Local v. Global Majority: An Edge Colouring Approach

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*“He that breaks a thing to find out what it is
has left the path of wisdom.”*

Gandalf (to Saruman)

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Last but not least, I would like to thank my best friend, Adriana, for always giving me the support to keep me going.

Declaration

This thesis stems from a problem of Caro, Lauri and Zarb, to which the author was introduced sometime in early 2023. Most of the material in this thesis appears in Caro et al. (2023a), written jointly between the author, Caro, Lauri, Yuster and Zarb. The author's main contributions were:

i. A generalisation of an argument of Yuster on the non-existence of $(2, 3)$-flip graphs, presented in Proposition 3.1.3.

ii. Establishing, together with Caro, the bound on $h(b, r)$ given in Proposition 3.2.4.

iii. Minor simplification of the original proof of Theorem 4.1.1, due to Lauri and Zarb, most notably in establishing the inequality (4.3).

iv. A generalisation of an argument of Caro and Yuster, presented in Proposition 5.5.1 (originally given for $k = 4$).

The author further extended this work in Mifsud (2024) with a number of original contributions. These results are presented in Section 2.3.1, Section 3.2.2, and Section 5.5.1. The results in Section 5.4 are unpublished and are the author's original contribution.

Together with Caro, the author also extended the work of Section 5.2 in Caro and Mifsud (2024a,b).

Abstract

We introduce a new problem on local v. global majority in graphs, concerning edge-colourings. In particular, we ask for which positive integers b and r, such that $b < r$, does a $b + r$ regular graph G exist with an edge colouring $f : E(G) \to \{\text{blue}, \text{red}\}$ satisfying the following:

i. The subgraphs induced by the blue and red edges are b and r regular respectively, resulting in a global majority ordering since $b < r$, where across the entire graph 'red' wins against 'blue'.

ii. On the other-hand, for every vertex v, the number of blue edges in the closed neighbourhood of v is *greater* than the number of red edges, resulting in a locally opposite majority ordering where locally 'blue' wins against 'red'.

We term such a graph as a (b,r)-flip graph, due to the local v. global majority flip they demonstrate. We show that such edge-coloured graphs do exist, namely for when the difference between b and r is not too great, as intuition would suggest. In particular we show that a (b,r)-flip graph exists if, and only if, $3 \leq b < r < \binom{b+1}{2}$.

We also establish a number of bounds on the number of vertices $h(b,r)$ of the smallest (b,r)-flip graph. Somewhat surprisingly we establish that such graphs can be very small, illustrating cases where $h(b,r)$ is $\Theta(b + r)$.

To establish these results, we provide a number of different construction techniques using: edge-coloured graph products, graph packings and Cayley graphs.

Two natural extensions of this problem are considered: extending the local flip up to the closed neighbourhood at a distance t from each vertex, and extending to more than two colours.

The extension to the closed t-neighbourhood is considered briefly, offering two distinct techniques of constructing such graphs. The extension to more than two colours is considered more exhaustively. Given a colour-degree sequence $(a_1, \ldots, a_k)$ such that $a_1 < a_2 < \cdots < a_k$, analogous to (b,r) in the two colour case, then if a_1 is sufficiently large and $a_k \leq a_1 + \left\lfloor \frac{1}{4}\left(a_1^2 - 10a_1^{\frac{3}{2}}\right) \right\rfloor$, it is shown that an $(a_1, \ldots, a_k)$-flip graph exists.

The proof is constructive, and relates to another (existence) problem of interest which we briefly consider. We show that for every integer $r \geq 1$ and integer c, $0 \leq c \leq \frac{r^2}{2} - 5r^{3/2}$, there exists an r regular graph with the property that for every vertex v, the open neighbourhood of v contains precisely c edges. Here we outline a number of connections with other well-known families of graphs, such as those with constant link and (r, b)-regular graphs.

We conclude with a somewhat surprising result, namely that unlike the case of two colours, for $k \geq 4$ colours, a_k need not necessarily be bounded in a_1. Formally, there is some positive integer $m = m(k)$ such that for every positive integer N, there exists a k-flip sequence $(m, a_2, \ldots, a_k)$ such that $a_k > N$. Consequently, when considering four or more colours, there exists constructions where the first colour can locally win against the k^{th} colour, no matter how big of a majority the k^{th} colour has across the entire graph.

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Introduction

The study of local v. global phenomena in graphs has been a fruitful and active area of research for decades. Whilst outlining the significant contributions in this area, we will introduce a new variation concerning local v. global majority in the edge-colourings of graphs.

1.1 | Local v. global phenomena in graphs

Many variations on local-global phenomena in graphs have been studied over the past 80 years, where typically some global graph parameter is studied in terms of a local parameter (eg. on some smaller induced subgraph). These variations include problems on vertex colouring, cliques, *et cetera*.

One simple example of such phenomena is as follows. Recall that a graph is said to be *Eulerian* if every component has a trail starting and ending at the same vertex, such that each edge is visited exactly once. A classical result is *Euler's theorem*, which states that a graph is Eulerian if, and only if, every vertex has even degree. The property that the graph is Eulerian is a property of the graph and hence a *global* property, whilst the property that every vertex has even degree is a *local* property.

A survey on local-global phenomena is given in Linial (1997), as well as an outline to various applications in distributed computing. Throughout the years, another variation which cropped up was related to *majority* problems.

In one of the earliest papers on local v. global majority, dating back to almost 40 years ago, Fishburn et al. (1986) study chains of white and black coloured beads on a necklace. In particular they demonstrate that for certain arrangements of beads, if certain sub-chains of fixed length have more white beads than black beads (a local majority), then the entire necklace has more white beads than black beads (a global majority). Different variations and applications of such majority problems have been explored throughout the years; we highlight a number of these here in chronological order: Woodall (1992), Broere et al. (1995), Peleg (2002), Abdullah and Draief (2015), Caro and Yuster (2018), and Gärtner and Zehmakan (2021).

Problems of local v. global majority in graphs concerning edge-colourings are of notable interest, such as in voting problems. Consider a graph G where the edges of G are coloured from the set $\{-1, 1\}$ and therefore are being assigned a *weighting*. The weight $w(G)$ of the graph G is the sum of all the weights assigned to the edges of G. Caro and Yuster (2018) consider the following (binary) voting problem: Let $k \in \mathbb{N}$. Suppose that for every connected k-edge subgraph H of G, we have that $w(H) > 0$. When is $w(G) > 0$?

Observe that the requirement that $w(H) > 0$ is a local majority (namely since it is restricted to connected subgraphs of given size), whilst $w(G) > 0$ establishes a global property. The local v. global majority problems considered in this thesis follow a similar flavour to the aforementioned, with surprising consequences.

1.2 | Definitions and nomenclature

By $\mathbb{Z}$ and $\mathbb{N}$ we will denote the set of integers and naturals (positive integers), respectively. The set difference of two sets A and B shall be denoted by $A - B$, while their intersection, union and Cartesian product shall be denoted by $A \cap B$, $A \cup B$ and $A \times B$, respectively. To emphasise that A and B are disjoint, we shall sometimes denote their union by $A \dot{\cup} B$. The cardinality of a set A shall be denoted by $|A|$.

By an increasing positive-integer sequence $(a_1, \ldots, a_k)$ we mean that the integers $a_1, \ldots, a_k$ satisfy the inequalities $1 \leq a_1 < a_2 < \cdots < a_{k-1} < a_k$.

A graph $G = (V(G), E(G))$ is a pair of sets $V(G)$, a non-empty set termed as the *vertex set*, and $E(G)$, a set of 2-element subsets of $V(G)$ termed as the *edge set*. Our graphs shall be *simple*, meaning without repeated edges and loops.

Two vertices u and v in $V(G)$ belonging to the same edge in $E(G)$ are said to be *adjacent*, which we sometimes denote by $u \sim v$. Similarly, if a vertex v in $V(G)$ belongs to an edge e in $E(G)$, then e is said to be *incident* to v. The *degree* of a vertex v in $V(G)$, denoted by $\deg^G(v)$, is the number of edges in $E(G)$ incident to v. A graph is said to be ρ *regular* if for every vertex v in $V(G)$, $\deg^G(v) = \rho$. The minimum and maximum degree of a graph are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

The *line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$, such that two vertices in $L(G)$ are adjacent if the corresponding edges in G are incident at a common vertex.

