Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2897 – 2915
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doi:10.3906/mat-1802-66

The diameter vulnerability of the generalized Petersen graph $GP[tk, k]$

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Received: 16.02.2018 • **Accepted/Published Online:** 02.03.2018 • **Final Version:** 27.11.2018

Abstract: The diameter of a graph gives the length of the longest path among all the shortest paths between any two vertices of the graph, and the diameter vulnerability problem measures the change in the diameter upon the deletion of edges. In this paper we determine the diameter vulnerability of the generalized Petersen graph $GP[tk, k]$, for integers $t \geq 2$ and $k \geq 1$, and show that (except for some small cases) the diameter remains unchanged upon the deletion of one edge. This work contributes towards a solution of the well-known (Δ, D, D', s)-problem, which attempts to find large graphs with maximum degree Δ and diameter D such that the subgraphs obtained by deleting any set of up to s edges have diameter at most D', preferably equal to D itself. In cases when the delay in communication across a network is directly related to the length of the paths between stations, network designers generally prefer to opt for graphs having the property of being resistant to drastic shocks upon the deletion of edges. This reliability property makes this class of graphs ideal to be used for modeling interconnection networks.

Key words: Diameter vulnerability, fault tolerance, generalized Petersen graph, paths

1. Introduction and definitions

Let G be an undirected simple graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges, respectively. For two vertices $x_1, x_2 \in V(G)$, x_1 and x_2 are adjacent if there is an edge $e = x_1x_2$. The *degree* $d(x)$ of a vertex x is the number of vertices adjacent to x. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ be the *minimum degree* and the *maximum degree*, respectively. When $\delta = \Delta = r$, for some $r \in \mathbb{N}$, the graph is *regular* of degree r. The *edge–connectivity* $\lambda = \lambda(G)$ is the minimum cardinality of a set S of edges whose deletion from G results either in a disconnected graph or in a trivial graph. The path on n vertices and $(n-1)$ edges is denoted by P_n. The *length* (that is, the number of edges) of the shortest path between any two vertices x_1 and x_2 of G is called the *distance* between x_1 and x_2, denoted by $d(x_1, x_2)$. The *diameter* $D(G)$ is given by $\max\{d(x_1, x_2) : x_1, x_2 \in V(G)\}$. The *girth* $g = g(G)$ is the length of the shortest cycle in G. Throughout this work we use $\lfloor x \rfloor$ to denote the maximum integer less than or equal to $x \in \mathbb{R}$.

The notion of fault diameter was introduced in [8]. This notion examines the difference between the diameter of a graph and the diameter of the subgraph obtained upon the deletion of some vertices, and is of particular relevance when the subgraph obtained remains connected. Given a graph G with diameter D and edge-connectivity λ, the diameter vulnerability problem considers the difference between D and the diameter

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2010 *AMS Mathematics Subject Classification:* 05C12, 05C12, 05C75

D'_s of the resultant graph G'_s obtained by deleting s $(< \lambda)$ edges from G. The general problem of determining the maximum possible diameter for a graph obtained from a general graph G by deleting s edges has been proved to be NP-complete [10]. The problem of finding large graphs with maximum degree Δ and diameter D such that the subgraphs obtained by deleting any set of up to s edges (or vertices) have diameter at most D' is known as the (Δ, D, D', s)-problem (see, for example, [1] and [2]).

The effect that the deletions of vertices and edges has on a graph carries a practical interest because these deletions can leave a drastic effect on the efficiency of a network modeled by these graphs, even in cases when the underlying graph remains connected. This is especially the case when the time delay or the interference in the transmission of signals within a network is directly related to the length of the shortest path between any two stations in the network. Due to the nature of many interconnection networks, the possibility of a failure in the links between stations is greater than that of a failure in the stations themselves. Thus, one important measure of the reliability of an interconnection network is how the distance between stations changes when links malfunction.

This leads to the problem of determining the diameter D'_s of the resultant subgraph when s edges are deleted from the original graph of diameter D. The main interest obviously lies in families of graphs whereby the difference between D and D'_s is small. The odd graphs, the n-cubes, the folded n-cubes and the enhanced hypercubes are four families of graphs that have been studied in this context (see, for example, [7–9, 12]). Shi and Lu [11] gave bounds of fault-tolerant diameter of three particular families of interconnection networks introduced by Chen and Tan [4]. In the present work we shift our attention to a subfamily of the generalized Petersen graphs and show its suitability for modeling reliable interconnection networks. The class of generalized Petersen graphs was introduced by Coxeter [6] in 1950 and its name was coined in 1969 by Watkins [13]. This class of graphs is defined as follows.

Definition 1 *The* ***generalized Petersen graph*** $GP[n,k]$, *for* $n \geq 3$ *and* $1 \leq k < n$, *is the graph on the* $2n$ *vertices* $V(GP[n,k]) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ *and whose edge-set* $E(GP[n,k])$ *is composed of the* $3n$ *edges* $\{u_iu_{i+1}, v_iv_{i+k}, u_iv_i\}$, *for* $i \in \{0, 1, \ldots, n-1\}$, *addition modulo* n.

In the rest of this work, it is implicitly assumed that all the subscripts of the vertices of $GP[n,k]$ are taken modulo n.

For $i \in \{0, 1, \ldots, n-1\}$, the vertices u_i are called the *outer vertices* and the vertices v_i are called the *inner vertices*. The generalized Petersen graph $GP[tk,k]$, for $t \geq 2$ and $k \geq 1$, has order $2tk$ and is made up of an *outer cycle* on the outer vertices, k *inner cycles* each of length t on the inner vertices, and *spokes* connecting an outer vertex u_i with an inner vertex v_i (for $i \in \{0, \ldots, tk-1\}$). The edges on the outer cycles are referred to as *outer edges* and those on the inner cycles as *inner edges*. When $t = 2$, we consider the k inner cycles of length 2 as simple edges, and thus, in this case, the inner vertices generate a union of k vertex-disjoint inner edges. We also note that for $k = 1$ we restrict the values of t such that $t \geq 3$ because $GP[2,1]$ is not a simple graph. In the sequel we adopt the convention of labeling the outer vertices $u_0, u_1, \ldots, u_{n-1}$ consecutively around a plane embedding of the outer cycle.

The rest of this paper is structured as follows. In Section 2, we establish the exact value of the diameter of $GP[tk,k]$ for all values of $t \geq 2$ and $k \geq 1$. For graphs, in general, it is known that D'_1 is at most $2D$, and there are graphs (for example, odd cycles) for which this bound is sharp [5]. It is a well-known fact that the generalized Petersen graphs $GP[n,k]$ are 3-connected and 3-edge-connected, except when $n = 2k$. Thus, at most two edges can be deleted from $GP[n,k]$ without disconnecting the graph, implying that the possible values that s can take in the corresponding (Δ, D, D', s)-problem are $s=1$ and $s=2$. In Section 3 we show that, except for some small cases, the diameter of the generalized Petersen graphs $GP[tk,k]$ remains unchanged on the deletion of one edge (that is, $D'_1 = D$). This result is synthesized in Theorem 10. It contributes further towards the reliability of interconnection networks modeled by these generalized Petersen graphs because it shows that they are very fault tolerant since the diameter remains unchanged on the deletion of one edge. We end Section 4 by presenting a conjectured value for D'_2 for this class of generalized Petersen graphs.

2. The diameter of $GP[tk,k]$

The diameter of $GP[t,1]$ for $t \geq 3$ is trivially $\lfloor \frac{t}{2} \rfloor + 1$. In Lemma 2 we establish an upper bound for the diameter by considering the distance between any two vertices in $GP[tk,k]$ for $t \geq 2$ and $k \geq 2$. We then show that we can always find a pair of vertices having this distance. For practicality reasons, in the proofs we fix a labeling of the vertices of $GP[tk,k]$ and take the vertex u_0 as our point of reference.

We start by remarking that for any two integers k and t we have

$$-\left(\left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \right) \leq \left\lfloor \frac{-k-t+1}{2} \right\rfloor. \tag{1}$$

This result follows by considering the parities of t and k and it eases some of our calculations in the sequel.

Lemma 2 *Let $\mathcal{G}$ denote the graph $GP[tk,k]$ for $t \geq 2$ and $k \geq 2$. The diameter $D = D(\mathcal{G})$ is such that*

$$D \leq \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

Proof We consider first the paths from u_0 to u_i for any $i < tk$. Let $m \equiv i \,(mod\, k)$ and $i = sk + m$, where $s \in \mathbb{Z}^+ \cup \{0\}$. We consider three cases, depending on the value of m.

