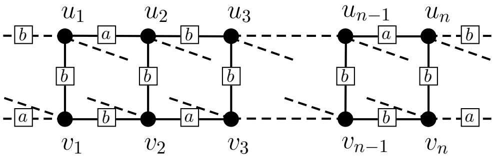

Figure 8.9: $A[n,k]$ when C is of Type A1

Since k is even, we also have $b \equiv \pm 3a \pmod{2n}$ (see for example the 4-circuit $(u_1, u_2, v_{2+k \pmod{n}}, v_{1+k \pmod{n}})$). This implies that $a \equiv \pm 9a \pmod{2n}$, that is, $8a \equiv 2n \pmod{2n}$, or $10a \equiv 2n \pmod{2n}$. Since $\gcd(2n, a) = 1$, the total number of vertices of G must be equal to 8 or 10, a contradiction, since $n \geq 8$. By using a very similar argument it can be shown that C cannot be of Type A2 (see Figure 8.10).

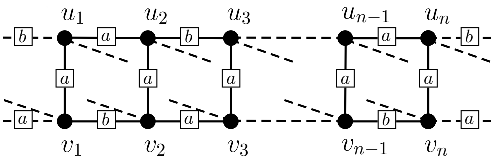

Figure 8.10: $A[n,k]$ when C is of Type A2

So assume that C is of Type A3. Then, $b \equiv \pm 3a \pmod{2n}$ and, in particular, the edge u_2u_3 has length b. Consequently, this edge must belong to some 4-circuit with the other three edges of the circuit having length a. We denote this 4-circuit by $C_4(u_2u_3)$. Since u_1u_2 and u_2v_2 are both of length a, we have the following cases:

- if $C_4(u_2u_3) = (u_3, u_2, u_1, v_{1+k})$, then $k = 2$, a contradiction;
- if $C_4(u_2u_3) = (u_3, u_2, u_1, u_n)$, then $n = 4$, a contradiction; and
- if $C_4(u_2u_3) = (u_3, u_2, v_2, v_1)$, then u_3 is adjacent to v_1. Consequently, we have that $k \equiv -2 \pmod{n}$, and as before, this implies that $n - 2 \leq n/2$, a contradiction, since $n \geq 8$.

Thus C cannot be of Type A3, and by using a similar argument, it can be shown that C cannot be of Type A4 either.

If C is of Type B1 or Type B2, then we have that $2a \equiv \pm 2b \pmod{2n}$, and since $1 \leq a < b \leq n - 1$, we can further assume that $2a \equiv -2b \pmod{2n}$. Consequently, we have that $a + b = n$, and so by Lemma 8.5.2, $G \simeq A[n, 2]$, a contradiction. Therefore, C and all other possible 4-circuits in G must be of Type B3, which is impossible, because then the edges having length a would induce two disjoint n-circuits, contradicting the fact that the edges having length a induce a Hamiltonian circuit (and thus a $2n$-circuit). As a consequence, $A[n,k]$ is not circulant, contradicting our initial assumption. □

Using the above two lemmas we can now prove the main result of this section.

Theorem 8.5.4. *The accordion graph $A[n,k]$ is not circulant if and only if both n and k are even, such that $k \geq 4$.*

Proof. By Lemma 8.5.3, it suffices to show that the accordion graph $A[n,k]$ is circulant if and only if either

(i) k is odd, or

(ii) k is even and n is odd, or

(iii) $k = 2$ and n is even.

Case (i). For k odd, we claim that the function $\phi : V(A[n, k]) \to V(\mathrm{Ci}[2n, \{2, k\}])$ defined by $\phi : u_i \mapsto x_{2i}$ and $\phi : v_i \mapsto x_{2i-k \pmod{2n}}$, where $i \in [n]$, is an isomorphism. Since $2i - k$ is odd, for every $i \in [n]$, one can deduce that the function ϕ is bijective. Also, $\mathrm{Ci}[2n, \{2, k\}]$ has the same number of edges as $A[n, k]$, and thus it suffices to show that an edge in $A[n, k]$ is mapped to an edge in $\mathrm{Ci}[2n, \{2, k\}]$.