A *path* is a sequence of distinct vertices $v_1, v_2, \ldots, v_s$ and edges $\{v_i, v_{i+1}\}$ for $1 \leq i < s$. If we include the additional edge $\{v_1, v_s\}$, we get a *cycle*. We say that a graph is connected if, and only if, there exists a path between every pair of vertices. Moreover, a graph is *acyclic* if it contains no cycles. A *tree* is a connected acyclic graph. A *forest* is an acyclic graph which is not necessarily connected.

Given two vertices u and v in $V(G)$, the *distance* between u and v, denoted by $d_G(u, v)$, is the length of the shortest path between u and v in G. If G is disconnected and the vertices belong to distinct components, then we let $d_G(u, v) = \infty$. The *diameter* of a graph, written as $\mathsf{diam}(G)$, is the length of the longest shortest-path between any two vertices. On the other-hand, the *girth* of a graph, denoted by $g(G)$, is the length of the shortest cycle in G.

For a vertex $v \in V(G)$ and $t \in \mathbb{N}$, $N_t^G(v) = \{u \in V(G) \colon 1 \leq d_G(u, v) \leq t\}$ is the *open* t-neighbourhood of v, whilst $N_t^G[v] = N_t^G(v) \cup \{v\}$ is its *closed* t-neighbourhood. In the case when $t = 1$ we shall simply write $N^G(v)$ and $N^G[v]$.

By $e^G[v]$ and $e^G(v)$ we will denote, respectively, the number of edges in the closed and open neighbourhoods of a vertex $v \in V(G)$.

Let S be a subset of $V(G)$. The vertex-induced subgraph of G induced by S is the graph with vertex set S and with an edge set being exactly all those edges in $E(G)$ incident to two vertices in S. By $E(S)$ we will denote the set of edges in the subgraph of G induced by the vertices in S.

Let $k \in \mathbb{N}$ and let $f\colon E(G) \to \{1, \ldots, k\}$ be an edge-colouring[1] of G. Under this edge-colouring, for $1 \leq j \leq k$,

i. Given a subset S of $V(G)$, $E_j(S)$ is the set of edges coloured j in the vertex-induced subgraph of G by S, and $e_j^G(S) = |E_j(S)|$.

ii. Moreover, $E_j(G)$ is the set of edges coloured j in G, and $e_j(G) = \left|E_j(G)\right|$.

iii. For a vertex $v \in V(G)$, let $e_j^G[v] = e_j^G\left(N^G[v]\right)$ and $e_j^G(v) = e_j^G\left(N^G(v)\right)$.

iv. Lastly, for a vertex $v \in V(G)$, the colour degree $\deg_j^G(v)$ is the number of edges coloured j incident to v.

The *colour degree sequence* of a vertex v in G is the sequence of colour degrees $\deg_1^G(v), \deg_2^G(v), \ldots, \deg_k^G(v)$, where k is the number of colours.

Whenever there is no room for ambiguity, we shall simplify our notation by removing any symbolic reference to a graph (such as sub-scripts and super-scripts). In other-words, we will write V instead of $V(G)$, $d(u,v)$ instead of $d_G(u,v)$, $e_j[v]$ instead of $e_j^G[v]$, *et cetera*.

Let Γ be a group. We shall denote the identity of a group Γ by 1_Γ. We say that a group is *Abelian* if it is endowed with a commutative group operation. Unless stated otherwise, all groups are assumed to be finite. By $\mathbb{Z}_n$ we denote the group of integers under addition modulo n.

Let Γ be an Abelian group and let $A, B \subseteq \Gamma$. The sum-set $A + B$ is the set $\{a + b\colon a \in A, b \in B\}$. By $2A$ we denote the set $A + A$ whilst by A^{-1} we denote the set of inverses of A. We say that A is sum-free if $2A \cap A = \emptyset$, and inverse-closed if $A = A^{-1}$.

Let S be a subset of a group Γ such that S is inverse-closed and does not contain the identity element. The *Cayley graph* $\mathrm{Cay}\,(\Gamma; S)$ is a graph with vertex set Γ and edge set $\{\{g, gs\}\colon s \in S, g \in \Gamma\}$. The set S is termed as the *connecting set* of the Cayley graph. Cayley graphs enjoy a number of properties, most notably that they are vertex-transitive.

Occasionally we shall use the Bachmann–Landau family of notations to describe asymptotics, namely Big-O, Big-Ω and Big-Θ.

[1] Not necessarily proper - indeed this will not be the case throughout this thesis.

1.3 | Local v. global majority in edge-colourings

We are now in a position to introduce a new variant of local v. global problems, concerning majorities in edge colourings of graphs, which we refer to as the *k-flip-colouring problem*.

Problem 1.3.1 (k-flip-colouring problem)**.** Given $k \geq 2$, a d regular graph G and an increasing positive-integer sequence $(a_1, \ldots, a_k)$ such that $d = \sum_{j=1}^{k} a_j$, does there exists an edge-colouring $f : E \to \{1, \ldots, k\}$ such that:

i. E_j spans an a_j regular subgraph, namely $\deg_j(v) = a_j$ for every $v \in V$, resulting in a global majority ordering,

ii. and for every vertex $v \in V$, $e_k[v] < e_{k-1}[v] < \ldots < e_1[v]$, resulting in a locally opposite majority ordering.

If such an edge-colouring exists then G is said to be an $(a_1, \ldots, a_k)$-flip graph, or more simply a k-flip graph, and $(a_1, \ldots, a_k)$ is called a k-flip sequence of G. Figure 1.1 is an example of a $(3, 4)$-flip graph.

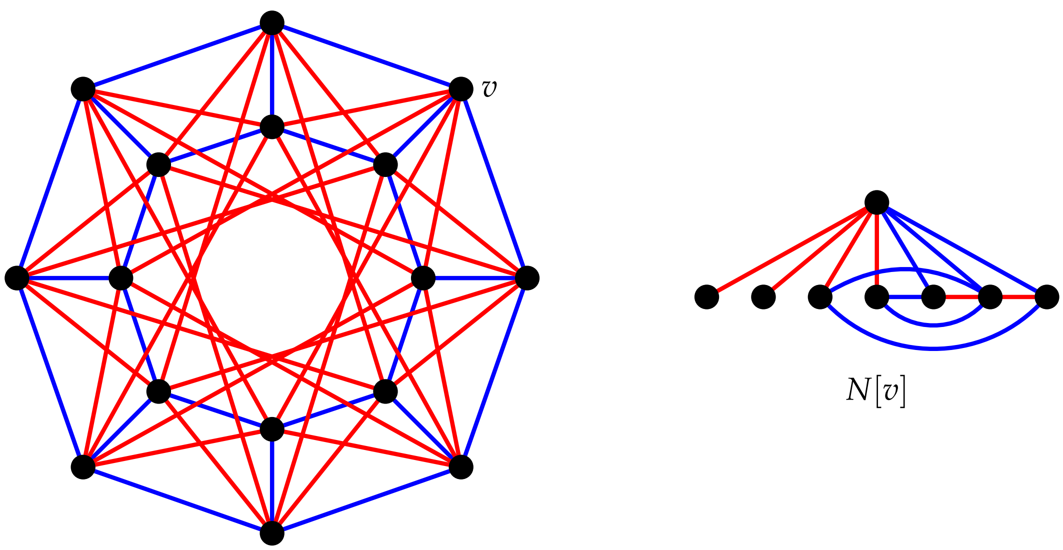

Figure 1.1: Smallest known $(3, 4)$-flip graph having 16 vertices, with the subgraph induced by $N[v]$ for any vertex v illustrated on the right.

1.3.1 | Major problems concerning k-flip graphs and sequences

In introducing this new kind of local v. global majority problem, as in Problem 1.3.1, a potpourri of other problems naturally crops up. We shall take the opportunity to chart out this landscape, highlighting the motivating problems behind the work carried out in this thesis.