Case 1 $m < \lfloor \frac{k+1}{2} \rfloor$

For $s \leq \lfloor \frac{t}{2} \rfloor$, we consider the path

$$u_0 \to u_1 \to u_2 \to \ldots \to u_m \to v_m \to v_{m+k} \to \ldots \to v_{m+sk} \to u_{m+sk}.$$

The number of the edges in this path is equal to $m+2+s$, and thus

$$d(u_0, u_i) \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 2 + \left\lfloor \frac{t}{2} \right\rfloor \leq \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

For $s > \lfloor \frac{t}{2} \rfloor$, we consider the path

$$u_0 \to u_1 \to u_2 \to \ldots \to u_m \to v_m \to v_{m+(t-1)k} \to \ldots \to v_{m+sk} \to u_{m+sk}.$$

The number of the edges in this path is equal to $m+2+t-s$, and since $-s \leq -\lfloor \frac{t+2}{2} \rfloor \leq \lfloor \frac{-t}{2} \rfloor$,

$$d(u_0, u_i) \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 2 + t + \left\lfloor \frac{-t}{2} \right\rfloor \leq \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

Case 2 $m = \left\lfloor \frac{k+1}{2} \right\rfloor$

For $s < \left\lfloor \frac{t}{2} \right\rfloor$, we consider the path

$$u_0 \to u_1 \to u_2 \to \ldots \to u_m \to v_m \to v_{m+k} \to \ldots \to v_{m+sk} \to u_{m+sk},$$

which has $m+2+s$ edges. Thus

$$d(u_0, u_i) \leq \left\lfloor \frac{k+1}{2} \right\rfloor + 2 + \left\lfloor \frac{t-2}{2} \right\rfloor \leq \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

For $s \geq \left\lfloor \frac{t}{2} \right\rfloor$, we let $m' = k - m$ and consider the path

$$u_0 \to u_{-1} \to u_{-2} \to \ldots \to u_{-m'} \to v_{-m'} \to v_{-m'+(t-1)k} \to \ldots \to v_{-m'+(s+1)k} \to u_{-m'+(s+1)k}.$$

This path has $(k-m)+2+t-s-1$ edges. Hence,

$$d(u_0, u_i) \leq k + t + 1 - \left\lfloor \frac{k+1}{2} \right\rfloor - \left\lfloor \frac{t}{2} \right\rfloor,$$

and by (1) we get $d(u_0, u_i) \leq \left\lfloor \frac{k+t+3}{2} \right\rfloor$.

Case 3 $m > \left\lfloor \frac{k+1}{2} \right\rfloor$

For $s < \left\lfloor \frac{t}{2} \right\rfloor$, we let $m' = k - m$ and consider the path

$$u_0 \to u_{-1} \to u_{-2} \to \ldots \to u_{-m'} \to v_{-m'} \to v_{-m'+k} \to \ldots \to v_{-m'+(s+1)k} \to u_{-m'+(s+1)k}.$$

The number of edges in this path is equal to $m' + 2 + s + 1$, and thus

$$d(u_0, u_i) \leq m' + 3 + s \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 2 + \left\lfloor \frac{t}{2} \right\rfloor \leq \left\lfloor \frac{k+t+3}{2} \right\rfloor.$$

For $s \geq \left\lfloor \frac{t}{2} \right\rfloor$, we again let $m' = k - m$ and consider the path

$$u_0 \to u_{-1} \to u_{-2} \to \ldots \to u_{-m'} \to v_{-m'} \to v_{-m'+(t-1)k} \to \ldots \to v_{-m'+(s+1)k} \to u_{-m'+(s+1)k}.$$

This path has $m' + 2 + t - s - 1 = k + t + 1 - m - s$ edges, implying that

$$d(u_0, u_i) \leq k + t - \left\lfloor \frac{k+1}{2} \right\rfloor - \left\lfloor \frac{t}{2} \right\rfloor,$$

and by (1) we get $d(u_0, u_i) \leq \left\lfloor \frac{t+k+3}{2} \right\rfloor$.

Thus, for $i \in \{1, 2, ..., tk-1\}$, there exists a path of length at most $\left\lfloor \frac{t+k+3}{2} \right\rfloor$ from u_0 to u_i. We next show that, for $i \in \{1, 2, \ldots, tk-1\}$, there are paths from u_0 to v_i and from v_0 to v_i having length at most $\left\lfloor \frac{t+k+3}{2} \right\rfloor$. For this end, we use the path from u_0 to u_i described above. In the case of a path from u_0 to v_i, we consider the corresponding path from u_0 to u_i and delete the spoke $v_i u_i$. Hence the path constructed in this way is one edge shorter than the path from u_0 to u_i described above. Similarly, the path from v_0 to v_i also has the spoke v_iu_i missing when compared to the corresponding path from u_0 to u_i described above, but has at most one additional spoke v_0u_0, and hence its length is not increased.

Therefore, $D \leq \left\lfloor \frac{t+k+3}{2} \right\rfloor$. □

We remark that in the next lemma we consider the case $t \geq 3$; the case when $t = 2$ is discussed separately in Remark 4(1).

Lemma 3 *Let $\mathcal{G}$ denote the graph $GP[tk, k]$ for $t \geq 3$ and $k \geq 2$. There are two vertices u_0 and u_N of $\mathcal{G}$ such that*

$$d(u_0, u_N) = \left\lfloor \frac{t+k+3}{2} \right\rfloor, \quad \text{where} \quad N = \begin{cases} (\frac{t}{2})k - \left\lfloor \frac{k-1}{2} \right\rfloor & \text{if } t \text{ is even} \\ (\frac{t-1}{2})k + \left\lfloor \frac{k}{2} \right\rfloor & \text{if } t \text{ is odd,} \end{cases}$$

except when $t = 3$ and $k = 2$.

Proof We let

$$N = \begin{cases} (\frac{t}{2})k - \left\lfloor \frac{k-1}{2} \right\rfloor & \text{if } t \text{ is even} \\ (\frac{t-1}{2})k + \left\lfloor \frac{k}{2} \right\rfloor & \text{if } t \text{ is odd,} \end{cases}$$

and start by considering the two paths from u_0 to u_N consisting only of outer edges, namely

$$P^{(0)} := u_0 \to u_1 \to ... \to u_N \quad \text{and} \quad P^{(1)} := u_0 \to u_{tk-1} \to ... \to u_N.$$

Since $|E(P^{(0)})| = N$ and $|E(P^{(1)})| = tk - N$, then $|E(P^{(1)})| - |E(P^{(0)})| = tk - 2N \geq 0$. Thus, $|E(P^{(0)})| \leq |E(P^{(1)})|$, which implies that the shortest path from u_0 to u_N consisting only of outer edges is $P^{(0)}$. We find a shorter path from u_0 to u_N by utilizing an inner edge to *bypass* sets of k consecutive outer edges of $P^{(0)}$, in which case two spokes are used. In the sequel:

(i) if k divides N, then $P^{(0)}$ is decomposed into $\frac{N}{k}$ sets of consecutive (outer) edges. This gives a path $P^{(2)}$ on the vertices

$$u_0 \to v_0 \to v_k \to ... \to v_N \to u_N.$$

(ii) if k does not divide N, then to go from u_0 to u_N we can

(a) either bypass $\left\lfloor \frac{N}{k} \right\rfloor$ outer edges of $P^{(0)}$ by using inner edges and then go **forward** (that is, follow the outer cycle in increasing order of the subscripts of the outer vertices) along the remaining (N mod k) outer edges of $P^{(0)}$; this gives a path $P^{(3)}$ on the vertices

$$u_0 \to v_0 \to v_k \to v_{2k} \to \ldots \to v_{\lfloor \frac{N}{k} \rfloor k} \to u_{\lfloor \frac{N}{k} \rfloor k} \to u_{\lfloor \frac{N}{k} \rfloor k+1} \to \ldots \to u_N;$$

(b) or bypass $\left\lfloor \frac{N}{k} \right\rfloor + 1$ outer edges by using inner edges and then go **backward** (that is, follow the outer cycle in decreasing order of the subscripts of the outer vertices) along ($k - N$ mod k) outer edges in order to get to u_N; this gives a path $P^{(4)}$ on the vertices

$$u_0 \to v_0 \to v_k \to v_{2k} \to \ldots \to v_{(\lfloor \frac{N}{k} \rfloor+1)k} \to u_{(\lfloor \frac{N}{k} \rfloor+1)k} \to u_{(\lfloor \frac{N}{k} \rfloor+1)k-1} \to \ldots \to u_N.$$

We consider two cases, according to the parity of t.