(a) We first take an edge $u_i u_j$ from the outer-circuit of $A[n, k]$, for some $i \in [n]$ and $j \equiv i + 1 \pmod{n}$, without loss of generality. Consider $\phi(u_i)\phi(u_j)$. The length of $\phi(u_i)\phi(u_j)$ can be calculated using $2(j - i)$ which is equivalent to $2 \pmod{2n}$. Since this belongs to $\{\pm 2, \pm k\}$, $\phi(u_i)\phi(u_j) \in E(\mathrm{Ci}[2n, \{2, k\}])$.

(b) By a similar reasoning to that used in (a), $\phi(v_i)\phi(v_j)$ is an edge in $\mathrm{Ci}[2n, \{2, k\}]$, for any $i \in [n]$ and $j \equiv i + 1 \pmod{n}$, without loss of generality.

(c) We now consider the spokes. Let $i \in [n]$ and $j \equiv i + k \pmod{n}$. The length of $\phi(u_i)\phi(v_i)$ can be calculated using $2i - k - 2i$, which is equal to $-k$. On the other hand, the length of $\phi(u_i)\phi(v_j)$ can be calculated using $2j - k - 2i$, which is equal to k. In both cases, the lengths obtained belong to $\{\pm 2, \pm k\}$, and so $\phi(u_i)\phi(v_i)$ and $\phi(u_i)\phi(v_j)$ are edges in $\mathrm{Ci}[2n, \{2, k\}]$.

Case (ii). For k even and n odd, we claim that the function $\phi : V(A[n, k]) \to V(\mathrm{Ci}[2n, \{2, n - k\}])$ defined by $\phi : u_i \mapsto x_{2i}$ and $\phi : v_i \mapsto x_{2i+n-k \pmod{2n}}$, where $i \in [n]$, is an isomorphism. As in Case (i), the function ϕ is bijective, since $n - k$ is odd. Moreover, $\mathrm{Ci}[2n, \{2, n - k\}]$ has the same number of edges as $A[n, k]$, and so it suffices to show that an edge in $A[n, k]$ is mapped to an edge in $\mathrm{Ci}[2n, \{2, n - k\}]$.

(a) By the same reasoning used in Case (i), $\phi(u_i)\phi(u_j)$ and $\phi(v_i)\phi(v_j)$ are edges in $\mathrm{Ci}[2n, \{2, n - k\}]$, for $i \in [n]$, and $j \equiv i + 1 \pmod{n}$, without loss of generality.

(b) We now consider the spokes. Let $i \in [n]$ and $j \equiv i + k \pmod{n}$. The length of $\phi(u_i)\phi(v_i)$ can be calculated using $2i + n - k - 2i$, which is equal to $n - k$. On the other hand, the length of $\phi(u_i)\phi(v_j)$ can be calculated using $2j + n - k - 2i$, which is equivalent to $-(n - k) \pmod{2n}$. In both cases, the lengths obtained belong to $\{\pm 2, \pm(n - k)\}$, and so $\phi(u_i)\phi(v_i)$ and $\phi(u_i)\phi(v_j)$ are edges in $\mathrm{Ci}[2n, \{2, n - k\}]$.

Case (iii). This was proven in Lemma 8.5.2 □

The following result follows immediately from the proof of Theorem 8.5.4.

Corollary 8.5.5. *The accordion graph* $A[n,k]$ *is isomorphic to the circulant graph*

(i) $Ci[2n, \{2,k\}]$*, when* k *is odd, and*

(ii) $Ci[2n, \{2, n-k\}]$*, when* n *is odd and* k *is even.*

8.6 CONCLUDING REMARKS AND OPEN PROBLEMS

Despite ruling out all accordion graphs $A[n,k]$ having $\gcd(n,k) \geq 5$, a complete characterisation of which accordion graphs have the PMH- or the PH-property is definitely of interest but still inaccessible. In Section 8.3, partial results were obtained for the cases when $\gcd(n,k) \leq 2$. These are portrayed in Table 8.1 together with other partial results obtained by a computer check conducted through Wolfram Mathematica [103] (see Appendix A). In particular, we identify which accordions are PMH and which are not, for $3 \leq k \leq 10$ and for $n \leq 21$. We remark that some values of n and k are marked as "unknown" due to problems with computation time and memory.

Additionally, as already remarked before, the main result in [11] gives more than just all the possible values of n_1 and n_2, for which $C_{n_1} \square C_{n_2}$ is circulant. In fact, the main result of the above paper is the following.

