symmetry

Review

Gravitational Light Bending in Weyl Gravity and Schwarzschild–de Sitter Spacetime

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Abstract: The topic of gravitational lensing in the Mannheim–Kazanas solution of Weyl conformal gravity and the Schwarzschild–de Sitter solution in general relativity has featured in numerous publications. These two solutions represent a spherical massive object (lens) embedded in a cosmological background. In both cases, the interest lies in the possible effect of the background non-asymptotically flat spacetime on the geometry of the local light curves, particularly the observed deflection angle of light near the massive object. The main discussion involves possible contributions to the bending angle formula from the cosmological constant Λ in the Schwarzschild–de Sitter solution and the linear term γr in the Mannheim–Kazanas metric. These effects from the background geometry, and whether they are significant enough to be important for gravitational lensing, seem to depend on the methodology used to calculate the bending angle. In this paper, we review these techniques and comment on some of the obtained results, particularly those cases that contain unphysical terms in the bending angle formula.

Keywords: Weyl gravity; geodesics; light bending

Citation: Sultana, J. Gravitational Light Bending in Weyl Gravity and Schwarzschild–de Sitter Spacetime. *Symmetry* **2024**, *16*, 101. https://doi.org/10.3390/sym16010101

Academic Editor: Kazuharu Bamba

Received: 26 October 2023
Revised: 26 December 2023
Accepted: 9 January 2024
Published: 14 January 2024

1. Introduction

During the last decade, the subject of the bending of light by a gravitational lens embedded in a non-asymptotically flat spacetime, particularly the effect of the background spacetime on the light bending angle, has attracted a lot of interest. One of the most discussed issues is that of whether the cosmological constant Λ in the Schwarzschild–de Sitter (SdS) spacetime contributes to the bending of light. It has been claimed [1,2] that since the constant Λ drops out of the null geodesic equations for the Schwarzschild–de Sitter spacetime then it should not feature in the formula for the bending angle, even though its effect would still be indirectly used in gravitational lensing, considering that the angular diameter distances in the lensing formula depend on Λ. However it was Rindler and Ishak [3,4] who first suggested that the bending angle of light in asymptotically non-flat spacetimes cannot be simply calculated by finding the angle between the asymptotes $r \to \infty$ of the photon's trajectories (commonly referred to as Weinberg's method [5]) as is normally done in Schwarzschild geometry, for example. So, they utilized a different method to calculate the angle of deflection and obtained a formula that contained a direct contribution from the cosmological constant. The reactions to this proposal were mixed [6–13] with some against and some in favor of Rindler and Ishak's method, while a few others still questioned whether Λ contributes effectively to lensing. This debate has since calmed down, with the recent general consensus being that the cosmological constant does indeed play a role in gravitational lensing, but that the effect is way too tiny to be significant and can thus be ignored for practical purposes, given that its magnitude is much smaller when compared with other lensing affects such as aberration and uncertainties in cosmological distances [14,15]. In the time that the debate surrounding the SdS spacetime has been going on, another spacetime that has also attracted a lot of interest with regards to light bending is the static and spherically symmetric vacuum solution to conformal Weyl gravity, which was first derived by Riegert [16] but is more commonly know as the Mannheim–Kazanas

(MK) solution [17,18]. Unlike general relativity, conformal Weyl gravity is a conformally invariant theory of gravitation, meaning that it has the added symmetry that, if a particular metric is a solution of the field equations of the theory, then any conformal transformation of this metric will also be a solution of Weyl gravity. The Mannheim–Kazanas solution contains a linear potential term γr in its lapse function. It has been shown that this can predict flat galactic rotational curves without the need to include dark matter [19–23], whose exact nature is still a mystery. Recent studies [24] have, however, criticized this claim. When calculating the deflection of light in this spacetime, Edery and Paranjape [25] (see also [26,27]) showed that the γr term in the metric leads to a negative contribution to the total bending angle that is linearly proportional to the impact parameter, meaning that the further away the light ray is from the lens, the greater is this contribution to the bending angle. This, therefore, renders the spacetime unphysical. Moreover, this contribution requires that the constant γ should have the opposite sign used for the prediction of flat galactic rotation curves. It turned out that this paradoxical result arises from the fact that when using the standard formula for bending of light given by Weinberg, the authors incorrectly assumed that the MK spacetime is asymptotically flat. It was later shown [28,29] by applying Rindler and Ishak's method that the contribution of the linear term in the MK metric is inversely proportional to the impact parameter and therefore is practically insignificant for lensing purposes, considering the small magnitude of constant γ derived earlier from the fitting of galactic rotational curves. However the issue of the bending of light in Weyl gravity, particularly the magnitude of the contribution of the linear term in the metric, remains unresolved. This has been revisited several times in recent literature [30–36], and so far remains inconclusive. The main disagreements arise from the different order of the approximations used to derive the bending angle formula, the different representations of the mass of the lens in terms of parameters in the MK-metric, and the application of various alternative geometric techniques for calculating the bending angle in non-asymptotically flat spacetimes. For example, another less commonly used method is that based on the Gauss–Bonnet theorem applied for the optical metric generated from the spacetime metric $g_{\mu\nu}$, which was originally used for asymptotically flat spacetimes [37,38] but has recently been generalized and applied to non-asymptotically flat spacetimes [39–41], specifically to the SdS and MK solutions. In a recent paper Kaşikçi and Deliduman [34], used Weinberg's method to obtain from first principles the angle of deflection of light in the MK-spacetime, taking into consideration the non-flat background by limiting the integration to the position of the cosmological event horizon in this spacetime. In the next section, we introduce the theory of Weyl gravity through its action and field equations and present different forms of the MK-metric. In Section 3, we present the different methods used to calculate the bending of light in the SdS and MK spacetimes and obtain the bending angle formula in each case. These results are discussed in detail, followed by a conclusion in Section 4. In this paper, we use geometric units in which $G = c = 1$.