Case 1. Let t be an even integer, $t \geq 4$.

If $k = 2$, then k divides N, in which case $N = t$ and we consider the path $P^{(2)}$ from u_0 to u_N. Now $|E(P^{(2)})| = 2 + \frac{t}{2}$ and

$$|E(P^{(0)})| - |E(P^{(2)})| = t - \left(2 + \frac{t}{2}\right) \geq 0,$$

and thus $|E(P^{(2)})| \leq |E(P^{(0)})|$. Hence, $P^{(2)}$ is the shortest path from u_0 to u_N and $|E(P^{(2)})| = 2 + \frac{t}{2} = \left\lfloor \frac{t+k+3}{2} \right\rfloor$.

If $k \geq 3$, then k does not divide N, and

(a) considering the path $P^{(3)}$ from u_0 to u_N, we get

$$\begin{aligned}|E(P^{(3)})| &= \left\lfloor \frac{N}{k} \right\rfloor + N \bmod k + 2 = \left(\frac{t}{2} - 1\right) + \left(\left(\frac{t}{2}\right)k - \left\lfloor \frac{k-1}{2} \right\rfloor\right) \bmod k + 2 \\ &= \frac{t}{2} + \left(k - \left\lfloor \frac{k-1}{2} \right\rfloor\right) + 1 \qquad \text{since} \quad \left\lfloor \frac{k-1}{2} \right\rfloor > 0 \quad \text{for} \quad k \geq 3;\end{aligned}$$

(b) considering the path $P^{(4)}$ from u_0 to u_N, we get

$$\begin{aligned}|E(P^{(4)})| &= \left\lfloor \frac{N}{k} \right\rfloor + 1 + (k - N \bmod k) + 2 \\ &= \left(\frac{t}{2} - 1\right) + \left(k - \left(k - \left\lfloor \frac{k-1}{2} \right\rfloor\right)\right) + 3 = \frac{t}{2} + \left\lfloor \frac{k-1}{2} \right\rfloor + 2.\end{aligned}$$

Now $|E(P^{(3)})| \leq |E(P^{(0)})|$ since

$$\begin{aligned}|E(P^{(0)})| - |E(P^{(3)})| &= \left(\frac{t}{2}\right)k - \left\lfloor \frac{k-1}{2} \right\rfloor - \left(\frac{t}{2} + k - \left\lfloor \frac{k-1}{2} \right\rfloor + 1\right) \\ &= \left(\frac{t}{2} - 1\right)(k-1) - 2 \geq 0,\end{aligned}$$

and $|E(P^{(4)})| \leq |E(P^{(3)})|$ since

$$|E(P^{(3)})| - |E(P^{(4)})| = k - 2\left\lfloor \frac{k-1}{2} \right\rfloor - 1 \geq 0.$$

Hence,

$$|E(P^{(4)})| \leq |E(P^{(3)})| \leq |E(P^{(0)})|.$$

Thus, for t even, $t \geq 4$, $P^{(4)}$ is the shortest path from u_0 to u_N and we have

$$d(u_0, u_N) = |E(P^{(4)})| = \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

Case 2. Let t be an odd integer and $t \geq 3$.

For k to divide N, then $\left\lfloor \frac{k}{2} \right\rfloor = 0$, which is not possible since $k \geq 2$. Thus k does not divide N and *(a)* considering the path $P^{(3)}$ from u_0 to u_N, we get

$$|E(P^{(3)})| = \left\lfloor \frac{N}{k} \right\rfloor + N \bmod k + 2 = \left(\frac{t-1}{2}\right) + \left(\left(\frac{t-1}{2}\right)k + \left\lfloor \frac{k}{2} \right\rfloor\right) \bmod k + 2$$
$$= \frac{t-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor + 2 = \frac{t+3}{2} + \left\lfloor \frac{k}{2} \right\rfloor;$$

(b) considering the path $P^{(4)}$ from u_0 to u_N, we get

$$|E(P^{(4)})| = \left\lfloor \frac{N}{k} \right\rfloor + 1 + (k - N \bmod k) + 2 = \frac{t-1}{2} + \left(k - \left\lfloor \frac{k}{2} \right\rfloor\right) + 3$$
$$= \frac{t+5}{2} + k - \left\lfloor \frac{k}{2} \right\rfloor.$$

Now $|E(P^{(4)})| > |E(P^{(3)})|$ since

$$|E(P^{(4)})| - |E(P^{(3)})| = k + 1 - 2\left\lfloor \frac{k}{2} \right\rfloor > 0.$$

Moreover, since

$$|E(P^{(0)})| - |E(P^{(3)})| = \left(\frac{t-1}{2}\right)k + \left\lfloor \frac{k}{2} \right\rfloor - \left(\frac{t+3}{2} + \left\lfloor \frac{k}{2} \right\rfloor\right) = (k-1)\left(\frac{t-1}{2}\right) - 2,$$

then $|E(P^{(0)})| \geq |E(P^{(3)})|$ if ($k = 2$ and $t \geq 5$) or $k \geq 3$.

Thus, for t odd, $t \geq 3$, $P^{(3)}$ is the shortest path from u_0 to u_N provided that $k \geq 3$ or $t \geq 5$ when $k = 2$, and we have

$$d(u_0, u_N) = |E(P^{(3)})| = \frac{t+3}{2} + \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

□

Remark 4 *It can be readily checked that:*

(1) in the case when $t = 2$ and $k \geq 2$, there are two vertices v_0 and v_N where $N = \left\lfloor \frac{k+1}{2} \right\rfloor$ such that a shortest path between them is the path

$$v_0 \rightarrow u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_N \rightarrow v_N$$

having $N + 2 = \left\lfloor \frac{k+5}{2} \right\rfloor = \left\lfloor \frac{t+k+3}{2} \right\rfloor$ edges (refer to Figure 1);

(2) in the case when $k = 2$ and $t = 3$, there are two vertices v_i and v_{i+3}, for $i \in \{0, \ldots, 5\}$, (for example, v_0 and v_3) such that $d(v_i, v_{i+3}) = 4 = \left\lfloor \frac{t+k+3}{2} \right\rfloor$ (refer to Figure 2).

Thus, the following theorem follows from Lemmas 2 and 3 and from the remark above.

Theorem 5 *The diameter D of $GP[tk, k]$ for $t \geq 2$ and $k \geq 2$ is given by*

$$D = \left\lfloor \frac{t+k+3}{2} \right\rfloor.$$

We note that the result of Theorem 5 agrees with that of Zhang et al. [16], which states that $D(GP[n,k]) = O\left(\frac{n}{2k}\right)$ for $k \geq 3$.

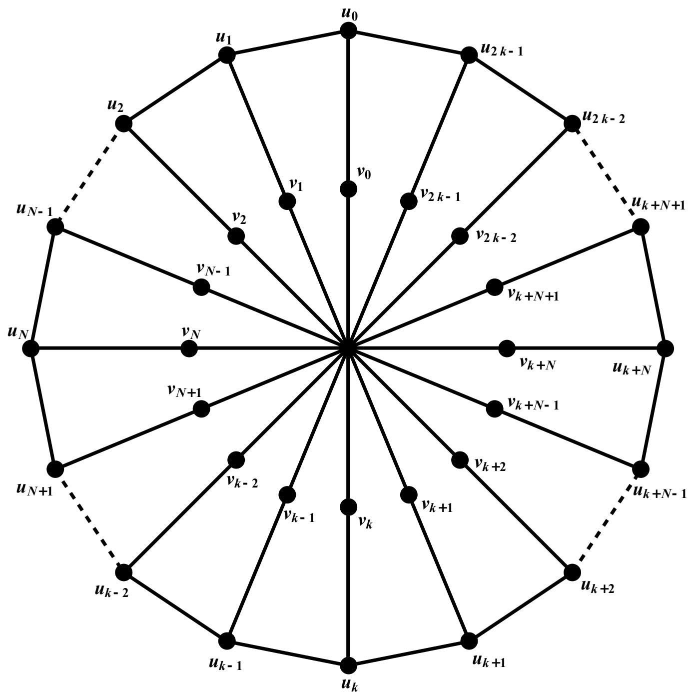

Figure 1. The graph $GP[2k, k]$ for $k \geq 2$.