Theorem 8.6.1. *[11] The circulant graph* $Ci[n', \{a_1, a_2\}]$ *is isomorphic to* $C_{n_1} \square C_{n_2}$ *if and only if:*

(i) $n' = n_1 n_2$,

(ii) $n_1 = \gcd(n', a_j)$ *and* $n_2 = \gcd(n', a_{3-j})$*, where* $j = 1$ *or* $j = 2$*, and*

(iii) $\gcd(n_1, n_2) = 1$.

In this sense, we think that it would be an interesting endeavour to give a necessary and sufficient condition for a quartic circulant graph to be isomorphic to some accordion.

$A[n,k]$		k									
		1	2	3	4	5	6	7	8	9	10
	3	▨	■	■	■	■	■	■	■	■	■
	4	▨	▨	■	■	■	■	■	■	■	■
	5	▨	▨	■	■	■	■	■	■	■	■
	6	▨	▨	▨	■	■	■	■	■	■	■
	7	▨	▨	▨	■	■	■	■	■	■	■
	8	▨	▨	▨	▨	■	■	■	■	■	■
	9	▨	▨	▨	▨	■	■	■	■	■	■
	10	▨	▨	▨	▨	⊥	■	■	■	■	■
	11	▨	▨	▨	▨	▨	■	■	■	■	■
n	12	▨	▨	▨	▨	▨	⊥	■	■	■	■
	13	▨	▨	⊥	▨	▨	▨	■	■	■	■
	14	▨	▨	⊥	▨	▨	▨	⊥	■	■	■
	15	▨	▨	⊥	⊥	⊥	▨	▨	■	■	■
	16	▨	▨	⊥	▨	▨	▨	⊥	⊥	■	■
	17	▨	▨	⊥	⊥	⊥	⊥	▨	▨	■	■
	18	▨	▨	⊥	▨	⊥	⊥	⊥	?	⊥	■
	19	▨	▨	⊥	⊥	⊥	⊥	⊥	⊥	⊥	■
	20	▨	▨	⊥	?	⊥	?	⊥	?	⊥	⊥
	21	▨	▨	⊥	⊥	⊥	⊥	⊥	?	?	?

▨	$A[n,k]$ PMH
⊥	$A[n,k]$ not PMH
?	Unknown

Table 8.1: Which accordions are PMH for $3 \leq n \leq 21$ and $1 \leq k \leq 10$

9

CUBIC PMH-GRAPHS

Unlike the majority of previous chapters this is still a work in its initial phases, and has not been submitted yet.

In what follows we present some results dealing with the properties discussed in the previous chapters of Part ii—this time with respect to cubic graphs. There is already a known and studied class of cubic graphs which are naturally PMH (as we shall see in Proposition 9.0.3). This is the class of cubic 2-factor Hamiltonian graphs. The term was coined by Martin Funk et al. in [33], where the authors study regular Hamiltonian graphs with the property that all their 2-factors are Hamiltonian, called 2-*factor Hamiltonian*. In [33], the authors prove that if a graph G is a bipartite k-regular 2-factor Hamiltonian graph, then G is either a circuit or $k = 3$, that is, G is cubic. In particular, they use the following proposition to construct an infinite family of bipartite cubic 2-factor Hamiltonian graphs.

Proposition 9.0.1. *[33] Let $G = G_1(v_1) * G_2(v_2)$ be a bipartite graph which is obtained by a 3-cut-connection on $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, both of degree 3. Then, G is 2-factor Hamiltonian if and only if G_1 and G_2 are both 2-factor Hamiltonian.*

We note that in the above proposition, G_1 and G_2 are not necessarily cubic graphs, and only need to admit a vertex of degree 3 each, say v_1 and v_2, respectively. For simplicity, and with some abuse of terminology and notation, in the above proposition we use "3-cut-connection" and "$G_1(v_1) * G_2(v_2)$", to describe the resulting graph obtained in a similar way to that described in Subsection 1.4.2. Moreover, we remark that the hypothesis that G is bipartite in the above proposition is needed, because although the complete graph K_4 is a 2-factor Hamiltonian graph, the graph obtained by applying a 3-cut-connection on two vertices from two copies of K_4 is not 2-factor Hamiltonian.