2. The Action of Weyl Gravity

Weyl (conformal) gravity is based on the principle of local conformal invariance of spacetime under local conformal deformations $g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x)$, where $\Omega(x)$ is a smooth strictly positive function of spacetime coordinates. Instead of requiring that the theory be no higher than second order, as in the case of the Einstein–Hilbert action, this is used as the supplementary condition that fixes the gravitational action. This results in fourth-order equations of motion for the gravitational field. Nonetheless, using this local invariance principle in order to determine the gravitational action, besides being in line with the way actions are chosen in field theory, uniquely selects the action of Weyl gravity amongst all other fourth-order theory actions. Moreover, it precludes from the action additional terms in the form of the Einstein–Hilbert terms, thereby providing no specific limit at which this theory will reduce to the "standard" general relativity, an approach normally used when considering the effects of higher-order terms in gravitational action.The restrictive conformal invariance leads to the unique conformally invariant action

$$\begin{aligned} I_W &= -\frac{1}{2\alpha}\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} \\ &= -\frac{1}{\alpha}\int d^4x(-g)^{1/2}[R_{\mu\kappa}R^{\mu\kappa} - (R^{\nu}_{\nu})^2/3] \\ &\quad +\text{a total derivative}, \end{aligned} \tag{1}$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and α is a purely dimensionless coefficient. Varying the action in (1) with respect to the metric leads to the field equations

$$\frac{1}{\sqrt{-g}}g_{\mu\alpha}g_{\nu\beta}\frac{\delta I_W}{\delta g_{\alpha\beta}} = -\frac{1}{\alpha}W_{\mu\nu}, \tag{2}$$

where

$$W_{\mu\nu} = 2C^{\alpha\ \ \beta}_{\ \mu\nu\ ;\beta\alpha} + C^{\alpha\ \ \beta}_{\ \mu\nu}R_{\alpha\beta} \tag{3}$$

is the Bach tensor. In the presence of a source $T_{\mu\nu}$, the full field equations are obtained through variation with respect to the metric of the total action $I = I_W + I_M$, where I_M is the action corresponding to the source. This gives

$$W_{\mu\nu} = \frac{\alpha}{2}T_{\mu\nu}, \tag{4}$$

where $T_{\mu\nu} = 2(-g)^{-1/2}\delta I_M/\delta g^{\mu\nu}$ is a conformally invariant stress–energy tensor. From the definition of $W_{\mu\nu}$, it can be seen that this vanishes when $R_{\mu\nu}$ is zero, and hence any vacuum solution of Einstein's field equations is also a vacuum solution of Weyl gravity. However, the converse is not necessarily true. Despite the fourth-order field equations of Weyl gravity and the fact that, in general, these are highly nonlinear, a number of exact solutions [42–47] to these equations have been found in cases of spherical symmetry, axisymmetry, and cylindrical symmetry. The exact static and spherically symmetric vacuum solution for Weyl gravity is given, up to a conformal factor, by the following line element [16,17]

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \frac{\Lambda}{3}r^2, \tag{6}$$

and β, γ, and Λ are integration constants. This solution includes the Schwarzschild metric $(\gamma = \Lambda = 0)$ and the Schwarzschild–de Sitter $(\gamma = 0)$ metric as special cases, with the latter requiring the presence of a cosmological constant in general relativity. The constant γ has dimensions of acceleration. Therefore, the solution provides a characteristic, constant Rindler-like acceleration, without the need to introduce one at the Lagrangian, such as in the relativistic implementation of MOND with TeVeS [48]. Although the magnitude and nature of γ remain uncertain, the fitting of galactic rotational curves suggests [17] that $\gamma \simeq 1/R_H$, where R_H is the Hubble length. This means that the effects of this acceleration are comparable to those due to the Newtonian potential term (with $\beta\gamma << 1$) $2\beta/r \equiv r_s/r$ (r_s is the Schwarzschild radius), on length scales given by

$$r_s/r^2 \simeq \gamma \simeq 1/R_H \ \text{ or } \ r \simeq (r_s\,R_H)^{1/2}. \tag{7}$$

By using the reparameterization $\beta = \frac{1-\sqrt{1-6\gamma m}}{3\gamma}$, the MK-solution can also be written in the form

$$B(r) = \sqrt{1-6m\gamma} - \frac{2m}{r} + \gamma r - \frac{\Lambda}{3}r^2. \tag{8}$$

Besides the bending of light (which is discussed in the next section) other classical tests such as the time delay [25] and perihelion precession [49] have also been obtained in Weyl gravity.