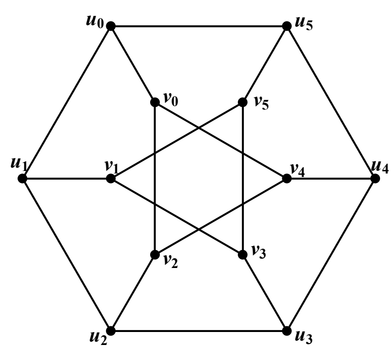

Figure 2. The graph $GP[6, 2]$.

3. Deleting one edge

For $k = 1$, it is easy to observe that the deletion of any edge e from $\mathcal{G} = GP[t, 1]$ leaves the diameter of the graph $G_1' = \mathcal{G} - e$ equal to that of $\mathcal{G}$ except when $t = 3$ and e is a spoke, in which case $D(GP[3, 1] - e) = 3$ instead of 2. Thus, in the sequel we will focus on values of $k \geq 2$.

We consider the deletion of the three different types of edges (i.e. outer edge, spoke, and inner edge) of the generalized Petersen graph $GP[tk, k]$ separately. We choose an arbitrary edge e and then fix a labeling of the graph such that:

1. if the chosen edge e is an outer edge, then $e = u_0u_1$;
2. if the chosen edge e is a spoke edge, then $e = u_0v_0$;
3. if the chosen edge e is an inner edge, then $e = v_0v_k$.

Further to the reasoning employed in Section 2, in each case we take the vertices u_0 and v_0 as our point of reference, and find the distances of all the remaining vertices from these two. To do this, we examine the lengths of paths from each of u_0 and v_0 to the other vertices u_i and v_i, where $i \in \{1, 2, \ldots, tk-1\}$. For $i = sk + m$, where $s \in \mathbb{Z}^+ \cup \{0\}$ and $m = i \bmod k$, we consider the paths:

$$u_0 \to v_0 \to v_k \to \ldots \to v_{sk} \to u_{sk} \to u_{sk+1} \to \ldots \to u_{m+sk} \ ,$$

$$u_0 \to u_1 \to v_1 \to v_{1+k} \to \ldots \to v_{1+sk} \to u_{1+sk} \to \ldots \to u_{m+sk} \ ,$$

$$\vdots$$

$$u_0 \to u_1 \to \ldots \to u_{m-1} \to v_{m-1} \to v_{m-1+k} \to \ldots \to v_{m-1+sk} \to u_{m-1+sk} \to u_{m+sk} \ ,$$

$$u_0 \to u_1 \to \ldots \to u_m \to v_m \to v_{m+k} \to \ldots \to v_{m+sk} \to u_{m+sk} \ .$$

It is clear that all these paths have the same length, namely $m+s+2$. However, we remark that the first path is the only one that includes the vertex v_0. Since in our work we are interested in finding the length of the shortest paths from u_0 and from v_0 to the other vertices, the first path plays a different role to that played by all the other paths. Thus, we consider the first path as being "essentially different" from all the others, which, for the purpose of our work, are **not** regarded as being essentially different from each other. Henceforth, when we say that we consider the possible candidates for a shortest path between two vertices, we implicitly imply that we consider the *essentially different* paths between the two vertices.

3.1. Deleting one outer edge

We let $e = u_0u_1$ be an outer edge of $\mathcal{G} = GP[tk, k]$, and consider the graph $G'_1 = \mathcal{G} - e$. Due to the argument used in determining the diameter D of $\mathcal{G}$, we note that in G'_1 only the distances from each of u_0 and v_0 to the vertices u_i and v_i, where $i \in \{1, \ldots, k-1\}$ can change, while all the other distances remain unchanged.

Lemma 6 *Let $\mathcal{G}$ denote the graph $GP[tk, k]$ for $t \geq 2$ and $k \geq 2$, and let e be an outer edge of $\mathcal{G}$. The diameter D'_1 of the graph $G'_1 = \mathcal{G} - e$ is given by*

$$D'_1 = \begin{cases} k+2 & \textit{if either } k=2 \textit{ and } t=2, \textit{ or } k=3 \textit{ and } t \in \{2,3\}, \\ D & \textit{if either } k=2 \textit{ and } t \geq 3, \textit{ or } k=3 \textit{ and } t \geq 4, \\ \lfloor \frac{k+1}{2} \rfloor + 4 & \textit{if } k \geq 4 \textit{ and either } k \textit{ is even and } t \leq 4, \textit{ or } k \textit{ is odd and } t \leq 5, \\ D & \textit{if } k \geq 4 \textit{ and either } k \textit{ is even and } t \geq 5, \textit{ or } k \textit{ is odd and } t \geq 6, \end{cases}$$

where D is the diameter of $\mathcal{G}$.

Proof We consider $G'_1 = \mathcal{G} - e$, where $e = u_0u_1$ is an outer edge of $\mathcal{G}$, and let its diameter be denoted by D'_1. Starting from $d(u_0, u_i)$, where $i \in \{1, \ldots, k-1\}$, there are three possible candidates for a shortest path from u_0 to u_i, namely:

$$P^{(1)} := u_0 \to u_{-1} \to \ldots \to u_{i-k} \to v_{i-k} \to v_i \to u_i \ ;$$

$$P^{(2)} := u_0 \to v_0 \to v_k \to u_k \to u_{k-1} \to \ldots \to u_i \ ;$$

$$P^{(3)} := u_0 \to v_0 \to v_k \to u_k \to u_{k+1} \to \ldots \to u_{k+i} \to v_{k+i} \to v_i \to u_i \ .$$

Thus, $|E(P^{(1)})| = |E(P^{(2)})| = k-i+3$ (and hence $P^{(1)}$ and $P^{(2)}$ can be used interchangeably) and $|E(P^{(3)})| = i+6$. We note that $|E(P^{(1)})| \leq |E(P^{(3)})|$ if and only if $i \geq \lfloor \frac{k-2}{2} \rfloor = \lfloor \frac{k}{2} \rfloor - 1$. Thus,

(i) if $i \leq \lfloor \frac{k}{2} \rfloor - 2$, then $d(u_0, u_i) = |E(P^{(3)})| = i+6 \leq \lfloor \frac{k}{2} \rfloor + 4$,

(ii) if $i \geq \lfloor \frac{k}{2} \rfloor - 1$, then $d(u_0, u_i) = |E(P^{(1)})| = |E(P^{(2)})| = k-i+3 \leq \lfloor \frac{k+1}{2} \rfloor + 4$.

However, since $\lfloor \frac{k+1}{2} \rfloor + 4 \geq \lfloor \frac{k}{2} \rfloor + 4$, the maximum distance from u_0 to u_i is $\lfloor \frac{k+1}{2} \rfloor + 4$ when $i = \lfloor \frac{k}{2} \rfloor - 1$ and $k \geq 4$ (since $i \geq 1$). For $k \in \{2,3\}$, then $|E(P^{(1)})| < |E(P^{(3)})|$ and $d(u_0, u_i) = k-i+3$. Thus the maximum distance from u_0 to u_i is $k+2$ when $i=1$.

We remark that from the way that $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ were defined, and recalling that $P^{(1)}$ and $P^{(2)}$ can be used interchangeably, we get that

- $d(u_0, v_i) \leq d(u_0, u_i) - 1$ by using $P^{(1)}$ and $P^{(3)}$,
- $d(v_0, v_i) \leq d(u_0, v_i) + 1 \leq d(u_0, u_i)$ by using $P^{(1)}$ and $P^{(3)}$, and
- $d(v_0, u_i) \leq d(u_0, u_i) - 1$ by using $P^{(2)}$ and $P^{(3)}$,

and thus it is sufficient to check when $d(u_0, u_i)$ exceeds D.