By using Proposition 9.0.1, the authors construct an infinite family of bipartite cubic 2-factor Hamiltonian graphs by taking repeated 3-cut-connections of $K_{3,3}$ and the Heawood graph. They also conjecture that these are the only such graphs, and this conjecture is still open.

In what follows we try to find ways how we can obtain cubic PMH-graphs other than the 3 cubic graphs admitting the PH-property, and cubic 2-factor Hamiltonian graphs. Before continuing, we define what we call a *good vertex*.

Definition 9.0.2. Let G be a cubic graph and let v be a vertex of G. The vertex v is said to be *good* if for every perfect matching M of G, there exist Hamiltonian circuits H_1 and H_2 both extending M, such that H_i contains e_i, for each $i \in [2]$, where $\partial v - M = \{e_1, e_2\}$.

Proposition 9.0.3. *Let G be a cubic 2-factor Hamiltonian graph. Then:*

(i) G is PMH, and

(ii) every vertex of G is good.

Proof. (i) Let M be a perfect matching, and let $\overline{M}$ be its complementary 2-factor. By our assumption, $\overline{M}$ is a Hamiltonian circuit, and since G is of even order, $E(\overline{M}) = N_1 \cup N_2$ with N_1 and N_2 perfect matchings. One can clearly see that for each $i \in [2]$, $M \cup N_i$ is a Hamiltonian circuit of G.

(ii) Clearly follows from (i). □

Despite the clear connection between cubic 2-factor Hamiltonian and cubic PMH-graphs, an analogous result to Proposition 9.0.1 for PMH-graphs is not possible, as the following result shows.

Proposition 9.0.4. *Let G_1 and G_2 be two cubic graphs and let $u \in V(G_1)$ and $v \in V(G_2)$.*

(i) If $G_1(u) * G_2(v)$ is PMH, then G_1 and G_2 are PMH.

(ii) The converse of (i) is not true.

Proof. (i) First assume that $G_1(u) * G_2(v)$ is PMH and let $X = \{u_1v_1, u_2v_2, u_3v_3\}$ be the principal 3-edge-cut of $G_1(u) * G_2(v)$, with u_1, u_2, u_3 and v_1, v_2, v_3, adjacent to $u \in V(G_1)$ and $v \in V(G_2)$, respectively. Let M be a perfect matching of G_1, and without loss of generality, assume that $u_1u \in M$. Let M' be a perfect matching of $G_1(u) * G_2(v)$ containing u_1v_1 and $M - \{u_1u\}$. We remark that such a perfect matching exists by [86]. Furthermore, since $G_1(u) * G_2(v)$ is PMH, M' (and every other perfect matching of this graph) intersects X in exactly one edge. Since $G_1(u) * G_2(v)$ is PMH, it admits a Hamiltonian circuit H containing $M - \{u_1u\}$ and one of u_2v_2 and u_3v_3. Assume $u_2v_2 \in E(H)$. This means that H induces a path in G_1 having end-vertices u_1 and u_2, passes through all the vertices in $V(G_1) - \{u\}$ and contains $M - \{u_1u\}$. This path together with the edges u_1u and u_2u forms a Hamiltonian circuit of G_1 containing M. By a similar reasoning, one can show that G_2 is also PMH.

(ii) Let G_1 and G_2 be two copies of $C_4 \square C_4$, that is, the 3-dimensional cube, and let $u \in V(G_1)$ and $v \in V(G_2)$. Both G_1 and G_2 are PMH (by [29]), but $G_1(u) * G_2(v)$ is not. In fact, consider the perfect matching of $G_1(u) * G_2(v)$ shown in Figure 9.1. One can clearly see that it cannot be extended to a Hamiltonian circuit. □

Corollary 9.0.5. *If G is a cubic PMH-graph having a 3-edge-cut, then G can be obtained by an appropriate 3-cut-connection between two cubic PMH-graphs G_1 and G_2.*

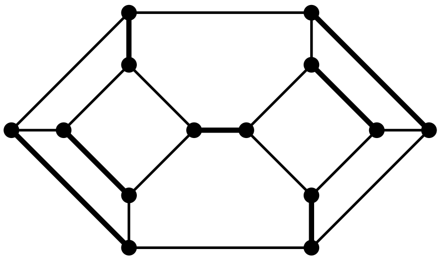

Figure 9.1: The bold edges cannot be extended to a Hamiltonian circuit

Analogous results to Proposition 9.0.4 and Corollary 9.0.5 can be obtained in terms of 2-cut-connections and cubic PMH-graphs having a 2-edge-cut.