3. Bending of Light in the SdS and MK Spacetimes

3.1. SdS Spacetime

The null geodesic equations $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$, where $\dot{x}^\alpha = dx^\alpha/d\lambda$; λ being an affine parameter, for the metric in Equation (5) reduce to [5]

$$\frac{du}{d\phi} = \sqrt{\frac{1}{R^2} - B(u)u^2}, \tag{9}$$

where $u \equiv 1/r$, $R \equiv L/E$ is the impact parameter, and E and L are the constants of motion representing the total energy and angular momentum, respectively. In an asymptotically flat spacetime such as the Schwarzschild solution, where the source and observer are assumed to be located at infinity ($r = \infty; u = 0$), the coordinate angle difference at the point of closest approach $r = r_0$ (or $u = u_0$ where $\frac{du}{d\phi}|_{u=u_0} = 0$ of the light ray from the lens with respect to the position of the source or observer is given by

$$\Delta\phi = \phi(u_0) - \phi(0) = \int_{u_0}^{0} \frac{du}{\sqrt{\frac{1}{R^2} - B(u)u^2}}. \tag{10}$$

Hence, the total bending angle is given by

$$\Delta\phi = 2\phi_\infty = 2|\phi(u_0) - \phi(0)| - \pi. \tag{11}$$

So, basically, in this case the total bending angle is obtained by finding the angle between the asymptotes $r \to \infty$ of the photon's trajectories, as shown in Figure 1.

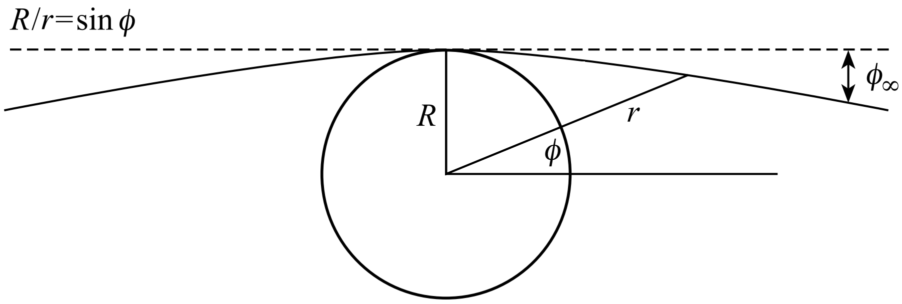

Figure 1. The deflected light trajectory with the one-sided bending angle ϕ_∞. (adapted from [50]).

This is the standard method for calculating the exact bending angle of light in asymptotically flat spacetimes and is commonly referred to as Weinberg's method. For example, applying this for the Schwarzschild metric with $B(u) = 1 - 2mu$ results [5] in a total bending angle of $\Delta\phi_{\rm sch} \approx 4m/R$ if one ignores higher-order terms in m/R. In this case, the relation between the impact parameter R and the distance of closest approach $r_0 = 1/u_0$ is obtained by equating $du/d\phi$ to zero in (9) to obtain $r_0^3 - R^2(r_0 - 2m) = 0$, whose largest root is given by [51]

$$r_0 = \frac{2R}{\sqrt{3}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{-3^{3/2}m}{R}\right)\right]. \tag{12}$$

This method has also been applied to obtain the light bending angle formula in non-asymptotically flat spacetimes, such as the SdS spacetime and the MK-solution of Weyl gravity [25–27], which is seen in the next subsection. For the case of the SdS spacetime where $B(r) = 1 - 2m/r - \Lambda r^2/3$ the cosmological constant Λ drops out from the bending angle formula, hence giving the same expression $\Delta\phi_{\rm sch}$ as in the case of the Schwarzschild solution. In Ref. [34], Weinberg's method was slightly modified by limiting the integration in (10) only up to the position of the cosmological event horizon $r = r_h$ (or $u = u_h$) instead of $u = 0$. This was based on the fact that this horizon marks the boundary between causally unconnected regions and so it would not make sense to extend the integration beyond this point. The position u_h is obtained by finding the largest root of $B(r) = 0$. By using this approach, the authors in Ref. [34] utilized an asymptotic expansion of the elliptic integral of the first kind in (10) to obtain the following approximation for the bending angle of light in the SdS spacetime written in terms of the parameters m and Λ and the distance of closest approach r_0

$$\begin{aligned}\Delta\phi &= -2\sqrt{\frac{\Lambda}{3}}r_0 + \frac{m}{r_0}\left(4 - 2\sqrt{\frac{\Lambda}{3}}r_0 - 2\frac{\Lambda}{3}r_0^2\right) + \frac{m^2}{r_0^2}\left(\frac{15}{4}\pi - 4\right) \\ &\quad - \frac{m^2}{r_0^2}\left(3\sqrt{\frac{\Lambda}{3}}r_0 + 2\frac{\Lambda}{3}r_0^2\right) + \cdots . \end{aligned} \tag{13}$$

Again, the point of closest approach $r = r_0$ (or $u = u_0$) is obtained by finding the largest root of $dr/d\phi = 0$ (or equivalently $du/d\phi = 0$.) in (9) with $B(r) = 1 - 2m/r - \Lambda r^2/3$ as was done in (12). Using the photon orbit equation, this can be approximated by [3]

$$\frac{1}{r_0} = \frac{1}{R} + \frac{m}{R^2}. \tag{14}$$

Rindler and Ishak [3] proposed a different method, which makes use of the local deflection of the photon at each point along its entire trajectory when calculating the total bending angle. So, following [3], one considers the subspace $\theta = \pi/2$, $t =$ const. in (5), and let ψ be the angle between the two directions d^α and δ^β in the photon planar trajectory, as shown in Figure 2.