We have already seen an example of a $(3,4)$-flip graph. Given an increasing positive integer sequence $(a_1,\ldots,a_k)$, a natural question to pose is whether there is an $(a_1,\ldots,a_k)$-flip graph which realises the sequence as a k-flip sequence.

Problem 1.3.2 (Recognising k-flip sequences)**.** Given an increasing positive integer sequence $(a_1,\ldots,a_k)$, determine if there exists an $(a_1,\ldots,a_k)$-flip graph.

The $(3,4)$-flip graph given in Figure 1.1 transpires to be the smallest known example of a graph which realises the 2-flip sequence $(3,4)$. An interesting problem is to determine, or at least bound, the smallest order of a graph which realises a given k-flip sequence.

Problem 1.3.3 (Smallest order of graphs realising a k-flip sequence)**.** For a given k-flip sequence $(a_1,\ldots,a_k)$, estimate

$$h(a_1,\ldots,a_k) = \min\{|V(G)|\colon G \text{ is an } (a_1,\ldots,a_k)\text{-flip graph}\}$$

It would be useful to devise systematic constructions of k-flip graphs, both for solving the aforementioned two problems, and for the purposes of demonstrating existence.

Problem 1.3.4. Devise constructions for flip graphs with two or more colours.

More so, in conjunction to devising constructions, it is of interest to find necessary and/or sufficient conditions for k-flip sequences.

Problem 1.3.5. Find necessary and/or sufficient conditions for an increasing positive-integer sequence $(a_1,\ldots,a_k)$ to be a k-flip sequence of some graph.

A special case of the previous problem is the case when the sequences under consideration form an integral interval. The importance of *interval flipping* will become all the more apparent in Chapter 5.

Problem 1.3.6 (Interval flipping)**.** Find long k-flip sequences $(a_1, \ldots, a_k)$, that form an interval, namely $a_j = a_1 + j - 1$, for $1 \leq j \leq k$.

Lastly, it is only natural to consider whether these problems can be generalised to the t-distance closed neighbourhoods in a graph.

Problem 1.3.7 (Extension to t-neighbourhood flipping)**.** Extend the flipping problem to the t-distance closed neighbourhood $N_t[v]$.

1.4 | Overview

We begin by introducing, in Chapter 2, the toolset required to construct edge-coloured graphs and in particular flip graphs from graphs satisfying particular properties. Here we consider a number of graph products and packing arguments in order to construct 'new' graphs from 'old'. Chapter 2 therefore lays down the foundations for all the subsequent chapters.

In Chapter 3 we then proceed to answer Problems 1.3.2, 1.3.4 and 1.3.5 for the case when $k = 2$. Intuitively one expects that for a sequence (a_1, a_2) to be a 2-flip sequence, the difference between a_1 and a_2 must not be too large for the flip to be possible. We show that this intuition is correct, namely that a_2 is quadratically bound in a_1. More so, we provide an in-depth treatment of Problem 1.3.3, where for certain 2-flip sequences we illustrate constructions which asymptotically use the smallest number of vertices.

Chapter 4 continues the work for the case when $k = 2$, by considering Problem 1.3.7 and extending our work to the t-distance closed neighbourhoods of a graph. We only concern ourselves with the problem of existence, providing two distinct and contrasting constructions.

In Chapter 5 we consider Problems 1.3.2, 1.3.4, 1.3.5 and 1.3.6 for the case when $k \geq 3$. For the case when $k = 3$ we give a necessary condition for (a_1, a_2, a_3) to be a 3-flip sequence, namely that a_3 is quadratically bound in a_1. In other words, the largest colour degree must be quadratically bound in terms of the smallest colour degree, just as in the case when $k = 2$. Indeed for the case when $k \geq 4$ we go on to show that such a condition is *sufficient* for $(a_1, \ldots, a_k)$ to be a k-flip sequence, but is it *necessary*?

We also introduce a new class of graphs called (r, c)-constant graphs, which are r regular graphs such that every open neighbourhood has exactly c edges. We discuss the motivation behind the introduction of such graphs as well as their relationship with numerous other well-studied families of graphs. We then proceed to consider the problem for which parameters r and c do these (r, c)-constant graphs exist.

The aforementioned results tantalisingly suggest that it is. We answer this question, with a somewhat surprising conclusion.

We conclude with a short discussion in Chapter 6, outlining a number of open problems to extend the landscape of what is currently known on the flip-colourings of graphs.

2

New edge-coloured graphs from old

We begin by a short exposition into a number of graph operations and their extension to edge-coloured graphs, allowing us to obtain new ones from old. In particular, we shall consider the Cartesian and strong product of edge-coloured graphs, as well as the union of edge-coloured graphs with identified vertex-sets.

Of interest will be the resulting colour degrees after applying these operations, and the number of edges of a given colour in the closed neighbourhoods.

Such operations will be a powerful constructive tool in many of our proofs, which we demonstrate towards the end of this chapter.

2.1 | Cartesian products of graphs

The Cartesian product of graphs enjoys a number of properties, which we will summarise below. We begin however by formally defining the Cartesian product of two graphs and a means of inheriting an edge-colouring from its factors.

Definition 2.1.1 (Cartesian product)**.** The Cartesian product $G \square H$ of the graphs G and H is the graph such that $V(G \square H) = V(G) \times V(H)$ and there is an edge $\{(u,v),(u',v')\}$ in $G \square H$ if, and only if, either $u = u'$ and $v \sim v'$ in H, or $v = v'$ and $u \sim u'$ in G.

We extend the edge-colourings of G and H to an edge-colouring of $G \square H$ as follows. Consider the edge $e = \{(u,v),(u',v')\}$ in $G \square H$; if $u = u'$ then e

inherits the colouring of the edge $\{v, v'\}$ in H, otherwise if $v = v'$ the colouring of the edge $\{u, u'\}$ in G is inherited. This colouring inheritance is illustrated in Figure 2.1, with its properties summarised in Lemma 2.1.2. We refer to $G \square H$ with its inherited edge-colouring as the *coloured Cartesian product* of G and H.

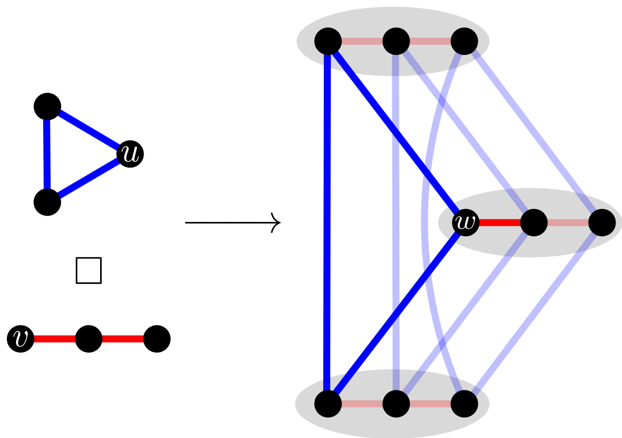

Figure 2.1: Illustration of Lemma 2.1.2, with the closed neighbourhood of $w = (u, v)$ in $K_3 \square P_3$ highlighted.

Lemma 2.1.2. *Let G and H be edge-coloured from $\{1, \ldots, k\}$. Then in the coloured Cartesian product $G \square H$, for any $1 \leq j \leq k$ and $(u, v) \in V(G \square H)$,*

i. $\deg_j\left((u, v)\right) = \deg_j^G(u) + \deg_j^H(v)$

ii. $e_j\left[(u, v)\right] = e_j^G[u] + e_j^H[v]$

Proof. Given any $u \in V(G)$, let H_u be the subgraph of $G \square H$ induced by the vertices (u, w) for $w \in V(H)$, which is isomorphic to H. Let $v \in V(H)$.

Clearly (u, v) has $\deg_j^H(v)$ incident edges coloured j in H_u. Moreover, for every edge $\{u, x\}$ coloured j in G, the edge $\{(u, v), (x, v)\}$ is coloured j. Hence there are a further $\deg_j^G(u)$ incident edges to (u, v) coloured j.

Therefore (i) follows, and by a similar argument so does (ii). $\square$

Since the Cartesian products of graphs is commutative and associative, one can easily observe that Lemma 2.1.2 naturally extends to the product of $s \geq 2$ factors $H_1, \ldots, H_s$.