Thus, if $k \in \{2,3\}$, then

$$\begin{aligned} D_1' = \max\{k+2,\ D\} &= \begin{cases} \max\{4,\ \lfloor \frac{t+5}{2} \rfloor\} & \text{if } k=2, \\ \max\{5,\ \lfloor \frac{t+6}{2} \rfloor\} & \text{if } k=3, \end{cases} \\ &= \begin{cases} 4 & \text{if } k=2 \text{ and } t=2, \\ D & \text{if } k=2 \text{ and } t \geq 3, \\ 5 & \text{if } k=3 \text{ and } t \in \{2,3\}, \\ D & \text{if } k=3 \text{ and } t \geq 4, \end{cases} \\ &= \begin{cases} k+2 & \text{if either } k=2 \text{ and } t=2, \text{ or } k=3 \text{ and } t \in \{2,3\}, \\ D & \text{if either } k=2 \text{ and } t \geq 3, \text{ or } k=3 \text{ and } t \geq 4, \end{cases} \end{aligned}$$

and if $k \geq 4$, then

$$\begin{aligned} D_1' &= \max\{\lfloor \tfrac{k+1}{2} \rfloor + 4,\ D\} \\ &= \begin{cases} \lfloor \frac{k+1}{2} \rfloor + 4 & \text{if either } k \text{ is even and } t \leq 4, \text{ or } k \text{ is odd and } t \leq 5, \\ D & \text{if either } k \text{ is even and } t \geq 5, \text{ or } k \text{ is odd and } t \geq 6. \end{cases} \end{aligned}$$

□

3.2. Deleting one spoke

We let $e = u_0v_0$ be a spoke of $\mathcal{G} = GP[tk, k]$, and consider the graph $G_1' = \mathcal{G} - e$. Further to the argument used in determining the diameter D of $\mathcal{G}$, we note that in G_1' the distances from u_0 to the vertices u_i and v_i, where $i \notin \{0, k, \ldots, k(t-1)\}$ remain unchanged, and the distances from v_0 to the vertices u_i and v_i, where $i \geq k$ and $i \notin \{k, \ldots, k(t-1)\}$ also remain unchanged. Thus, in the sequel we deal with the remaining cases, and due to the symmetry property of the generalized Petersen graphs we need only consider

- $d(u_0, u_i)$ and $d(u_0, v_i)$ for $i \in \{0, k, \ldots, \lfloor \frac{t}{2} \rfloor k\}$;

- $d(v_0, u_i)$ and $d(v_0, v_i)$ for $i \in \{1, \ldots, k-1\}$; and
- $d(v_0, u_i)$ and $d(v_0, v_i)$ for $i \in \{k, \ldots, \lfloor \frac{t}{2} \rfloor k\}$.

As a consequence of the difference in the structure of G_1' when $k = 2$ and when $k \geq 3$, we consider the two cases separately in Lemma 7 and Lemma 8, respectively.

Lemma 7 *Let $\mathcal{G}$ denote the graph $GP[2t, 2]$ for $t \geq 2$, and let e be a spoke of $\mathcal{G}$. The diameter D_1' of the graph $G_1' = \mathcal{G} - e$ is given by*

$$D_1' = \begin{cases} 4 & \text{if } t = 2, \\ 3 + \lfloor \frac{t}{2} \rfloor & \text{if } t \geq 6 \text{ and } t \text{ is even}, \\ D & \text{if either } t \in \{3,4,5\}, \text{ or } t \geq 7 \text{ and } t \text{ is odd}, \end{cases}$$

where D is the diameter of $\mathcal{G}$.

Proof We consider $G_1' = \mathcal{G} - e$, where $e = u_0v_0$ is a spoke of $\mathcal{G}$, and let its diameter be denoted by D_1'. Starting from $d(u_0, v_0)$, the shortest path is given by

$$P^{(1)} := u_0 \to u_1 \to u_2 \to v_2 \to v_0,$$

and thus $d(u_0, v_0) = 4$. Similarly, $d(v_0, u_1) < d(v_0, v_1) = 4$.

For $i \in \{k, \ldots, \lfloor \frac{t}{2} \rfloor k\}$, we consider all the possible candidates for the shortest paths from u_0 and from v_0 to each of the vertices u_i and v_i. The distance from u_0 to u_i is greatest when i is largest, that is, $i = \lfloor \frac{t}{2} \rfloor k$. Thus, we let $x = 2\lfloor \frac{t}{2} \rfloor$ and consider $d(u_0, u_x)$, $d(u_0, v_x)$, $d(v_0, u_x)$, and $d(v_0, v_x)$.

For $d(u_0, u_x)$, there are three possible candidates for a shortest path from u_0 to u_x, namely:

$$P^{(2)} := u_0 \to u_1 \to \ldots \to u_x \ ;$$

$$P^{(3)} := u_0 \to u_1 \to v_1 \to v_{1+k} \to \ldots \to v_{x-1} \to u_{x-1} \to u_x \ ;$$

$$P^{(4)} := u_0 \to u_1 \to u_2 \to v_2 \to v_{2+k} \to \ldots \to v_x \to u_x \ .$$

Thus, $|E(P^{(2)})| = 2\lfloor \frac{t}{2} \rfloor$ and $|E(P^{(3)})| = |E(P^{(4)})| = 3 + \lfloor \frac{t}{2} \rfloor$. We note that $|E(P^{(2)})| < |E(P^{(3)})|$ if and only if $t \leq 5$.

For $d(u_0, v_x)$, we again consider the three possible candidates for a shortest path from u_0 to v_x, as follows:

$$P^{(5)} := P^{(2)} + u_xv_x \ ; \qquad P^{(6)} := P^{(3)} + u_xv_x \ ; \qquad P^{(7)} := P^{(4)} - v_xu_x \ .$$

Thus, $|E(P^{(5)})| = 2\lfloor \frac{t}{2} \rfloor + 1$, $|E(P^{(6)})| = 4 + \lfloor \frac{t}{2} \rfloor$ and $|E(P^{(7)})| = 2 + \lfloor \frac{t}{2} \rfloor$, noting that $P^{(7)}$ is the shortest path for all values of t.

For $d(v_0, u_x)$, the shortest path is clearly

$$P^{(8)} := v_0 \to v_2 \to \ldots \to v_x \to u_x$$

having $\lfloor \frac{t}{2} \rfloor + 1$ edges. Thus, $d(v_0, v_x) = d(v_0, u_x) - 1$.

Thus, if $t \leq 5$, then

$$\begin{aligned} D_1' &= \max\{|E(P^{(1)})|, |E(P^{(2)})|, |E(P^{(7)})|, |E(P^{(8)})|, D\} \\ &= \max\{4,\ 2\lfloor \tfrac{t}{2} \rfloor,\ 2+\lfloor \tfrac{t}{2} \rfloor,\ 1+\lfloor \tfrac{t}{2} \rfloor,\ \lfloor \tfrac{t+5}{2} \rfloor\} \\ &= \begin{cases} 4 & \text{if } t = 2, \\ D & \text{if } t \in \{3,4,5\}, \end{cases} \end{aligned}$$

and if $t \geq 6$, then

$$\begin{aligned} D_1' &= \max\{|E(P^{(1)})|, |E(P^{(3)})|, |E(P^{(7)})|, |E(P^{(8)})|, D\} \\ &= \max\{4,\ 3+\lfloor \tfrac{t}{2} \rfloor,\ 2+\lfloor \tfrac{t}{2} \rfloor,\ 1+\lfloor \tfrac{t}{2} \rfloor,\ \lfloor \tfrac{t+5}{2} \rfloor\} \\ &= \begin{cases} D & \text{if } t \geq 7 \text{ and } t \text{ is odd}, \\ 3+\lfloor \tfrac{t}{2} \rfloor & \text{if } t \geq 6 \text{ and } t \text{ is even}. \end{cases} \end{aligned}$$

□

Lemma 8 *Let $\mathcal{G}$ denote the graph $GP[tk,k]$ for $t \geq 2$ and $k \geq 3$, and let e be a spoke of $\mathcal{G}$. The diameter D_1' of the graph $G_1' = \mathcal{G} - e$ is given by*

$$D_1' = \begin{cases} 5 & \text{if } k = 3 \text{ and } t \in \{2,3\}, \\ 6 & \text{if } k = 4 \text{ and } t \in \{2,3,4\}, \\ 4+\lfloor \frac{t}{2} \rfloor & \text{if either } k = 3 \text{ and } t \geq 4, \text{ or } k = 4, t \geq 6 \text{ and } t \text{ is even}, \\ D & \text{if } k = 4, t \geq 5 \text{ and } t \text{ is odd}, \\ 7 & \text{if } k \in \{5,6\} \text{ and } t \in \{2,3\}, \\ 5+\lfloor \frac{t}{2} \rfloor & \text{if either } k = 5 \text{ and } t \geq 4, \text{ or } k = 6, t \geq 4 \text{ and } t \text{ is even}, \\ D & \text{if } k = 6, t \geq 5 \text{ and } t \text{ is odd}, \\ \frac{k+7}{2} & \text{if } k \geq 7, k \text{ is odd and } t \in \{2,3\}, \\ \frac{k+6}{2} & \text{if } k \geq 8, k \text{ is even and } t = 2, \\ D & \text{if either } k \geq 8, k \text{ is even and } t = 3, \text{ or } k \geq 7 \text{ and } t \geq 4, \end{cases}$$

where D is the diameter of $\mathcal{G}$.