Despite the discouraging statement of Proposition 9.0.4, one can still obtain cubic PMH-graphs from two smaller cubic PMH-graphs by using 3-cut-connections and some additional assumptions, as the following result shows.

Proposition 9.0.6. *Let G_1 be a cubic PMH-graph admitting a good vertex u and let G_2 be a cubic PMH-graph. If there exists a vertex $v \in V(G_2)$ such that every perfect matching of $G_1(u) * G_2(v)$ intersects the principal 3-edge-cut exactly once, then $G_1(u) * G_2(v)$ is PMH.*

Proof. Let $X = \{u_1v_1, u_2v_2, u_3v_3\}$ be such a principal 3-edge-cut of $G_1(u) * G_2(v)$, and let M be a perfect matching of $G_1(u) * G_2(v)$. As before, u_1, u_2, u_3 and v_1, v_2, v_3, are adjacent to $u \in V(G_1)$ and $v \in V(G_2)$, respectively. Without loss of generality, assume that $M \cap X = \{u_1v_1\}$. Consequently, M induces perfect matchings M_1 and M_2 in G_1 and G_2, such that $u_1u \in M_1 \cap E(G_1)$ and $v_1v \in M_2 \cap E(G_2)$. Since G_2 is PMH, M_2 can be extended to a Hamiltonian circuit H_2 of G_2. Without loss of generality, we assume that $v_2v \in E(H_2)$. Since u is a good vertex, M_1 can be extended to a Hamiltonian circuit H_1 of G_1 which intersects u_2u. One can easily see that $(E(H_1) - \{u_1u, u_2u\}) \cup (E(H_2) - \{v_1v, v_2v\}) \cup \{u_1v_1, u_2v_2\}$ is a Hamiltonian circuit of $G_1(u) * G_2(v)$ containing M, as required. □

Corollary 9.0.7. *Let G_1 be a cubic PMH-graph admitting a good vertex u and let G_2 be a cubic PMH-graph. If at least one of G_1 and G_2 is bipartite, then $G_1(u) * G_2(v)$ is PMH, for every vertex v of G_2.*

Proof. Since at least one of G_1 and G_2 is bipartite, any perfect matching of $G_1(u) * G_2(v)$ intersects the principal 3-edge-cut in exactly one edge. Result follows by Proposition 9.0.6. □

Finally, we remark that the resulting graph after applying a Y-extension to a vertex u of a cubic graph G is equivalent to $G(u) * K_4(v)$, for any vertex v of K_4. By keeping this in mind, one can obtain the following easy consequence of Proposition 9.0.6.

Corollary 9.0.8. *The graph obtained by applying a Y-extension to a vertex of a bipartite cubic PMH-graph is again PMH.*

Analogously, the graph obtained after applying a Y-extension to a bipartite cubic 2-factor Hamiltonian graph, is again 2-factor Hamiltonian. Although applying a Y-extension (or Y-reduction) is a very simple graph operation, we still have no complete answer to the following question: when do Y-extensions (Y-reductions) preserve the PMH-property?

9.0.1 *Obtaining cubic PMH-graphs with girth at least 4*

Corollary 9.0.8 presents a way of obtaining a new cubic PMH-graph from a smaller (bipartite) cubic PMH-graph. In this section we devise a similar method but in such a way that the resulting PMH-graph has girth at least 4.

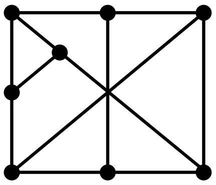

Figure 9.2: Applying a Y-extension to $F_0 = K_{3,3}$

Let F be the graph obtained by applying a Y-extension to a bipartite cubic 2-factor Hamiltonian graph F_0 (see for example Figure 9.2). An easy way to obtain cubic PMH-graphs with girth at least 4 is by using the following proposition.