2.1.1 | The coloured Cartesian product lemmas

The coloured Cartesian product (CCP) of graphs shall be an important tool throughout this thesis. As a hint at its utility, we consider the following results, which we collectively refer to as the *CCP Lemmas*.

The first CCP Lemma (Lemma 2.1.3) describes how to construct a k-flip graph from a family of regular graphs satisfying certain prescribed conditions.

Lemma 2.1.3 (CCP Lemma I). *Let $k \in \mathbb{N}, k \geq 2$. Suppose $H_1, \ldots, H_k$ are $a_1, \ldots, a_k$ regular graphs respectively, with $a_i < a_j$ for $1 \leq i < j \leq k$. Furthermore, suppose that for $1 \leq i < k$,*

$$\max_{u \in V(H_{i+1})} e^{H_{i+1}}[u] < \min_{v \in V(H_i)} e^{H_i}[v] \,.$$

Then $G = \square_{j=1}^{k} H_j$ is an $(a_1, \ldots, a_k)$-flip graph.

Proof. Colour the edges of H_j using colour j, for $1 \leq j \leq k$. Let $G = \square_{j=1}^{k} H_j$ be the corresponding CCP. Clearly for every vertex w in G, given any $1 \leq i < k$, we have that $\deg_i(w) = a_i < a_{i+1} = \deg_{i+1}(w)$ and

$$e_{i+1}[w] \leq \max_{u \in V(H_{i+1})} e^{H_{i+1}}[u] < \min_{v \in V(H_i)} e^{H_i}[v] \leq e_i[w].$$

Consequently G is an $(a_1, \ldots, a_k)$-flip graph. □

The second CCP Lemma (Lemma 2.1.4) describes how to construct k-flip graphs (sequences) from other k-flip graphs (sequences).

Lemma 2.1.4 (CCP Lemma II). *Let $k \in \mathbb{N}, k \geq 2$. If, for $1 \leq j \leq q$, $(a_{j,1}, \ldots, a_{j,k})$ are k-flip sequences then $(a_1, \ldots, a_k)$ is a k-flip sequence where $a_i = \sum_{j=1}^{q} a_{j,i}$.*

Proof. For $1 \leq j \leq q$, let H_j be a k-flip graph that realises the k-flip sequence $(a_{j,1}, \ldots, a_{j,k})$. Consider the CCP graph $G = \square_{j=1}^{q} H_j$.

By Lemma 2.1.2, for $1 \leq i_1 < i_2 \leq k$ and a vertex $v = (v_1, \ldots, v_q)$, we have

$$\deg_{i_2}(v) = \sum_{j=1}^{q} \deg_{i_2}^{H_j}(v_j) > \sum_{j=1}^{q} \deg_{i_1}^{H_j}(v_j) = \deg_{i_1}(v)$$

and

$$e_{i_2}[v] = \sum_{j=1}^{q} e_{i_2}^{H_j}[v_j] < \sum_{j=1}^{q} e_{i_1}^{H_j}[v_j] = e_{i_1}[v]$$

since the H_j's are k-flip graphs. Hence G is a $(a_1, \dots, a_k)$-flip graph. $\square$

The third and final CCP Lemma (Lemma 2.1.5) illustrates how, for certain constructions which realise a k-flip sequence and make use of the Cartesian product, any sub-sequence of length $s \geq 2$ is also a flip-sequence. Therefore from a single construction we may realise many different flip-sequences.

Lemma 2.1.5 (CCP Lemma III)**.** *Let $k, s \in \mathbb{N}$ such that $2 \leq s \leq k$. Let G be some k-coloured Cartesian product of graphs, where each factor is either monochromatic or triangle-free. If a k-flip sequence $(a_1, \dots, a_k)$ is realised by G, then any sub-sequence $(a_{j_1}, \dots, a_{j_s})$ is an s-flip sequence realised by the subgraph in G induced by the edges coloured $j_1, \dots, j_s$.*

Proof. Let $H_1, \dots, H_t$ be triangle-free graphs with edges coloured from $\{1, \dots, k\}$ and let $H_{t+1}, \dots, H_r$ be graphs monochromatically edge-coloured from $\{1, \dots, k\}$.

Consider the coloured Cartesian product $G = \square_{i=1}^{r} H_i$, and let $(a_{j_1}, \dots, a_{j_s})$ be a sub-sequence of $(a_1, \dots, a_k)$. By Lemma 2.1.2, we have that for $1 \leq j \leq k$ and $v = (v_1, \dots, v_r) \in V(G)$,

$$\deg_j^G(v) = \sum_{i=1}^{r} \deg_j^{H_i}(v_i) \quad \text{and} \quad e_j^G[v] = \sum_{i=1}^{r} e_j^{H_i}(v_i)$$

Without loss of generality, let $H_1, \dots, H_{t_0}$ and $H_{t+1}, \dots, H_{t+r_0}$ be all the factors which have some edge coloured from $\{j_1, \dots, j_s\}$. In particular, for $1 \leq i \leq t_0$, let H_i^* be the subgraph of H_i induced by the edges coloured using $j_1, \dots, j_s$. Since each H_i is triangle-free, it follows that $e_j^{H_i^*}(u) = e_j^{H_i}(u)$ and $\deg_j^{H_i^*}(u) = \deg_j^{H_i}(u)$, for $1 \leq i \leq t_0$ and $j \in \{j_1, \dots, j_s\}$.

From the definition of the Cartesian product, the subgraph of G induced by the edges coloured $j_1, \dots, j_s$ is $G^* = \left(\square_{i=1}^{t_0} H_i^*\right) \square \left(\square_{i=1}^{r_0} H_{t+i}\right)$.

Every vertex $v' = (v_1, \dots, v_{t_0}, v_{t+1}, \dots, v_{t+r_0})$ of $V(G^*)$ is a sub-sequence of some vertex $v = (v_1, \dots, v_{t+r})$ in $V(G)$, where for v_i in v but not in v' we have that v_i is in a factor H_i of G not coloured using any one of the colours $j_1, \dots, j_s$ and hence $\deg_j^{H_i}(v_i) = 0 = e_j^{H_i}[v_i]$ for $j \in \{j_1, \dots, j_s\}$.

By Lemma 2.1.2, we have that for $j \in \{j_1, \dots, j_s\}$ and $v' \in V(G^*)$,

$$\begin{aligned}
\deg_j^{G^*}(v') &= \sum_{i=1}^{t_0} \deg_j^{H_i^*}(v_i) + \sum_{i=1}^{r_0} \deg_j^{H_{t+i}}(v_{t+i}) \\
&= \sum_{i=1}^{t_0} \deg_j^{H_i}(v_i) + \sum_{i=1}^{r_0} \deg_j^{H_{t+i}}(v_{t+i}) && \because \deg_j^{H_i^*}(v_i) = \deg_j^{H_i}(v_i) \\
&= \sum_{i=1}^{t} \deg_j^{H_i}(v_i) + \sum_{i=1}^{r} \deg_j^{H_{t+i}}(v_{t+i}) && \because \deg_j^{H_i}(v_i) = 0, v_i \notin v' \\
&= \deg_j^{G}(v) && \text{for some } v \in V(G)
\end{aligned}$$

By a similar argument, we have that $e_j^{G^*}[v'] = e_j^G[v]$. Since G is an $(a_1, \ldots, a_k)$-flip graph, it follows that G^* is an $(a_{j_1}, \ldots, a_{j_s})$-flip graph as required. $\square$

We briefly note the necessity for multi-coloured factors to be triangle-free in Lemma 2.1.5: Consider a factor H such as in the proof of Lemma 2.1.5, and suppose that we consider the subgraph H^* induced by the edges of some colour j. Then there might be a vertex v such that $e_j^H[v] > e_j^{H^*}[v]$, as illustrated in Figure 2.2 (where $e_j^H[v] = 2 > 1 = e_j^{H^*}[v]$), potentially resulting in the flipping constraint to not hold any longer in the graph G^* as constructed in the proof.