Proof We consider $G_1' = \mathcal{G} - e$ where $e = u_0v_0$ is a spoke of $\mathcal{G}$, and let its diameter be denoted by D_1'. Starting from $d(u_0, v_0)$, there are two possible candidates for a shortest path from u_0 to v_0, namely:

$$P^{(1)} := u_0 \to u_1 \to \ldots \to u_k \to v_k \to v_0 \ ;$$

$$P^{(2)} := u_0 \to u_1 \to v_1 \to v_{1+k} \to u_{1+k} \to u_k \to v_k \to v_0 \ .$$

Thus, $|E(P^{(1)})| = k+2$ and $|E(P^{(2)})| = 7$, implying that $|E(P^{(1)})| < |E(P^{(2)})|$ if and only if $k \leq 4$.

For $i \in \{k, \ldots, \lfloor \frac{t}{2} \rfloor k\}$, we consider all the possible candidates for the shortest paths from u_0 to each of the vertices u_i and v_i. The distance from u_0 to either one of u_i or v_i is greatest when i is largest, that is, $i = \lfloor \frac{t}{2} \rfloor k$. Thus, we let $x = \lfloor \frac{t}{2} \rfloor k$ and consider $d(u_0, u_x)$ and $d(u_0, v_x)$.

For $d(u_0, u_x)$, there are three possible candidates for a shortest path from u_0 to u_x, namely:

$$P^{(3)} := u_0 \to u_1 \to \ldots \to u_x \ ;$$

$$P^{(4)} := u_0 \to u_1 \to v_1 \to v_{1+k} \to \ldots \to v_{x+1} \to u_{x+1} \to u_x \ ;$$

$$P^{(5)} := u_0 \to u_1 \to \ldots \to u_k \to v_k \to v_{2k} \to \ldots \to v_x \to u_x \ .$$

Thus, $|E(P^{(3)})| = \lfloor \frac{t}{2} \rfloor k$, $|E(P^{(4)})| = 4 + \lfloor \frac{t}{2} \rfloor$ and $|E(P^{(5)})| = k + \lfloor \frac{t}{2} \rfloor + 1$. We note that $|E(P^{(4)})| \leq |E(P^{(5)})|$ since $k \geq 3$, and that $|E(P^{(3)})| < |E(P^{(4)})|$ if and only if $t \in \{2,3\}$ and $k \in \{3,4\}$.

For $d(u_0, v_x)$, the three possible candidates for a shortest path from u_0 to v_x are

$$P^{(6)} := P^{(3)} + u_x v_x \ ; \quad P^{(7)} := P^{(4)} + u_x v_x \ ; \quad P^{(8)} := P^{(5)} - v_x u_x \ .$$

Thus, $|E(P^{(6)})| = \lfloor \frac{t}{2} \rfloor k + 1$, $|E(P^{(7)})| = 5 + \lfloor \frac{t}{2} \rfloor$ and $|E(P^{(8)})| = k + \lfloor \frac{t}{2} \rfloor$. Here we note that since $t \geq 2$ and $k \geq 3$, $|E(P^{(8)})| \leq |E(P^{(6)})|$, and $|E(P^{(7)})| < |E(P^{(8)})|$ if and only if $k > 5$.

We consider next the distance $d(v_0, u_i)$ for $i \in \{1, \ldots, k-1\}$. There are two possible candidates for a shortest path from v_0 to u_i, namely:

$$P^{(9)} := v_0 \to v_k \to u_k \to u_{k-1} \to \ldots \to u_i \ ;$$

$$P^{(10)} := v_0 \to v_k \to u_k \to u_{k+1} \to \ldots \to u_{k+i} \to v_{k+i} \to v_i \to u_i \ .$$

Thus, $|E(P^{(9)})| = 2+k-i$ and $|E(P^{(10)})| = 5+i$, implying that $|E(P^{(9)})| < |E(P^{(10)})|$ if and only if $i \geq \lfloor \frac{k-1}{2} \rfloor$. Hence, $d(v_0, u_i) > D$ if and only if

(i) k is even, $t = 2$, and $i = \frac{k-2}{2}$, in which case $d(v_0, u_i) = |E(P^{(9)})| = \frac{k+6}{2}$,

(ii) or k is even, $t = 2$, and $i = \frac{k-4}{2}$, in which case $d(v_0, u_i) = |E(P^{(10)})| = \frac{k+6}{2}$,

(iii) or k is odd, $t \in \{2,3\}$, and $i = \frac{k-3}{2}$, in which case $d(v_0, u_i) = |E(P^{(10)})| = \frac{k+7}{2}$.

Taking into consideration the distance $d(v_0, v_i)$ for $i \in \{1, \ldots, k-1\}$, there are two possible candidates for a shortest path from v_0 to v_i, namely:

$$P^{(11)} := P^{(9)} + u_i v_i \ ; \quad P^{(12)} := P^{(10)} - v_i u_i \ .$$

In this case, $3 + k - i = |E(P^{(11)})| < |E(P^{(12)})| = 4 + i$ if and only if $i \geq \lfloor \frac{k+1}{2} \rfloor$. Hence, $d(v_0, v_i) > D$ if and only if

(i) k is even, $t = 2$, and $i = \frac{k-2}{2}$, in which case $d(v_0, v_i) = |E(P^{(12)})| = \frac{k+6}{2}$,

(ii) or k is even, $t = 2$, and $i = \frac{k}{2}$, in which case $d(v_0, v_i) = |E(P^{(11)})| = \frac{k+6}{2}$,

(iii) or k is odd, $t \in \{2,3\}$, and $i = \frac{k-1}{2}$, in which case $d(v_0, v_i) = |E(P^{(12)})| = \frac{k+7}{2}$.

Finally, we consider the distance from v_0 to each of the vertices u_i and v_i for $i \in \{k, \ldots, \lfloor \frac{t}{2} \rfloor k\}$. Again, this distance is greatest when i is largest, that is, $i = \lfloor \frac{t}{2} \rfloor k$. Thus, we let $x = \lfloor \frac{t}{2} \rfloor k$ and consider $d(v_0, u_x)$ and $d(v_0, v_x)$.

For $d(v_0, u_x)$, the shortest path is clearly

$$P^{(13)} := v_0 \to v_2 \to \ldots \to v_x \to u_x$$

having $\lfloor \frac{t}{2} \rfloor + 1$ edges, and thus $|E(P^{(13)})| < \lfloor \frac{t+k+3}{2} \rfloor = D$ since $k \geq 3$. Moreover, $d(v_0, v_x) = d(v_0, u_x) - 1 < D$.

Thus, collating the above results, we get:

- if $k = 3$ and $t = 2$ or 3, then $D'_1 = \max\{k+2,\ \lfloor \frac{t}{2} \rfloor k,\ k + \lfloor \frac{t}{2} \rfloor,\ \frac{k+7}{2}\} = 5$;
- if $k = 3$ and $t \geq 4$, then $D'_1 = \max\{k+2,\ 4 + \lfloor \frac{t}{2} \rfloor,\ k + \lfloor \frac{t}{2} \rfloor,\ D\} = \lfloor \frac{t}{2} \rfloor + 4$;
- if $k = 4$ and $t = 2$, then $D'_1 = \max\{k+2,\ \lfloor \frac{t}{2} \rfloor k,\ k + \lfloor \frac{t}{2} \rfloor,\ \frac{k+6}{2}\} = 6$;
- if $k = 4$ and $t = 3$, then $D'_1 = \max\{k+2,\ \lfloor \frac{t}{2} \rfloor k,\ k + \lfloor \frac{t}{2} \rfloor,\ D\} = 6$;
- if $k = 4$ and $t \geq 4$, then

$$D'_1 = \max\{k+2,\ 4 + \lfloor \tfrac{t}{2} \rfloor,\ k + \lfloor \tfrac{t}{2} \rfloor,\ D\} = \begin{cases} 6 & \text{if } t = 4, \\ 4 + \frac{t}{2} & \text{if } t \geq 6 \text{ and } t \text{ is even}, \\ D & \text{if } t \geq 5 \text{ and } t \text{ is odd;} \end{cases}$$

- if $k \geq 5$, k is odd, and $t \in \{2, 3\}$, then

$$D'_1 = \max\{7,\ 4 + \lfloor \tfrac{t}{2} \rfloor,\ 5 + \lfloor \tfrac{t}{2} \rfloor,\ \tfrac{k+7}{2}\} = \begin{cases} 7 & \text{if } k = 5, \\ \frac{k+7}{2} & \text{if } k \geq 7; \end{cases}$$

- if $k \geq 5$, k is even, and $t = 2$, then

$$D'_1 = \max\{7,\ 4 + \lfloor \tfrac{t}{2} \rfloor,\ 5 + \lfloor \tfrac{t}{2} \rfloor,\ \tfrac{k+6}{2}\} = \begin{cases} 7 & \text{if } k = 6, \\ \frac{k+6}{2} & \text{if } k \geq 8; \end{cases}$$

- if $k \geq 5$, and either k is odd and $t \geq 4$ or k is even and $t \geq 3$, then $D'_1 = \max\{7,\ 4 + \lfloor \frac{t}{2} \rfloor,\ 5 + \lfloor \frac{t}{2} \rfloor,\ D\}$. We note that $4 + \lfloor \frac{t}{2} \rfloor < 5 + \lfloor \frac{t}{2} \rfloor$ for all t, and that $7 > 5 + \lfloor \frac{t}{2} \rfloor$ if and only if $t = 3$ and k is even. Hence,
  - for $t = 3$ and k even, $7 > D$ if and only if $k = 6$,
  - for $t \geq 4$, $5 + \lfloor \frac{t}{2} \rfloor > D$ if and only if either t is even and $k \leq 6$ or t is odd and $k = 5$.