Proposition 9.0.9. *Let G be a bipartite cubic PMH-graph and let v be a vertex of F lying on its triangle. Then, for any $u \in V(G)$, $G(u) * F(v)$ has girth 4 and is PMH.*

Proof. Let u be a vertex of $V(G)$. Since G is bipartite and every vertex of F is good, $G(u) * F(v)$ is clearly PMH by Corollary 9.0.7. Moreover, since G and F_0 are both bipartite, a circuit of length 3 in $G(u) * F(v)$ can only occur if the edges of the circuit intersect (twice) the principal 3-edge-cut of $G(u) * F(v)$. This is impossible, proving our result. □

Graphs obtained by using Proposition 9.0.9 are not necessarily 2-factor Hamiltonian. In fact, by letting G be equal to Q_3, the 3-dimensional cube, and $F_0 = K_{3,3}$, we note that the resulting graph depicted in Figure 9.3 is not 2-factor Hamiltonian, since the blue and black perfect matchings do not form a Hamiltonian circuit.

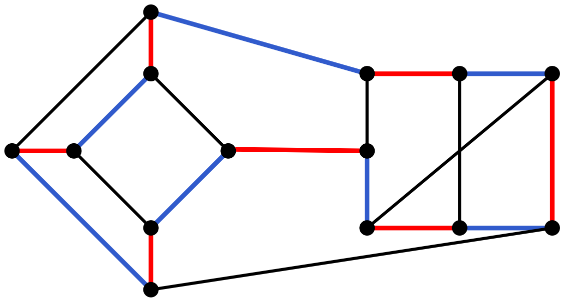

Figure 9.3: A non-bipartite cubic PMH-graph having girth 4 which is not 2-factor Hamiltonian

9.1 FINAL REMARKS

All the methods discussed above on how to generate new cubic PMH-graphs give graphs which are either 2-edge-connected or 3-edge-connected. Consequently, we think that it would be very intriguing to find other cyclically 4-edge-connected cubic PMH-graphs, or, more interestingly, a general method (if such a method exists) giving a construction of how to obtain cyclically 4-edge-connected cubic PMH-graphs from smaller ones.

We finally suggest the following. If a cubic graph G is PMH, then it would also mean that every perfect matching of G corresponds to one of the colours of a 3-edge-colouring of the graph. When this occurs we say that every perfect matching can be extended to a 3-edge-colouring. However, this is not the only time that this occurs. In fact, there are cubic graphs which are not PMH, but with every one of their perfect matchings capable of being extended to a 3-edge-colouring (see for example Figure 9.4). In fact the following theorem is an easy observation.

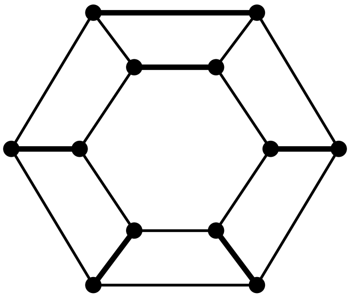

Figure 9.4: The bold edges cannot be extended to a Hamiltonian circuit

Theorem 9.1.1. *Let G be a bridgeless cubic graph. Every perfect matching of G can be extended to a 3-edge-colouring of G if and only if every 2-factor of G is made up of circuits of even length only.*

If a graph G has the PMH-property, then, every one of its 2-factors is made up of even circuits. In particular, this also applies for cubic

2-factor Hamiltonian graphs, since these graphs are PMH-graphs (see Proposition 9.0.3). For, suppose not, and let $\overline{M}$ be such a 2-factor, with $M = E(G) - E(\overline{M})$. Since G is PMH, there must exist a perfect matching N in $\overline{M}$ such that $M \cup N$ is a Hamiltonian circuit. However, this is impossible since $\overline{M}$ contains odd circuits.

If a cubic graph is bipartite, then trivially, each of its perfect matchings can be extended to a 3-edge-colouring, since every 2-factor is clearly made up of even circuits. But what about non-bipartite cubic graphs? In Table 9.1, we show that the number of non-bipartite cubic graphs G (having girth at least 4) such that each one of their perfect matchings can be extended to a 3-edge-colouring, is not insignificant, and the data suggests that this number increases considerably with the order of G. The numbers shown in this table were obtained thanks to a computer check done by Jan Goedgebeur, and the data is sorted according to the cyclic connectivity of the graphs considered.