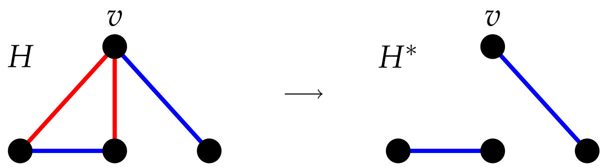

Figure 2.2: Illustration of how the number of blue edges in a closed neighbourhood of a vertex v can decrease when considering a monochromatic subgraph.

2.2 | Strong products of graphs

Another useful graph product shall be the *strong product*, which we now define and illustrate how an edge-colouring is inherited from its factors.

Definition 2.2.1 (Strong product). The strong product $G \boxtimes H$ of two graphs G and H is the graph such that $V(G \boxtimes H) = V(G) \times V(H)$ and there is an edge

$\{(u,v),(u',v')\}$ in $G \boxtimes H$ if, and only if, either $u = u'$ and $v \sim v'$ in H, or $v = v'$ and $u \sim u'$ in G, or $u \sim u'$ in G and $v \sim v'$ in H.

We extend the edge-colourings of G and H to an edge-colouring of $G \boxtimes H$ as follows. Consider the edge $e = \{(u,v),(u',v')\}$ in $G \boxtimes H$; if $u = u'$ then e inherits the colouring of the edge $\{v,v'\}$ in H, otherwise if $u \neq u'$ the colouring of the edge $\{v,v'\}$ in G is inherited. This colouring inheritance is illustrated in Figure 2.3, with its properties summarised in Lemma 2.2.2. We refer to $G \boxtimes H$ with its inherited edge-colouring as the *coloured strong product* of G and H.

Note that the inherited colourings of $G \boxtimes H$ and $H \boxtimes G$ are different, even though the two uncoloured graphs are isomorphic.

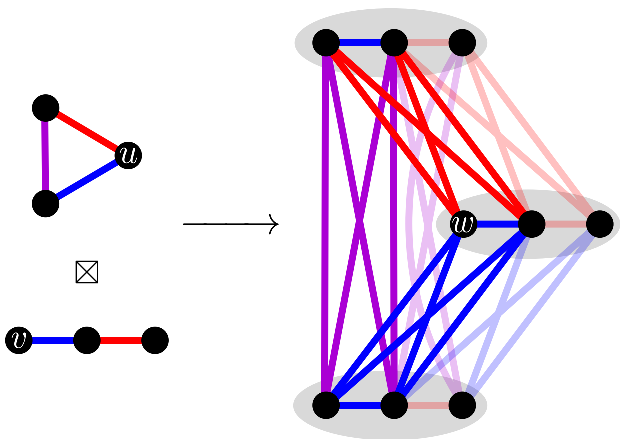

Figure 2.3: Illustration of Lemma 2.2.2, with the closed neighbourhood of $w = (u,v)$ in $K_3 \boxtimes P_3$ highlighted.

Lemma 2.2.2. *Let G and H be edge-coloured from $\{1,\ldots,k\}$. Then in the coloured strong product $G \boxtimes H$, for any $1 \leq j \leq k$ and $(u,v) \in V(G \boxtimes H)$,*

i. $\deg_j\big((u,v)\big) = \deg_j^H(v) + \deg_j^G(u)\left(1 + \deg^H(v)\right)$

ii. $e_j\big[(u,v)\big] = e_j^H[v]\left(1 + \deg^G(u)\right) + e_j^G[u]\left(1 + \deg^H(v) + 2\sum_{i=1}^{k} e_i^H[v]\right)$

Proof. Given any $u \in V(G)$, let H_u be the subgraph of $G \boxtimes H$ induced by the vertices (u,w) for $w \in V(H)$, which is isomorphic to H. Consider any vertex (u,v)

in $G \boxtimes H$. We will find the colour degrees and coloured closed neighbourhood sizes of this vertex

Let $\{x, y\}$ be an edge in G coloured j. By the definition of the strong product, in $G \boxtimes H$, the vertex (x, v) in H_x has $\left|N^H[v]\right|$ edges incident to some vertex in H_y which will be coloured j. Hence these contribute $\deg_j^G(u)\left(1 + \deg^H(v)\right)$ to $\deg_j\left((u, v)\right)$. More over, (u, v) has $\deg_j^H(u)$ additional edges coloured j incident to it in H_u. Therefore (i) follows.

Let H^* be the vertex-induced subgraph of H by $N^H[v]$, inheriting its edge-colouring. Between the closed neighbourhoods of (x, v) in H_x and (y, v) in H_y there are $\sum_{w \in V(H^*)} 1 + \deg^{H^*}(w)$ edges coloured j, by the definition of the strong product. By the hand-shaking lemma and the fact that an edge-colouring is a partition of the edge set, we have that

$$\begin{aligned}\sum_{w \in V(H^*)} 1 + \deg^{H^*}(w) &= |V(H^*)| + 2\,|E(H^*)| \\ &= |V(H^*)| + 2\sum_{i=1}^{k} e_i^{H^*}[v] \\ &= 1 + \deg^H(v) + 2\sum_{i=1}^{k} e_i^H[v]\end{aligned}$$

with the last equality following from the definition of H^*. Hence the edges coloured j in $N^G[u]$ contribute

$$e_j^G[u]\left(1 + \deg^H(v) + 2\sum_{i=1}^{k} e_i^H[v]\right)$$

edges coloured j in $e_j\left[(u, v)\right]$, between $1 + \deg^G(u)$ copies of H in $G \boxtimes H$.

Each of these $1 + \deg^G(u)$ copies of H contribute an additional $e_j^H[v]$ edges coloured j amongst the closed neighbourhood of (u, v). Hence (ii) follows. □

For special classes of graphs, the expressions in Lemma 2.2.2 simplify greatly, as illustrated in the following corollary.

Corollary 2.2.3. *Let G be a ρ regular triangle-free graph and let H be the complete graph on n vertices. Let G and H be edge-coloured from $\{1, \ldots, k\}$. Then in the coloured strong product $G \boxtimes H$, for any $1 \leq j \leq k$ and $(u, v) \in V(G \boxtimes H)$,* *i.* $\deg_j\big((u,v)\big) = \deg_j^H(v) + n\deg_j^G(u)$

ii. $e_j\big[(u,v)\big] = \dfrac{n\deg_j^H(v)\,(1+\rho)}{2} + n^2\deg_j^G(u)$

Proof. Since G is triangle-free then $e_j^G[u] = \deg_j^G(u)$. Also, for the complete graph H on n vertices we have that $1+\deg^H(v) = n$ and $e_j^H[v] = \dfrac{n\deg_j^H(v)}{2}$. Substituting into Lemma 2.2.2, the result follows. □

2.3 | Packing of edge-coloured graphs

Another useful operation that we will consider is the *packing* of graphs with an identified vertex set, which we define next.

Definition 2.3.1 (Graph packing)**.** Two graphs G and H are said to pack if there exists bijections $g\colon V(G) \to \{1,\ldots,n\}$ and $h\colon V(H) \to \{1,\ldots,n\}$ such that the images of $E(G)$ and $E(H)$ under g and h, respectively, do not intersect.

The packing of G and H is the graph with vertex set $\{1,\ldots,n\}$ and edge set being union of the images of $E(G)$ and $E(H)$ under g and h, respectively.

An edge-colouring is naturally inherited by a packing of G and H, by keeping the original colour of every single edge. Note that by the definition of a packing, the inherited edge-colouring is well-defined.

2.3.1 | Packing of Cayley graphs

If S and T are two disjoint inverse-closed subsets of Γ not containing 1_Γ, then the Cayley graph $\mathsf{Cay}(\Gamma; S\cup T)$ is a packing of $\mathsf{Cay}(\Gamma; S)$ and $\mathsf{Cay}(\Gamma; T)$. Counting the colour-degree of every vertex in a packing of two graphs G and H, in terms of the colour-degrees in G and H, is straight-forward. However counting the coloured closed neighbourhood sizes in this manner is more difficult.

In certain cases, such as the case when G and H are monochromatically edge-coloured Cayley graphs on an Abelian group, we can do such counting. This is summarised in Proposition 2.3.2.