  Thus

$$D'_1 = \begin{cases} 5 + \lfloor \frac{t}{2} \rfloor & \text{if either } k = 5 \text{ and } t \geq 4, \text{ or } k = 6, t \geq 4, \text{ and } t \text{ is even}, \\ 7 & \text{if } k = 6 \text{ and } t = 3, \\ D & \text{if } k \geq 8,\ k \text{ is even, and } t = 3, \\ D & \text{if either } k = 6, t \geq 5, \text{ and } t \text{ is odd, or } k \geq 7 \text{ and } t \geq 4. \end{cases}$$

□

3.3. Deleting one inner edge

We let $e = v_0 v_k$ be an inner edge of the generalized Petersen graph $\mathcal{G} = GP[tk, k]$, and consider the graph $G'_1 = \mathcal{G} - e$. In this case, we only need to consider the distances from each of u_0 and v_0 to the inner vertices on the inner cycle containing v_0 and to the outer vertices adjacent to them. Due to the symmetry property, in Lemma 9 we consider $d(u_0, u_{mk})$, $d(u_0, v_{mk})$, $d(v_0, u_{mk})$, and $d(v_0, v_{mk})$, where $m \in \{1, 2, \ldots, \lfloor \frac{t}{2} \rfloor\}$. We treat the cases $m = 1$ and $2 \leq m \leq \lfloor \frac{t}{2} \rfloor$ separately.

Lemma 9 *Let $\mathcal{G}$ denote the graph $GP[tk,k]$ for $t \geq 2$ and $k \geq 2$, and let e be an inner edge of $\mathcal{G}$. The diameter D_1' of the graph $G_1' = \mathcal{G} - e$ is given by*

$$D_1' = \begin{cases} k+2 & \text{if } k \leq 4 \text{ and } t = 2, \\ 7 & \text{if } 5 \leq k \leq 8 \text{ and } t = 2, \\ D & \text{if either } k \leq 8 \text{ and } t \geq 3, \text{ or } k \geq 9, \end{cases}$$

where D is the diameter of $\mathcal{G}$.

Proof We consider $G_1' = \mathcal{G} - e$, where $e = v_0 v_k$ is an inner edge of $\mathcal{G}$, and let its diameter be denoted by D_1'. In the two cases below, we consider $d(u_0, u_{mk})$, $d(u_0, v_{mk})$, $d(v_0, u_{mk})$, and $d(v_0, v_{mk})$, where $m \in \{1, 2, \ldots, \lfloor \frac{t}{2} \rfloor\}$, for the different values of m.

Case 1. $m = 1$

We treat the cases $t = 2$ and $t \geq 3$ separately.

Case 1.1. $t = 2$

The two possible candidates for a shortest path from v_0 to v_k are

$$P^{(1)} := v_0 \to u_0 \to u_1 \to \ldots \to u_k \to v_k \ ;$$

$$P^{(2)} := v_0 \to u_0 \to u_1 \to v_1 \to v_{1+k} \to u_{1+k} \to u_k \to v_k \ .$$

Thus, $|E(P^{(1)})| = k + 2$ and $|E(P^{(2)})| = 7$ and hence $|E(P^{(1)})| < |E(P^{(2)})|$ if and only if $k \leq 4$. Clearly, $d(u_0, u_k) < d(u_0, v_k) = d(v_0, u_k) < d(v_0, v_k)$, and thus it is sufficient to check when $d(v_0, v_k)$ exceeds $D = \lfloor \frac{t+k+3}{2} \rfloor$. Hence, when $t = 2$, we have

$$D_1' = \begin{cases} \max\{k+2,\ D\} & \text{if } k \leq 4, \\ \max\{7,\ D\} & \text{if } k \geq 5, \end{cases} = \begin{cases} k+2 & \text{if } k \leq 4, \\ 7 & \text{if } 5 \leq k \leq 8, \\ D & \text{if } k \geq 9. \end{cases}$$

Case 1.2. $t \geq 3$

There are three possible candidates for a shortest path from u_0 to u_k, namely:

$$P^{(3)} := u_0 \to u_1 \to \ldots \to u_k \ ;$$

$$P^{(4)} := u_0 \to u_1 \to v_1 \to v_{1+k} \to u_{1+k} \to u_k \ ;$$

$$P^{(5)} := u_0 \to v_0 \to v_{-k} \to \ldots \to v_k \to u_k \ .$$

Thus, $|E(P^{(3)})| = k$, $|E(P^{(4)})| = 5$, and $|E(P^{(5)})| = t + 1$. We note that $|E(P^{(3)})| < |E(P^{(4)})|$ if and only if $k \leq 4$. Thus, for $k \leq 4$, we consider $P^{(3)}$ and $P^{(5)}$ and get that

(i) if $k \leq t$, then $|E(P^{(3)})| < |E(P^{(5)})|$ and $d(u_0, u_k) = |E(P^{(3)})| = k < D$,

(ii) if $k \geq t + 1$, then $|E(P^{(5)})| \leq |E(P^{(3)})|$ and $d(u_0, u_k) = |E(P^{(5)})| = t + 1 < D$.

For $k \geq 5$, we consider $P^{(4)}$ and $P^{(5)}$ and get that

(iii) if $t \leq 4$, then $|E(P^{(5)})| \leq |E(P^{(4)})|$ and $d(u_0, u_k) = |E(P^{(5)})| = t + 1 < D$,

(iv) if $t \geq 5$, then $|E(P^{(4)})| < |E(P^{(5)})|$ and $d(u_0, u_k) = |E(P^{(4)})| = 5 < D$.

We remark that:

- when $P^{(5)}$ is the shortest path from u_0 to u_k (corresponding to (ii) and (iii) above), then $d(v_0, v_k) < d(u_0, v_k) = d(v_0, u_k) < d(u_0, u_k) < D$;
- when $P^{(3)}$ or $P^{(4)}$ is the shortest path from u_0 to u_k (corresponding to (i) and (iv) above, respectively), then $d(u_0, v_k) = d(v_0, u_k) \leq d(u_0, u_k) + 1 \leq D$.

Thus, we still need to consider $d(v_0, v_k)$ when either $P^{(3)}$ or $P^{(4)}$ is the shortest path from u_0 to u_k.

- When $k \leq 4$ and $k \leq t-1$, then $d(u_0, u_k) = |E(P^{(3)})| = k \leq D-2$, and thus $d(v_0, v_k) \leq d(u_0, u_k)+2 = D$.
- When $k \leq 4$ and $k = t$, we consider $P^{(5)}$ and obtain $d(v_0, v_k) \leq |E(P^{(5)})| - 2 < D$.
- When $k \geq 5$ and $t = 5$, we consider $P^{(5)}$ and obtain $d(v_0, v_k) \leq |E(P^{(5)})| - 2 < D$.
- When $k \geq 5$ and $t \geq 6$, then $d(u_0, u_k) = |E(P^{(4)})| = 5 \leq D - 2$, and thus $d(v_0, v_k) \leq d(u_0, u_k) + 2 = D$.

Hence, when $t \geq 3$, we have that $d(u_0, u_k)$, $d(u_0, v_k)$, $d(v_0, u_k)$, and $d(v_0, v_k)$ are always at most equal to D.