	Cyclic connectivity				Number of graphs
Number of vertices	3	4	5	6	TOTAL
8	/	1	/	/	1
10	/	/	/	/	0
12	2	5	2	/	9
14	2	2	2	/	6
16	35	56	4	/	95
18	84	21	9	/	114
20	926	655	15	2	1598
22	2978	331	17	6	3332

Table 9.1: The number of non-bipartite cubic 3-connected graphs with girth at least 4 having the property that every 2-factor consists of only even circuits

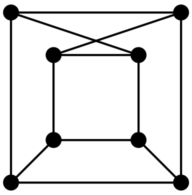

Figure 9.5: The smallest non-bipartite cubic graph such that every one of its perfect matchings can be extended to a 3-edge-colouring

As we have seen, a complete characterisation of which cubic graphs are PMH is still intangible, so considering the Class I cubic graphs having the property that each one of their perfect matchings can be extended to a 3-edge-colouring may look presumptuous. As far as we know this property and the corresponding characterisation problem were never considered before, and in this sense we suggest the following.

Problem 9.1.2. Characterise the Class I cubic graphs for which each one of their perfect matchings can be extended to a 3-edge-colouring.

Despite being a very natural problem to tackle, so far, very little is known. Such a characterisation may look daunting, and the above table suggests that such a problem does not seem trivial, however, we still very much believe that the mentioned property is worth pursuing. Finally, we remark that although the PMH-property is an appealing property in its own right, Problem 9.1.2 continues to justify its study in relation to cubic graphs.

Part III

APPENDIX

A

APPENDIX

Here we give the computer programs used in Chapter 8 to define accordion graphs (Listing A.1), and to check whether an accordion graph has the PMH-property (Listing A.2) or the PH-property (Listing A.3). We remark that in order for the latter two programs to work, one first needs to define what accordions are, that is, Listing A.1 needs to be evaluated before. In Listings A.2 and A.3 one just needs to enter the values of n and k, corresponding to the accordion that is to be checked. These computer checks were conducted through Wolfram Mathematica [103].

Listing A.1: Defining accordion graphs in Mathematica

```
A[n_, k_] := Graph[Tuples[{Range[2], Range[0, n - 1]}],
   For[j = 0, j < n, j++,
    OuterCircuit[j + 1] = {1, j}
    \[UndirectedEdge] {1, Mod[j + 1, n]}];
   For[j = 0, j < n, j++,
    InnerCircuit[j + 1] = {2, j}
    \[UndirectedEdge] {2, Mod[j + 1, n]}];
   For[j = 0, j < n, j++,
    Spoke1[j + 1] = {1, j}
    \[UndirectedEdge] {2, Mod[j, n]}];
   For[j = 0, j < n, j++,
    Spoke2[j + 1] = {1, j}
    \[UndirectedEdge] {2, Mod[j + k, n ]}];
   Join[Array[OuterCircuit, n], Array[InnerCircuit, n],
    Array[Spoke1, n], Array[Spoke2, n]], VertexLabels -> "Name"];
```

Listing A.2: Checking the PMH-property for a particular accordion

```
n =(***INPUT n HERE***);
k =(***INPUT k HERE***);
Print["n=", n];
Print["k=", k];
g = A[n, k];
lg = LineGraph[g];
pmall = EdgeList[g][[#]] & /@

   FindIndependentVertexSet[lg,
```

```
   Length /@ FindIndependentVertexSet[lg], All];
Print["Number of perfect matchings of A[", n,",", k,"] is ",
 Length[pmall]];
GoodPM = {};
BadPM = 0;
For[i = 1, i <= Length[pmall], i++,
 If[Divisible[i, 500], Print[i," perfect matchings checked!"]];
 If[SubsetQ[GoodPM, {i}] == False,
  For[j = 1, j <= Length[pmall], j++,
   If[IntersectingQ[Map[Sort, Extract[pmall, i]],
       Map[Sort, Extract[pmall, j]]] == False &&
     SubsetQ[GoodPM, {i}] == False,
    If[Length@
       ConnectedComponents@
        Graph[Join[Extract[pmall, i], Extract[pmall, j]]]] == 1,
     AppendTo[GoodPM, i] && AppendTo[GoodPM, j]; Break[],
     If[j == Length[pmall] && SubsetQ[GoodPM, {i}] == False,
      BadPM = i; i = Length[pmall] + 1 ]],
    If[j == Length[pmall] &&
      IntersectingQ[Map[Sort, Extract[pmall, i]],
       Map[Sort, Extract[pmall, j]]] == True, BadPM = i;
     i = Length[pmall] + 1]]]]];
finalGoodPM = DeleteDuplicates[GoodPM];
If[Length[finalGoodPM] == Length[pmall],
 Print["The accordion A[", n,",", k,"] is PMH!"],
 Print["!!!The accordion A[", n,",", k,"] is NOT PMH!!! The first bad
     perfect matching is ", BadPM,"."]]
```