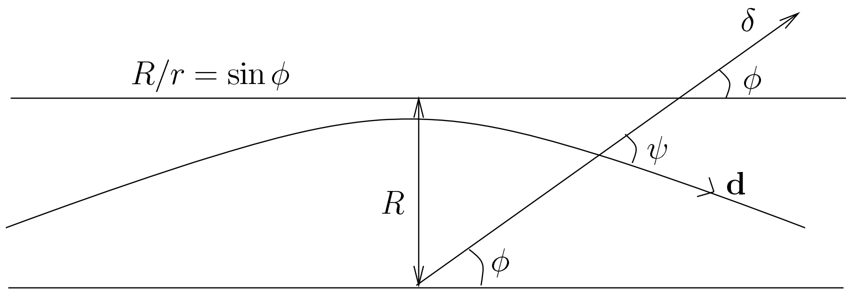

Figure 2. The deflected photon trajectory with the one-sided bending angle given by $\epsilon = \psi - \phi$ (adapted from [3]).

Then, the angle ψ that the photon orbit makes with the position $\phi =$ const. is given by

$$\cos\psi = \frac{g_{\alpha\beta} d^\alpha \delta^\beta}{\sqrt{g_{\alpha\beta} d^\alpha d^\beta}\sqrt{g_{\alpha\beta}\delta^\alpha\delta^\beta}}, \tag{15}$$

where $d^\alpha = (dr, d\phi)$, and $\delta^\alpha = (\delta r, 0)$ and $g_{\alpha\beta}$ denote the metric on the $\theta = \pi/2$, $t =$ const. submanifold. Substituting the metric (5) into (15) gives

$$\cos\psi = \frac{|A(r,\phi)|}{(A^2 + B(r) r^2)^{1/2}}, \tag{16}$$

or equivalently

$$\tan\psi = \frac{B(r)^{1/2} r}{|A(r,\phi)|}, \tag{17}$$

where $A(r, \phi) = \frac{dr}{d\phi}$. At a general point (r, ϕ) along the photon's orbit, the one-sided bending angle is given by $\epsilon = \psi - \phi$. The total one-sided bending angle occurs at $\phi = 0$ or $\epsilon = \psi_0$. For the SdS metric, this formula gives the following expression for the total bending angle, expressed in terms of the impact parameter R instead of the distance of the closest approach r_0 [4],

$$\Delta\phi = 2\psi_0 = \frac{4m}{R} + \frac{15\pi}{4}\frac{m^2}{R^2} + \frac{305}{12}\frac{m^3}{R^3} - \frac{\Lambda R^3}{6m}. \tag{18}$$

A third method for calculating the bending of light using a gravitational lens involves the application of the Gauss–Bonnet theorem in the spatial domain defined by the optical metric, in which the light ray is represented by a spatial curve. Without loss of generality, the spatial domain of the photon trajectory is taken to be the equatorial plane $\theta = \pi/2$, so that using $ds^2 = 0$ in (5), the optical metric for this two dimensional surface is given by

$$dt^2 = \frac{dr^2}{B(r)^2} + \frac{r^2}{B(r)}d\phi^2. \tag{19}$$

This method was first proposed by Gibbons and Werner [37] for asymptotically flat spacetimes and was later generalized by Ishihara et al. [38] and Takizawa et al. [40,41] to non-asympototically flat spacetimes in which the source and observer are at finite distances from the lens. Unlike the Rindler–Ishak method, in this case, the source, lens, and observer are not assumed to be co-aligned, as shown in Figure 3.

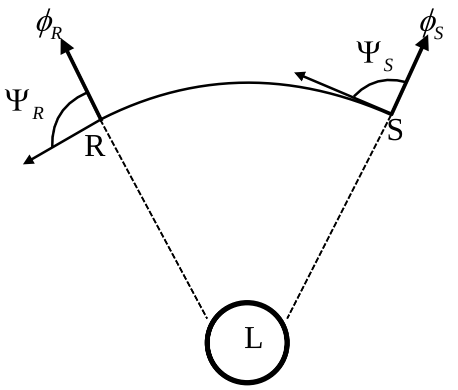

Figure 3. The observer R, source S, and lens L, with Ψ_R and Ψ_S being the angles between the light ray and the corresponding radial directions at these positions. The coordinate angular separation between the source and observer is $\phi_{RS} = \phi_R - \phi_S$. (adapted from [40]).