Case 2. $2 \leq m \leq \lfloor \frac{t}{2} \rfloor$

There are three possible candidates for a shortest path from u_0 to u_{mk}, namely:

$$P^{(6)} := P^{(3)} + (u_k \to v_k \to v_{2k} \to \ldots \to v_{mk} \to u_{mk}) \;;$$

$$P^{(7)} := P^{(4)} + (u_k \to v_k \to v_{2k} \to \ldots \to v_{mk} \to u_{mk}) \;;$$

$$P^{(8)} := u_0 \to v_0 \to v_{-k} \to \ldots \to v_{mk} \to u_{mk} \;.$$

Thus, $|E(P^{(6)})| = k+m+1$, $|E(P^{(7)})| = m+6$ and $|E(P^{(8)})| = 2+t-m$. We note that $|E(P^{(6)})| < |E(P^{(7)})|$ if and only if $k \leq 4$.

Case 2.1. $k \leq 4$

We consider $P^{(6)}$ and $P^{(8)}$ and note that $|E(P^{(6)})| \leq |E(P^{(8)})|$ if and only if $m \leq \frac{t-k+1}{2}$, in which case $t \geq 3 + k$ since $m \geq 2$. Thus,

- when $t \geq 3 + k$, we note that $|E(P^{(6)})|$ is greatest when m is maximum, say $m = \widehat{m}$, where $\widehat{m} = \frac{t-k+1}{2}$ when the parity of t and k is different, while $\widehat{m} = \frac{t-k}{2}$ when t and k have the same parity. Hence,

$$\begin{aligned} d(u_0, u_{mk}) = |E(P^{(6)})| &\leq d(u_0, u_{\widehat{m}k}) = k + \widehat{m} + 1 \\ &= \begin{cases} k + \frac{t-k+1}{2} + 1 = D & \text{if } t \text{ and } k \text{ have a different parity,} \\ k + \frac{t-k}{2} + 1 = D & \text{if } t \text{ and } k \text{ have the same parity,} \end{cases} \end{aligned}$$

- when $t \leq 2 + k$, we note that $|E(P^{(8)})|$ is greatest when m is minimum, i.e. $m = 2$. Hence,

$$d(u_0, u_{mk}) = |E(P^{(8)})| \leq t \leq D.$$

Thus, in either of these two cases, $d(u_0, u_{mk}) \leq D$.

Case 2.2. $k \geq 5$

We consider $P^{(7)}$ and $P^{(8)}$, and note that $|E(P^{(7)})| \leq |E(P^{(8)})|$ if and only if $m \leq \frac{t-4}{2}$, in which case $t \geq 8$ since $m \geq 2$. Thus,

- when $t \geq 8$, we note that $|E(P^{(7)})|$ is greatest when m is maximum, say $m = \widehat{m}$, where $\widehat{m} = \frac{t-4}{2}$ when t is even, while $\widehat{m} = \frac{t-5}{2}$ when t is odd. Hence

$$d(u_0, u_{mk}) = |E(P^{(7)})| \leq d(u_0, u_{\widehat{m}k}) = \widehat{m} + 6 = \begin{cases} \frac{t-4}{2} + 6 \leq D & \text{if } t \text{ is even,} \\ \frac{t-5}{2} + 6 \leq D & \text{if } t \text{ is odd,} \end{cases}$$

- if $t \leq 7$, we note again that $|E(P^{(8)})|$ is greatest when m is minimum, i.e. $m = 2$. Hence

$$d(u_0, u_{mk}) = |E(P^{(8)})| \leq t \leq D.$$

As before, in either of these two cases, $d(u_0, u_{mk}) \leq D$.

Finally, to conclude Case 2, we remark that from the way that $P^{(6)}$, $P^{(7)}$, and $P^{(8)}$ were defined we get that

- $d(u_0, v_{mk}) \leq d(u_0, u_{mk}) - 1 < D$;
- $d(v_0, v_{mk}) \leq d(u_0, v_{mk}) + 1 \leq d(u_0, u_{mk}) \leq D$;
- when $P^{(8)}$ is the shortest path from u_0 to u_k, then $d(v_0, u_{mk}) \leq d(u_0, u_{mk}) - 1 < D$;
- when either $P^{(6)}$ or $P^{(7)}$ is the shortest path from u_0 to u_k and $m \neq \widehat{m}$, then $d(u_0, u_{mk}) \leq D - 1$ and $d(v_0, u_{mk}) = d(u_0, u_{mk}) + 1 \leq D$. On the other hand, when $m = \widehat{m}$ (and thus $d(u_0, u_{mk}) = D$), we note that
  - if $k \leq 4$ and $\widehat{m} = \frac{t-k+1}{2}$ (corresponding to Case 2.1 when t and k have a different parity), then $|E(P^{(8)})| = 2 + t - \left(\frac{t-k+1}{2}\right) = \frac{t+k+3}{2}$, and thus $d(v_0, u_{mk}) = |E(P^{(8)})| - 1 < D$;
  - if $k \leq 4$ and $\widehat{m} = \frac{t-k}{2}$ (corresponding to Case 2.1 when t and k have the same parity), then $|E(P^{(8)})| = 2 + t - \left(\frac{t-k}{2}\right) = \frac{t+k+4}{2}$, and thus $d(v_0, u_{mk}) = |E(P^{(8)})| - 1 \leq D$;
  - if $k \geq 5$ and $\widehat{m} = \frac{t-4}{2}$ (corresponding to Case 2.2 when t is even), then $|E(P^{(8)})| = 2 + t - \left(\frac{t-4}{2}\right) = \frac{t+8}{2}$, and thus $d(v_0, u_{mk}) = |E(P^{(8)})| - 1 < D$;
  - if $k \geq 5$ and $\widehat{m} = \frac{t-5}{2}$ (corresponding to Case 2.2 when t is odd), then $|E(P^{(8)})| = 2 + t - \left(\frac{t-5}{2}\right) = \frac{t+9}{2}$, and thus $d(v_0, u_{mk}) = |E(P^{(8)})| - 1 \leq D$.

Hence, in either of the above cases, $d(v_0, u_{mk}) \leq D$, completing our proof. □

3.4. Threshold values

From the results discussed above, we note that for some values of k there are threshold values $T_1(k)$ of t such that, upon deleting one edge e from $\mathcal{G} = GP[tk, k]$, the diameter of $G_1' = \mathcal{G} - e$ is equal to the diameter D of $\mathcal{G}$ for all $t \geq T_1(k)$. This result is presented in the next theorem, the proof of which is a direct consequence of Lemmas 6 to 9.

Theorem 10 *Let $\mathcal{G}$ denote the graph $GP[tk, k]$ for $t \geq 2$ and $k \geq 1$, and let e be any edge of $\mathcal{G}$. The threshold values $T_1(k)$ of t for which the diameter D_1' of the graph $G_1' = \mathcal{G} - e$ is equal to the diameter D of $\mathcal{G}$ are given by*

- $T_1(1) = 4$,
- $T_1(k) = 6$ *if* $k \geq 7$ *and* k *is odd,*
- $T_1(k) = 5$ *if* $k \geq 8$ *and* k *is even.*

4. Conclusion

In this work, we have determined that graphs belonging to the family of generalized Petersen graphs are a solution to the $(\Delta, D, D, 1)$-problem. In particular, we have shown that for sufficiently large values of t and k (as given in Theorem 10) the diameter of $GP[tk, k]$ remains unchanged upon deleting one edge.

Interconnection networks are frequently modeled by graphs that have a small girth (less than the diameter) [14] and that are 3-regular and 3-connected [16]. Among the graphs having the same connectivity parameters, the most reliable ones are those that are super-connected and super-edge-connected (that is, graphs in which every smallest set of vertices, or, respectively, edges, that disconnects the graph isolates a vertex). In [3] it is shown that the class of generalized Petersen graphs $GP[tk, k]$ are super-connected and super-edge-connected for all $k \geq 1$ and $n > k$ except when $n \in \{2k, 3k\}$. All the above-mentioned properties, coupled with the main result of this work (Theorem 10) showing that the diameter generally remains unchanged on the deletion of an edge, contribute towards making $GP[tk, k]$ one of the families of graphs that are the most ideal candidates for modeling interconnection networks.

Furthermore, it is known that, in general, $D'_2 \leq 3D - 1$ and the bound is attainable [15]. We conjecture that for $GP[tk, k]$ the deletion of two edges changes the value of the diameter by at most one, that is $D'_2 = D+1$, except for some small cases. This result would make this class of graphs even more reliable.

Acknowledgment

The authors would like to thank the referees for their comments and suggestions, which helped us improve this paper.

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