Listing A.3: Checking the PH-property for a particular accordion

```
n =(***INPUT n HERE***);
k =(***INPUT k HERE***);
Print["n=", n];
Print["k=", k];
g = A[n, k];
lg = LineGraph[g];
pmall = EdgeList[g][[#]] & /@

  FindIndependentVertexSet[lg,
   Length /@ FindIndependentVertexSet[lg], All];
Print["Number of perfect matchings of A[", n, ",", k,"] is ",
 Length[pmall]];
kn = K[n];
lkn = LineGraph[kn];
pmknall =
 EdgeList[kn][[#]] & /@

  FindIndependentVertexSet[lkn,
   Length /@ FindIndependentVertexSet[lkn], All];
Print["Number of pairings of A[", n,",", k,"] is ", Length[pmknall]];
GoodPM = {};
```

```
BadPM = 0;
For[i = 1, i <= Length[pmknall], i++,
 If[Divisible[i, 500], Print[i," pairings checked!"]];
 If[SubsetQ[GoodPM, {i}] == False,
  For[j = 1, j <= Length[pmall], j++,
   If[IntersectingQ[Map[Sort, Extract[pmknall, i]],
      Map[Sort, Extract[pmall, j]]] == False &&
    SubsetQ[GoodPM, {i}] == False,
   If[Length@
      ConnectedComponents@
       Graph[Join[Extract[pmknall, i], Extract[pmall, j]]]] == 1,
    AppendTo[GoodPM, i] && AppendTo[GoodPM, j]; Break[],
    If[j == Length[pmall] && SubsetQ[GoodPM, {i}] == False,
     BadPM = i; i = Length[pmknall] + 1 ]],
   If[j == Length[pmall] &&
     IntersectingQ[Map[Sort, Extract[pmknall, i]],
      Map[Sort, Extract[pmall, j]]] == True, BadPM = i;
    i = Length[pmknall] + 1]]]]]];
finalGoodPM = DeleteDuplicates[GoodPM];
If[Length[finalGoodPM] == Length[pmknall],
 Print["The accordion A[", n,",", k,"] is PMH!"],
 Print["!!!The accordion A[", n,",", k,"] is NOT PMH!!! The first bad
      perfect matching is ", BadPM,"."]]
```

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INDEX

IF—

If you can keep your head when all about you
Are losing theirs and blaming it on you,
If you can trust yourself when all men doubt you,
But make allowance for their doubting too;
If you can wait and not be tired by waiting,
Or being lied about, don't deal in lies,
Or being hated, don't give way to hating,
And yet don't look too good, nor talk too wise:

If you can dream—and not make dreams your master;
If you can think—and not make thoughts your aim;
If you can meet with Triumph and Disaster
And treat those two impostors just the same;
If you can bear to hear the truth you've spoken
Twisted by knaves to make a trap for fools,
Or watch the things you gave your life to, broken,
And stoop and build 'em up with worn-out tools:

If you can make one heap of all your winnings
And risk it on one turn of pitch-and-toss,
And lose, and start again at your beginnings
And never breathe a word about your loss;
If you can force your heart and nerve and sinew
To serve your turn long after they are gone,
And so hold on when there is nothing in you
Except the Will which says to them: "Hold on!"

If you can talk with crowds and keep your virtue,
Or walk with Kings—nor lose the common touch,
If neither foes nor loving friends can hurt you,
If all men count with you, but none too much;
If you can fill the unforgiving minute
With sixty seconds' worth of distance run,
Yours is the Earth and everything that's in it,
And—which is more—you'll be a Man, my son!

—RUDYARD KIPLING

And all that you give

is in tune
But the sun is eclipsed by the moon

—Pink Floyd, from *The Dark Side of the Moon*