The bending angle is then defined by [38]

$$\Delta\phi = \Psi_R - \Psi_S + \phi_{RS}, \tag{20}$$

where Ψ_R and Ψ_S are the angles between the spatial trajectory of the light ray and the radial directions of the observer and source, respectively, while ϕ_{RS} is the angular coordinate separation of the source and observer. These angles are obtained by applying the Gauss–Bonnet theorem on the spatial domain of the photon trajectory shown in Figure 3. In general, for a two dimensional orientable surface T with a piecewise smooth boundary consisting of smooth curves ∂T_a joined together, the Gauss–Bonnet theorem [52] states

$$\int_T K\, dS + \sum_{a=1}^{N} \int_{\partial T_a} \kappa_g\, dl + \sum_{a=1}^{N} \theta_a = 2\pi, \tag{21}$$

where K denotes the Gaussian curvature of the surface T, κ_g is the geodesic curvature of the boundary curves ∂T_a, and θ_a denotes the jump angles between the curves making up

∂T_a. So, using this method, the authors in Ref. [40] obtained the following expression for the bending angle of light in the SdS spacetime:

$$
\begin{aligned}
\Delta\phi &= \frac{2m}{R}\left(\sqrt{1-R^2u_S^2}+\sqrt{1-R^2u_R^2}\right)-\frac{R\Lambda}{6}\left(\frac{\sqrt{1-R^2u_S^2}}{u_S}+\frac{\sqrt{1-R^2u_R^2}}{u_R}\right) \\
&\quad +\frac{mR\Lambda}{6}\left(\frac{1}{\sqrt{1-R^2u_S^2}}+\frac{1}{\sqrt{1-R^2u_R^2}}\right)+\mathcal{O}(m^2,\Lambda^2),
\end{aligned} \tag{22}
$$

where u_R and u_S are the reciprocals of the radial coordinates of the observer and source, respectively. As expected, irrespective of the method used, the leading term in the bending angle formulae given by (13), (18) and (22) coincides with $\Delta\phi_{\rm Sch} = 4m/R$. However, the leading contribution from the cosmological constant Λ is different in all three cases. This is discussed in Section 4.

3.2. MK Spacetime

Applying Weinberg's method for the the MK-spacetime (8), the authors in Ref. [25] obtained the following bending angle formula (ignoring higher-order terms in β/r or m/r)

$$
\Delta\phi = \frac{4m}{R} - \gamma R. \tag{23}
$$

The second term in (23) is negative and so diminishes the amount of bending when compared with the Schwarzschild or SdS spacetimes. It is also linear in the impact parameter R and therefore its effect increases with the distance of the light ray from the lens, such that for sufficiently large R it will completely cancel the Schwarzschild contribution, resulting in a negative deflection. This unexpected and unrealistic result is attributed to the fact that the authors in [25] used a method for calculating the deflection angle that assumes that the spacetime is asymptotically flat, with both source and observer located at infinity.

Using the modified Weinberg method and assuming that the source, lens and observer lie within the cosmological horizon of the MK spacetime, the authors in Ref. [34] obtained the following bending angle formula in terms of the distance of closest approach r_0 (which is related to the impact parameter R by (14))

$$
\begin{aligned}
\Delta\phi &= m_0\left(4-2\sqrt{\frac{\Lambda_0}{3}}-2\frac{\Lambda_0}{3}\right)-2\sqrt{\frac{\Lambda_0}{3}}+\gamma_0\sqrt{\frac{\Lambda_0}{3}} \\
&\quad +m_0^2\left(\frac{15\pi}{4}-4-3\sqrt{\frac{\Lambda_0}{3}}-2\frac{\Lambda_0}{3}\right)+m_0\gamma_0\left(2+\frac{\Lambda_0}{3}\right) \\
&\quad +m_0^2\gamma_0\left(\frac{15\pi}{4}-4-\frac{3}{2}\sqrt{\frac{\Lambda_0}{3}}\right)+\cdots,
\end{aligned} \tag{24}
$$

where the dimensionless parameters are defined by $m_0 = m/r_0$, $\gamma_0 = \gamma r_0$, and $\Lambda_0 = \Lambda r_0^2$.

In Ref. [28], the Rindler–Ishak method is used to obtain the bending angle for the MK-metric given in (6), in which the source, lens, and observer are assumed to be co-aligned. This was performed using the formula in (17) evaluated at $\phi = 0$ corresponding to the position $r_{\phi=0} = 2R^2/(2\beta(2-3\beta\gamma)-\gamma R^2)$, which in terms of the parameter m introduced in (8) can be written as $r_{\phi=0} = 2R^2/(4m - R^2\gamma)$. The resulting bending angle to first-order in γ and Λ for the metric (6) is given by [28]

$$
\Delta\phi = \frac{4\beta}{R} - \frac{2\beta^2\gamma}{R} - \frac{\Lambda R^3}{6\beta}. \tag{25}
$$

The first and last terms in the above formula are just the Schwarzschild and cosmological contributions, respectively, obtained for the SdS spacetime as given in (18). One notes that the contribution to the bending angle from the linear term in the metric is now inversely proportional to the impact parameter, and like the cosmological term it also has a negative sign, meaning that the amount of bending is suppressed and the effect diminishes with a closer approach distance (or impact parameter) from the lens. Considering that it had been shown earlier that the parameter γ is related to the gravitational source (see Equation (18) in Ref. [45] and also Ref. [21–23]), the authors in Ref. [31] pointed out that this term should be positive, so that it enhances the bending angle, in the same way that it provides an explanation of the flat galactic rotational curves in the absence of dark matter. This issue was settled [53,54] when it was shown that the sign of the linear term γr in the metric (6) can be reversed using a conformal transformation $ds^2 \to \tilde{\Omega} ds^2$ followed by a coordinate transformation

$$r' = r\sqrt{\tilde{\Omega}}, \tag{26}$$

where in the weak field approximation $\beta/r << 1$, $\beta\gamma << 1$, the necessary conformal factor is given by $\tilde{\Omega} = 1 - 2\gamma r$. In this gauge, the metric potential in (5) takes the form

$$B(r') = 1 - \frac{\beta(2 - 3\beta\gamma)}{r'} - 3\beta\gamma - \gamma r' - \frac{\Lambda}{3} r'^2. \tag{27}$$