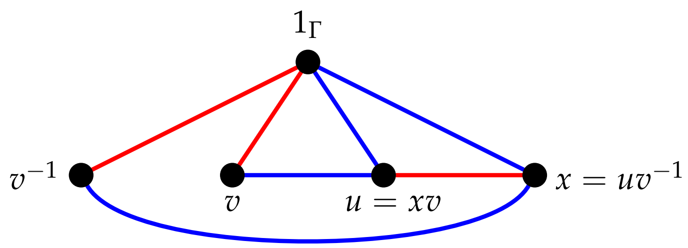

Figure 2.4: Illustration of the counting argument in the proof of Proposition 2.3.2, where the red edge $\{u, x\}$ between two blue neighbours of 1_Γ corresponds to two blue edges, each incident to a blue and red neighbour of 1_Γ.

Proposition 2.3.2. *Let Γ be an Abelian group and let R, B be disjoint inverse-closed subsets of Γ which do not contain 1_Γ. Let $G = \mathsf{Cay}(\Gamma; B)$ and $H = \mathsf{Cay}(\Gamma; R)$ be monochromatically edge-coloured using colours 1 and 2, respectively.*

Then in $\mathsf{Cay}(\Gamma; B \cup R)$, for $v \in \Gamma$,

i. $\deg_1(v) = \deg^G(v)$ *and* $\deg_2(v) = \deg^H(v)$

ii. $e_1[v] - e_2[v] = \left(e_1^G[v] - e_2^H[v]\right) + \left(e_2^H\left(N^G(v)\right) - e_1^G\left(N^H(v)\right)\right)$

iii. Furthermore, if $(R + B) \cap R = \emptyset$ and $e_1^G[v] > e_2^H[v]$, then $e_1[v] > e_2[v]$.

Proof. It suffices to consider a single vertex, say 1_Γ, by virtue of the vertex-transitivity of Cayley graphs. Note that $B = N^G(1_\Gamma)$ and $R = N^H(1_\Gamma)$. More so, since R and B are disjoint, the edge-colouring of the packing is well-defined and $N(1_\Gamma) = B \mathbin{\dot\cup} R$. Clearly an edge incident to 1_Γ in the packing of G and H must be incident to 1_Γ in either G or H. Therefore $\deg_1(v) = \deg^G(v)$ and $\deg_2(v) = \deg^H(v)$. We now count the number of edges coloured 1 in $N(1_\Gamma)$. We have three cases for an edge $\{u, v\}$ coloured 1:

i. Both u and v are in B, of which there are $e_1^G(B)$ such edges.

ii. Both u and v are in R, of which there are $e_1^G(R)$ such edges.

iii. The vertex u is in B and the vertex v is in R. We show that the number of such edges is $2e_2^H(B)$, *i.e.* twice the number of edges coloured 2 amongst the neighbours of 1_Γ in G.

Since $\{u, v\}$ is coloured 1 then it is an edge in the Cayley graph G. Therefore there is some $x \in B$ such that $u = xv$. Since $v^{-1} \in R$, then $\{u, x\}$ is an edge in H. Hence $x = uv^{-1}$ and since Γ is Abelian and $u \in B$, $\{x, v^{-1}\}$ is an edge in G.

In other words, for every edge $\{u, x\}$ in H, where $u, x \in B$ and $u = xv$ for some $v \in R$, there are two edges $\{u, v\}$ and $\{x, v^{-1}\}$ in G with one vertex in R and one vertex in B. This counting argument is illustrated in Figure 2.4.

Hence, $e_1[1_\Gamma] = e_1^G[1_\Gamma] + e_1^G(R) + 2e_2^H(B)$. Repeating the argument for $e_2[1_\Gamma]$ and subtracting, we get (ii) as required.

Now, suppose that $(R + B) \cap R = \emptyset$ and $e_1^G[v] > e_2^H[v]$. Then given any $u \in R$ and $v \in B$, we have $uv \notin R$ and therefore $\{u, uv\}$ is not an edge in the subgraph of $\mathsf{Cay}(\Gamma; B \cup R)$ induced by R. In other-words, this subgraph has no edges coloured 1 and therefore $e_1^G(R) = 0$. Therefore (iii) follows from (ii). $\square$

3

Existence and construction of 2-flip graphs

In this chapter we shall completely settle Problems 1.3.2, 1.3.4 and 1.3.5 for the case when $k = 2$, whilst also exploring in depth Problem 1.3.3. In particular, we ask for which positive integers b and r, such that $b < r$, does a $b + r$ regular graph G exist with edges coloured from $\{\text{blue, red}\}$ satisfying the following:

i. The subgraphs induced by the blue and red edges are b and r regular respectively, where across the entire graph 'red' wins against 'blue' since $r > b$.

ii. On the other-hand, for every vertex v, the number of blue edges in the closed neighbourhood of v is *greater* than the number of red edges, and therefore locally 'blue' wins against 'red'.

Somewhat intuitively, one expects that for (b, r) to be a 2-flip sequence the disparity between b and r must not be too large, as otherwise if r is much greater than b then the more difficult it is to have a flip. We shall see that this intuition is indeed correct — and that such flip graphs can be constructed using a small number of vertices.

3.1 | Existence of 2-flip graphs and sequences

Before proceeding any further, it is useful to define a vocabulary for describing edge-coloured triangles rooted at some vertex.

In a graph G with edges coloured from $\{1,2\}$ and $X, Y, Z \in \{1,2\}$, a triangle rooted at a vertex v is said to be of *type* XYZ *at* v if the two edges incident to v are coloured X and Y respectively, and the third edge is coloured Z. We wish to count the triangles in G based on these types; since a triangle has three vertices, each will be counted three times.

Let $T_{XYZ}(v)$ be the number of triangles of type XYZ rooted at v. Note that the types XYZ and YXZ are indistinguishable, and thus we count only one of them. We shall establish a simple convention, namely that the first two symbols of the type are written in increasing order. Therefore a triangle rooted at v can have one of six possible types, illustrated in Figure 3.1.

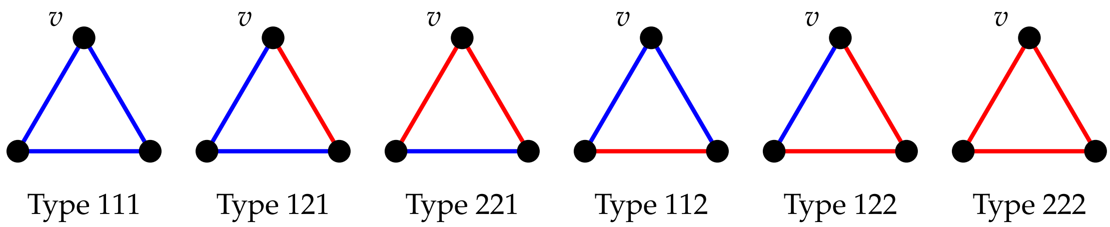

Figure 3.1: All six possible triangle types rooted at a vertex v, with *blue* representing the colour 1 and *red* representing the colour 2.

We shall make use of the following two straightforward lemmas.

Lemma 3.1.1. *In a graph G with edges coloured from $\{1,2\}$, we have that*

$$2\sum_{v\in V} T_{221}(v) = \sum_{v\in V} T_{122}(v) \quad \text{and} \quad 2\sum_{v\in V} T_{112}(v) = \sum_{v\in V} T_{121}(v)$$

Proof. Consider a triangle with two edges coloured 2 and a single edge coloured 1. Since the edges coloured 2 are incident to two vertices, call them u and w, then rooted at these vertices the triangle must be of type 122 in both cases. Rooted at the remaining vertex v, the triangle must be of type 221, since the blue edge is not incident to v.

Consequently, every such triangle is counted once in the summation $\sum_{v\in V} T_{122}(v)$ and twice in the summation $\sum_{v\in V} T_{221}(v)$. A similar argument follows for the relation between types 112 and 121. The result follows. $\square$

Lemma 3.1.2. *Let $b \in \mathbb{N}$ and let G be a graph with edges coloured from $\{1,2\}$. If v is a vertex such that v is incident to b edges coloured using 1, then*

$$T_{111}(v) + T_{112}(v) \leq \binom{b}{2}$$

Proof. Consider the set $N^{(1)}(v)$ of neighbours of v via an edge coloured 1. For any pair of distinct vertices $u, w \in N^{(1)}(v)$, either $\{u, w\}$ is in G or not. Since the graph is simple, we have that $\left|N^{(1)}(v)\right| = b$ and therefore there are at most $\binom{b}{2}$ edges amongst the vertices in $N^{(1)}(v)$.