Then, since null geodesics are insensitive to conformal transformations, it follows that the γ contribution to the light bending angle in (25) in this gauge will have the opposite sign and thus leads to a deflection towards the lens. One has to point out that the conformally related metrics in Equation (5) with the lapse function given by (6) and (27), respectively, are not the same metric, but considering that Weyl gravity is conformally invariant, both are vacuum solutions of the theory. A few other slightly different formulae for the bending angle that were also derived using Rindler and Ishak's method can be found in Refs. [29,31,32]. The main differences in these alternative formulae arise from the different order of the approximations taken, the association of different parameters (m vs. β) with the geometric mass of the lens, and the different points in the derivations where higher powers of m/R, β/R, γ and Λ are discarded. For example, in Ref. [31] the authors used Rindler and Ishak's method for the MK-metric written in the form (8) and obtained the following formula

$$\Delta\phi = \frac{4m}{R} + \frac{15m^2\gamma}{R} + \frac{2mR\Lambda}{3} - \frac{\Lambda R^3}{6m}. \tag{28}$$

However, the most important thing is that, in all these formulae, the first-order contribution from γ to the bending angle is always inversely proportional to the impact parameter R, so that just like the Einstein contribution $4m/R$, it decreases with distance from the lens. This would be expected if γ is really associated with the gravitational lens itself.

In Ref. [40], the authors also applied the method based on the Gauss–Bonnet theorem to obtain the bending angle of light in the MK-spacetime. In this case, using the fact that $\beta\gamma << 1$, the authors used a simplified form of the lapse function in (5) given by $B(r) = 1 - 3\beta\gamma - 2\beta/r + \gamma r - \Lambda r^2/3$, and to simplify their analysis they also considered the case $\Lambda = 0$. The obtained bending angle formula is given by

$$\begin{aligned}\Delta\phi \quad = \quad & \frac{2\beta}{R}\left(\sqrt{1 - R^2u_S^2} + \sqrt{1 - R^2u_R^2}\right) - \beta\gamma\left(\frac{Ru_S}{\sqrt{1 - R^2u_S^2}} + \frac{Ru_R}{\sqrt{1 - R^2u_R^2}}\right) \\ & + \mathcal{O}(\beta^2, \gamma^2),\end{aligned} \tag{29}$$

where u_R and u_S are again the reciprocals of the radial coordinates of the observer and source, respectively, as shown in Figure 3. As in the SdS case, the leading terms in all light bending angle formulae, irrespective of the method used, coincide with the Schwarzschild bending angle $\Delta\phi_{\text{Sch}}$, and the main difference lies in the way the linear term γr contributes to the bending and the magnitude of this contribution with respect to $\Delta\phi_{\rm Sch}$. This is discussed in the next section.

4. Discussion

In this section, we will compare the bending angle formulae obtained using the different methods as described in the previous section. So, starting with the SdS spacetime and comparing (18) with (13), we note that the majority of the bending in both cases equates to the Schwarzschild contribution represented by the first term in both expressions, which apart from the difference between R and r_0 takes the same form in both cases, as expected. In addition, in both cases, the first-order contribution from the cosmological constant is negative, so that the cosmological constant Λ diminishes the bending (see also Ref. [9,55]). However, due to the different methods used to calculate the bending angle, this term is different in the two cases. As already mentioned above, Kaşikçi and Deliduman based their calculation on Weinberg's method given by Equation (10) through restricting the upper limit of integration to the position of the cosmological event horizon, which in the case of the SdS spacetime is given by $u_h = 1/r_h \sim \sqrt{\Lambda/3}$. In Rindler and Ishak's derivation, it is assumed that the observer, lens, and source are collinear. The position $\phi = 0$ corresponding to the one-sided total bending angle occurs at $r_{\phi=0} = R^2/2m$, where $\epsilon = \psi_0$. They also assumed that the photon trajectory intersects the optic axis (the line through the coaligned source, lens, and observer where $\phi = 0$) within the cosmological horizon, such that $R^2/2m < \sqrt{\Lambda/3}$. Otherwise the trajectory would connect causally unconnected regions of spacetime. In fact, this is the same reason why Kaşikçi and Deliduman assumed that the deflection of the photon trajectory happens entirely within the cosmological horizon.

Considering the small magnitude of the cosmological constant, one would expect that the effects of the first-order terms in Λ in Equations (13) and (18) would be much smaller than the first-order Schwarzschild contribution $\Delta\phi_{\rm sch} \sim 4m/R$, so that in reality the presence of these terms in the bending angle formulae would not effect gravitational lensing [14,15]. However, by considering examples of galaxies and galaxy clusters and obtaining the magnitudes of these cosmological contributions (as was done in Table 1 of Ref. [4]), one finds that this is not always the case. Therefore, as an example, if we take the case of the galaxy cluster Abell 2744 [56,57] (see table 1 in Ref. [4]) having Einstein radius $R_E = 96.4$ Kpc and geometric mass $m = 1.97 \times 10^{13}\ \mathrm{M_\odot h^{-1}}$, and use the following values of the cosmological parameters $\Lambda = 1.1056 \times 10^{-52}\ \mathrm{m^{-2}}$ (obtained using $H_0 = 67.66 \pm 0.42\ \mathrm{kms^{-1}/Mpc}$, $\Omega_\Lambda = 0.6889 \pm 0.0056$ [58]), we obtain

$$\frac{4m}{R} = 5.510 \times 10^{-5}; \quad \frac{15\pi}{4}\frac{m^2}{R^2} = 2.235 \times 10^{-9}; \quad \frac{\Lambda R^3}{6m} = 1.184 \times 10^{-5};$$
$$2\sqrt{\frac{\Lambda}{3}}r_0 = 3.612 \times 10^{-5}. \tag{30}$$

Surprisingly, from the above numerical values, it can be seen that the first-order contributions from the cosmological constant to the bending angle in Equations (13) and (18) are greater than the second-order term m^2/R^2 and are indeed of the same order of magnitude as $\Delta\phi_{\rm sch}$. Considering that the SdS solution is not a realistic model of a gravitational lens embedded in a cosmological background, this in no way contradicts the conclusions reached in Ref. [14,15] about the insignificant effect from the cosmological constant in practical gravitational lensing. In another paper, Ishak et al. [4] applied the same method that was used earlier for (18) to the case of a SdS vacuole matched to a Friedmann–Robertson–Walker (FRW) background (also known as the Einstein–Straus model [59]), where the source and observer are assumed to lie inside the SdS vacuole, so that the deflection of the light trajectory happens entirely within the vacuole (the SdS vacuole model was also studied in

Ref. [60]). In this case, the source, lens, and observer were also assumed to be coaligned and the obtained bending angle formula was given by

$$\Delta\phi = \frac{4m}{R} + \frac{15\pi}{4}\frac{m^2}{R^2} + \frac{305}{12}\frac{m^3}{R^3} - \frac{\Lambda R r_b}{3}. \tag{31}$$

The contribution from Λ now involves the radial coordinate r_b at the boundary of the vacuole, which can be obtained using the appropriate matching conditions there, i.e.,

$$r_{b\,(\mathrm{SdS})} = a(t) r_{b\,(\mathrm{FRW})} \tag{32}$$

and

$$m_{\mathrm{SdS}} = \frac{4\pi}{3} r^3_{b\,(\mathrm{SdS})} \rho_{\mathrm{m}}, \tag{33}$$

where $a(t)$ is the scale factor in the FRW metric, and ρ_{m} is the density of the Universe at the instance when light passes by the lens, which is positioned at the center of the SdS vacuole. From (32), it is evident that while the size of the hole $r_{b\,(\mathrm{FRW})}$ is fixed in comoving coordinates, the physical size of the hole $r_{b\,(\mathrm{SdS})}$ increases in static coordinates, due to the expansion of the Universe. Taking the same example of the Abell 2744 galaxy cluster mentioned above, we find that $\Lambda R r_b / 3 = 1.425 \times 10^{-8}$. This is still larger than the second-order term m^2/R^2, but it is now significantly smaller than the Schwarzschild term $\Delta\phi_{\mathrm{sch}}$. In a way, this would be expected when considering that the position of the source and observer with respect to the lens determines the total bending angle of light. For this particular example, one finds that the three coordinate radii satisfy the inequalities $r_b < r_{\phi=0} < r_h$. Hence, although the exact de-Sitter spacetime does not cause any bending of null trajectories, since it is conformally flat, the SdS being *asymptotically* conformally flat would still cause a deflection of light even at large distances from the lens itself. Having said this, one should point out the fact that the SdS vacuole matched to FRW spacetime considered by Ishak et al. [4] is still far from being a realistic model of a gravitational lens embedded in a cosmological background. In a recent paper, Hu et al. [15] (see also Ref. [14]) considered an improved variation of this model by allowing the source and observer to be within the FRW background, so that they are comoving with the expansion of the universe. They also included the effect of the change in the size of the SdS vacuole as light propagates through it from the source to the observer. In this case, it was found that the contribution from Λ was even smaller than that obtained earlier by Ishak et al. [4]. This contribution can be almost entirely attributed to the Λ dependence of the angular diameter distances in the lensing equation $\Delta\phi = D_S \theta_E / D_{LS}$, where D_S, D_{LS} are the angular diameter distances of the source from the observer and the source from the lens, respectively, and θ_E is the Einstein angle that is related to the Einstein radius by $R = \theta_E D_L$; with D_L being the angular diameter distance of the lens from the observer. The expression for the bending angle based on the Gauss–Bonnet theorem in (22) also contains a coupling term between the mass and the cosmological constant. This is the third term on the RHS of this expression, which for the case $u_R = u_S = U$ reduces to $mR\Lambda/(3\sqrt{1 - R^2U^2})$. This is obviously greater in magnitude than the pure de Sitter term (the second term in this expression) and is very similar to the last term in (31), albeit having the opposite sign, which is immaterial considering its tiny magnitude. In a sense, this was to be expected, since both formulae consider a situation where the source and receiver are at a finite distance from the lens.