Each of these edges is coloured using either 1 or 2; more so any triangle of type 111 or 112 at v corresponds to one of these edges. The result follows. $\square$

Having well understood this vocabulary for describing edge-coloured triangles, we are now in a position to establish a necessary condition for 2-flip graphs.

Proposition 3.1.3. *Let $r, b \in \mathbb{N}$ such that $b < r$. Let G be a graph with edges coloured using $\{1,2\}$, such that each vertex is incident to b and r edges coloured 1 and 2, respectively. If $r \geq \binom{b+1}{2}$, then G is not a (b,r)-flip graph.*

Proof. Suppose, for contradiction, that G has a suitable edge colouring using $\{1,2\}$ such that it is a (b,r)-flip graph. Note that in a (b,r)-flip graph, for every vertex $v \in V$, since $e_1[v] = e_1(v) + b$ and $e_2[v] = e_2(v) + r$, we require that $e_1(v) - e_2(v) > r - b$. It suffices to show, for contradiction, that there exists a vertex u such that $e_1(u) - e_2(u) \leq r - b$.

Expressing the sizes of $e_1(v)$ and $e_2(v)$ in terms of the counts of the six triangle types rooted at v, we obtain the inequality

$$\sum_{v \in V} (e_1(v) - e_2(v)) \tag{3.1}$$

$$= \sum_{v \in V} \Big(T_{111}(v) + T_{121}(v) + T_{221}(v)\Big) - \sum_{v \in V} \Big(T_{222}(v) + T_{112}(v) + T_{122}(v)\Big) \tag{3.2}$$

$$= \sum_{v \in V} \Big(T_{111}(v) - T_{222}(v)\Big) + \sum_{v \in V} \Big(T_{121}(v) - T_{112}(v)\Big) + \sum_{v \in V} \Big(T_{221}(v) - T_{122}(v)\Big) \tag{3.3}$$

$$\leq \sum_{v \in V} T_{111}(v) + \sum_{v \in V} \Big(T_{121}(v) - T_{112}(v)\Big) + \sum_{v \in V} \Big(T_{221}(v) - T_{122}(v)\Big) \tag{3.4}$$

$$= \sum_{v \in V} T_{111}(v) + \sum_{v \in V} T_{112}(v) - \sum_{v \in V} T_{221}(v) \tag{3.5}$$

$$\leq \sum_{v \in V} T_{111}(v) + T_{112}(v) \tag{3.6}$$

where (3.5) follows from Lemma 3.1.1.

Now, by Lemma 3.1.2 and the fact that $r \geq \binom{b+1}{2}$, we have that for every vertex v in G:

$$T_{111}(v) + T_{112}(v) \leq \binom{b}{2} = \binom{b+1}{2} - b \leq r - b$$

and therefore by inequality (3.6) we get that

$$\sum_{v \in V} (e_1(v) - e_2(v)) \leq |V|(r - b).$$

By the pigeon-hole principle, it follows that there exists a vertex $u \in V$ such that $e_1(v) - e_2(v) \leq r - b$, and therefore G is not a (b, r)-flip. □

As an immediate consequence of this result comes the necessity that $b \geq 3$.

Corollary 3.1.4. *If G is a (b, r)-flip graph, then $b \geq 3$.*

Proof. Since G is a (b, r)-flip graph, then $b < r$. Suppose that $b < 3$. By Proposition 3.1.3, when b is 1 it follows that $r \leq 0$, which is a contradiction since $b < r$. Likewise when $b = 2$, by Proposition 3.1.3 we have that $r \leq 2$, which is a contradiction once again. Therefore $b \geq 3$. □

We are now in a position to prove our main result for this section, outlining a full characterisation of 2-flip graphs and sequences.

Theorem 3.1.5. *Let $r, b \in \mathbb{N}$. If $3 \leq b < r \leq \binom{b+1}{2} - 1$ then there exists a (b, r)-flip graph, and both the upper bound and lower bound are sharp*

Proof. That $3 \leq b < r \leq \binom{b+1}{2} - 1$ follows immediately from Proposition 3.1.3 and Corollary 3.1.4. We show that given such r and b, a (b, r)-flip graph exists.

Consider the coloured Cartesian product $G = K_{r,r} \square K_{b+1}$, where the edges of K_{b+1} are coloured using 1 and the edges of $K_{r,r}$ are coloured using 2.

By virtue of Lemma 2.1.2, it follows that every vertex v in G has $\deg_1(v) = b$ and $\deg_2(v) = r$. Moreover we have that $e_1[v] = \binom{b+1}{2}$ and $e_2[v] = r$. Hence G is a (b, r)-flip graph. □

Remark 3.1.6. Recall that $h(b, r)$ is the number of vertices of the smallest (b, r)-flip graph. The graph constructed in the proof of Theorem 3.1.5 is a (b, r)-flip graph having $2r(b + 1)$ vertices. Hence $h(b, r) \leq 2r(b + 1)$.

3.1.1 | The cases $b = 1$ and $b = 2$

As a consequence of Corollary 3.1.4, it turns out that $(2, r)$-flip graphs do not exist, and $(3, 4)$ are the smallest parameters such that a (b, r)-flip graph exists. However, weakening the flipping constraint from $e_1[v] > e_2[v]$ to $e_1[v] \geq e_2[v]$, then an admissible colouring can be found such that the edges coloured 1 span a b regular subgraph and the edges coloured 2 span an r regular subgraph. We term such graphs as *weak*-flip graphs.

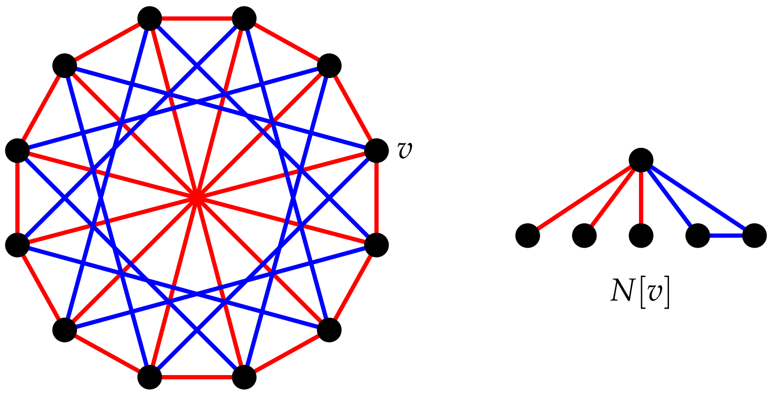

Figure 3.2: Smallest known $(2, 3)$-weak-flip graph having 12 vertices, with the subgraph induced by the closed neighbourhood of any vertex v illustrated on the right. It is impossible to construct a $(2, 3)$-flip graph.

The coloured Cartesian product $K_{3,3} \square K_3$, with $K_{3,3}$ coloured red and K_3 coloured blue, is a $(2, 3)$-weak-flip graph on 18 vertices. Figure 3.2 illustrates the existence of a smaller $(2, 3)$-weak-flip graph, having just 12 vertices.

Likewise, $(1, r)$-flip graphs do not exist. However, in this case, not even $(1, r)$-weak-flip graphs exist.

Proposition 3.1.7. *Let $r \in \mathbb{N}$ such that $r > 1$. Neither a $(1, r)$-flip graph nor a $(1, r)$-weak-flip graph exists.*

Proof. From Corollary 3.1.4, a $(1, r)$-flip graph certainly does not exist. Consider the case when $r > 2$. Suppose that a $(1, r)$-weak-flip graph G exists. Let v be a vertex in such a graph. Since each vertex has a single incident edge coloured 1, then $e_1[v] \leq 1 + \lfloor \frac{r}{2} \rfloor < r \leq e_2[v]$ since $r > 2$.