Considering now the MK-spacetime given by (6) or (8), we can say that this provides a more realistic example of a lens in a cosmological background, because unlike the Einstein–Straus model, in which the spacetime changes abruptly from SdS to FRW at r_b, the MK spacetime provides a smooth transition from a Schwarzschild-like metric in the vicinity of the lens to a general FRW-like metric in the asymptotic background. In Ref. [45], the authors studied the interior structure of a static and spherically symmetric conformally invariant source in Weyl gravity and showed that the parameter γ can be expressed solely in terms of this source. Moreover, for large r, when the β terms in (6) can be ignored, the resulting spacetime is conformally related to the FRW metric, having an arbitrary scale factor $a(t)$ and spatial curvature $\kappa = -\Lambda/3 - \gamma^2/4$. This is obtained by applying the coordinate transformation [17]

$$\rho = \frac{4r}{2(1+\gamma r - \frac{\Lambda r^2}{3})^{1/2} + \gamma r + 2} \quad \text{and} \quad \tau = \int a(t)dt, \tag{34}$$

such that the line element in (5) with $\beta = 0$ reduces to

$$ds^2 = \frac{1}{a^2(\tau)} \frac{[1-\rho^2(\gamma^2/16+\Lambda/12)]^2}{[(1-\gamma\rho/4)^2+\Lambda\rho^2/12]^2}\Big(-d\tau^2 + \frac{a^2(\tau)}{[1-\rho^2(\gamma^2/16+\Lambda/12)]^2}(d\rho^2+\rho^2 d\Omega)\Big). \tag{35}$$

Since in the asymptotic form of the MK-spacetime given by (35) the spatial curvature of the FRW background depends on γ, one can also conclude that the γr term in the metric has an effect on the cosmological background, even when $\beta = 0$ (or $m = 0$). Therefore, in a way, one can say that the presence of the linear term in the MK solution facilitates the embedding of the gravitational source in a cosmological background. Using the static coordinates in (5) for the MK-metric, one is again faced with the same issue mentioned above for SdS spacetime, namely the position of the source and observer with respect to the lensing object, since the total light bending will depend on these positions. The bending angle formula in (24) obtained by Kaşikçi and Deliduman [34] contains the term $\gamma_0\sqrt{\Lambda_0/3} = \gamma r_0^2\sqrt{\Lambda/3}$. This is the first-order and main contribution from the linear term in the MK-metric and it increases with the distance of closest approach r_0 (which is related to the impact parameter R). Therefore, this is similar to but smaller in magnitude than the contribution $-\gamma R$ in (23) obtained by Edery and Paranjape (see also Equation (21) in Ref. [25])), which has been termed unphysical due to the fact that it increases linearly with the impact parameter or the distance of closest approach r_0 from the lens. Thus, both Edery & Paranjape's and Kaşikçi and Deliduman's derivations utilize Weinberg's method to calculate the bending angle, but the latter derivation takes into consideration the asymptotic non-flatness of the MK spacetime and limits the integration (and so the deflection of the null trajectory) to the position of the cosmological event horizon $r = r_h$ in this spacetime, while in the former case the integration is extended to infinity, thereby yielding a larger contribution. On the other hand, one can easily check that the leading contribution from γ in (24) is significantly higher than that in (25). So, if we take again the example of the galaxy cluster Abell 2744 and use the value $\gamma \sim 1/R_H = 10^{-26}\mathrm{m}^{-1}$, we obtain $\gamma r_0^2\sqrt{\Lambda/3} = 5.37 \times 10^{-10}$ and $2\beta^2\gamma/R = 1.128 \times 10^{-14}$. These are still both insignificant for practical gravitational lensing when compared to the Schwarzschild bending angle. On another note, it is also interesting to see that in (24) the leading γ contribution to the bending angle is coupled to the cosmological constant instead of the geometric mass of the lens as in (25) and in the other similar formulae obtained in Ref. [29,31,32]). This would lead us to rethink the exact nature of the linear γr term in the MK-metric, i.e., whether this term is derived from the gravitational source as previously claimed or whether it is associated with the asymptotic region of the spacetime. This issue is still open and has not been settled so far. In the formula based on the Gauss–Bonnet theorem given by (29), the contribution to the bending angle from the linear term γr in the MK-metric is also negative, as in (25). However, in this case, one cannot comment about the possible coupling between γ and the cosmological constant Λ, because in deriving this formula the authors considered the case $\Lambda = 0$. Assuming that $u_S = u_R = U$ in (29), this contribution takes the form $-2\beta\gamma RU/\sqrt{1-R^2U^2}$, and so considering that the position of the source (or observer) is typically greater than the impact parameter R, i.e., $1/U >> R$, its effect diminishes with distance from the lens, as in the case of the formula obtained using the Rindler–Ishak method. For Abell 2744 with $r_R = r_S = 100R$, the magnitude of this term is 8.196×10^{-12}, which although greater than the corresponding leading γ term in (25) is still insignificant for practical gravitational lensing.

In this paper, we have reviewed the methods used for calculating the bending angle of light by a source embedded in a non-asymptotically flat background, such as the SdS spacetime, the Einstein–Straus vacuole, and the MK-spacetime. Apart from the obvious dependence on the mass of the lens, we noted that the total bending angle depends on the position of the source and observer relative to the lens and whether they are stationary or moving with respect to the lens. With the exception of the SdS spacetime, it was shown that for practical gravitational lensing the contribution to the bending angle from the cosmological background is practically insignificant and so one can safely state that the Schwarzschild bending angle formula still applies.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflicts of interest.

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