In the case when $r = 2$, suppose that a $(1, 2)$-weak-flip graph G exists. Then by the above argument, it follows that $e_1[v] = e_2[v] = 2$.

Consider a vertex v with neighbours u, w and x such that $\{v, w\}$ and $\{v, u\}$ are coloured 2 and $\{v, x\}$ is coloured 1. Clearly $\{u, w\}$ must be coloured 1 since $e_1[v] = e_2[v] = 2$. But then u has some neighbour y different from v and w, such that $\{u, y\}$ is coloured 2. Hence $e_2[u] \geq 3$, a contradiction. □

3.2 | Constructing (small) 2-flip graphs

For all valid parameters (b, r) for which a (b, r)-flip graph exists, as in Theorem 3.1.5, we have outlined a construction on $2r(b + 1)$ vertices as highlighted in Remark 3.1.6. This raises the question of whether this bound is sharp — already for $h(3, 4)$ we have seen in Figure 1.1 an example of a $(3, 4)$-flip graph with 16 vertices, smaller than $2r(b + 1) = 32$.

In light of this, this section shall be devoted to developing different constructions for 2-flip graphs which significantly improve on this bound.

3.2.1 | Bounding $h(b, r)$ using coloured products

We begin with a generalisation of our construction in the proof of Theorem 3.1.5.

Theorem 3.2.1. *Let $b, r \in \mathbb{N}$ such that $b < r$. Then,*

$$h(b, r) \leq \min \left\{ 2(r + x)(b + 1 - x) \colon x \in \mathbb{Z}, 0 \leq x \leq b, x + \binom{b + 1 - x}{2} > r \right\}$$

Proof. Suppose we are given (b, r). Clearly $h(b, r) \leq 2r(b + 1)$ and $r < \binom{b+1}{2}$ by Theorem 3.1.5. Let x be an integer for which $x + \binom{b+1-x}{2} > r$; by our previous remark, at least one such x exists (namely $x = 0$). Consider an edge-colouring of $K_{r+x,r+x}$ such that an x-factor is coloured using colour 1 and an r-factor is coloured using 2. Also consider K_{b+1-x} where all the edges are coloured by 1.

By Lemma 2.1.2, every vertex v in the coloured Cartesian product of $K_{r+x,r+x}$ and K_{b+1-x} has $\deg_1(v) = b + x - x = b$ and $\deg_2(v) = r$.

Moreover, $e_1[v] = x + \binom{b+1-x}{2}$ and $e_2[v] = r$ (since $K_{r+x,r+x}$ is bipartite). By our choice of x, it follows that $e_1[v] > e_2[v]$. $\square$

The upper-bound in Theorem 3.2.1 warrants further analysis. Considering the case when $r = b+1$, the coloured Cartesian product construction $K_{r,r} \,\square\, K_r$ gave an upper bound on $h(b, b+1) \leq 2(b+1)^2$. Perhaps surprisingly, we shall see that explicit analysis on Theorem 3.2.1 significantly improves the upper bound to $h(b, b+1) \leq 16(b-1)$. We first require the following two lemmas.

Lemma 3.2.2. *Let $b, r \in \mathbb{N}$ such that $b < r$. Define $f(x) = 2(r+x)(b+1-x)$. Then, $f(x) < f(x-1)$ and, in particular, $f(x) < 2r(b+1)$.*

Proof. Equating $f(x)$ to $f(x-1)$, it suffices to show that $r + 2x > b + 2$. This follows immediately since $r > b$ and $x \geq 1$. Hence $f(x)$ is monotonically decreasing on $\mathbb{N}$. In particular, $f(x) \leq f(1) < f(0) = 2r(b+1)$. $\square$

As an immediate consequence of Lemma 3.2.2, we have that the upper bound in Theorem 3.2.1 always improves upon or (at worst) is equal to the bound $2r(b+1)$. We are now in a position to explicitly compute the minimum.

Lemma 3.2.3. *Let $b, r \in \mathbb{N}$ such that $b < r$. Let $x_0 = \left\lceil b - \dfrac{1 + \sqrt{1 + 8(r-b)}}{2} \right\rceil - 1$. Then, $2(r + x_0)(b + 1 - x_0)$*

$$= \min\left\{2(r+x)(b+1-x) : x \in \mathbb{Z}, 0 \leq x \leq b, x + \binom{b+1-x}{2} > r\right\}$$

Proof. Let $g(z) = z + \binom{b+1-z}{2} - r$ be a real-valued function. By Lemma 3.2.2 the minimum, in integer value, is attained for the largest possible integer $0 \leq x_0 \leq b$, such that $g(x_0) > 0$. Re-arranging, $g(z)$ can be written as a quadratic in z,

$$g(z) = \binom{b+1}{2} - r + \left(\frac{1}{2} - b\right) z + \frac{z^2}{2}$$

which has a minimum, as well as distinct roots $z_{\pm} = b - \frac{1 \mp \sqrt{1+8(r-b)}}{2}$.

Then $g(z) > 0$ whenever $z < z_-$ or $z > z_+$. Since the integer x_0 we are seeking must satisfy $x_0 \leq b$, then the only admissible case when $g(x_0) > 0$ is when $x_0 < z_-$. Since we seek the largest such integer, then

$$x_0 = \left\lceil b - \frac{1+\sqrt{1+8(r-b)}}{2} \right\rceil - 1$$

which we now show is non-negative.

It suffices to show that $b > \frac{1+\sqrt{1+8(r-b)}}{2}$. Re-arranging, we require that $\frac{(2b-1)^2-1}{8} > r - b$. Indeed,

$$\frac{(2b-1)^2-1}{8} = \frac{b^2}{2} - \frac{b}{2} = \binom{b+1}{2} - b > r - b$$

by Theorem 3.1.5. Hence x_0 as derived is the largest integer for which the minimum is attained, as required. $\square$

Proposition 3.2.4. *Let $b, r \in \mathbb{N}$ such that $3 \leq b < r$. Then*

$$h(b,r) \leq 2\left(r+b+1-\left\lfloor \frac{5+\sqrt{1+8(r-b)}}{2} \right\rfloor\right)\left\lfloor \frac{5+\sqrt{1+8(r-b)}}{2} \right\rfloor$$

Proof. Substituting in Theorem 3.2.1 for Lemmas 3.2.2 and 3.2.3, we get that

$$\begin{aligned}
&h(b,r)\\
&\leq 2\left(r-1+\left\lceil b - \frac{1+\sqrt{1+8(r-b)}}{2} \right\rceil\right)\left(b+2-\left\lceil b - \frac{1+\sqrt{1+8(r-b)}}{2} \right\rceil\right)\\
&= 2\left(r+b-1+\left\lceil - \frac{1+\sqrt{1+8(r-b)}}{2} \right\rceil\right)\left(2-\left\lceil - \frac{1+\sqrt{1+8(r-b)}}{2} \right\rceil\right)\\
&= 2\left(r+b-1-\left\lfloor \frac{1+\sqrt{1+8(r-b)}}{2} \right\rfloor\right)\left(2+\left\lfloor \frac{1+\sqrt{1+8(r-b)}}{2} \right\rfloor\right)\\
&= 2\left(r+b+1-\left\lfloor \frac{5+\sqrt{1+8(r-b)}}{2} \right\rfloor\right)\left\lfloor \frac{5+\sqrt{1+8(r-b)}}{2} \right\rfloor
\end{aligned}$$

as required. $\square$

Indeed for the case when $r = b+1$, the minimum is obtained at $x_0 = r - 4$. Then the flip graph $G = K_{2r-4,2r-4} \,\square\, K_4$ from Theorem 3.2.1 is a $(b, b+1)$-flip graph with $16r - 32$ vertices, as per our introductory remarks.

3.2.2 | Bounding $h(b, r)$ through Cayley flip graphs

It transpires that our recurring example in Figure 1.1 is the packing of two Cayley graphs for the dihedral group D_8, resulting in $\mathsf{Cay}\,(D_8; R \cup B)$ where the edges labelled in $R = \{ab, a^{-1}b, a^3b, a^{-3}b\}$ are coloured *red* whilst those in $B = \{a, a^{-1}, b\}$ are coloured *blue*. With the toolset outlined in Section 2.3.1 on Cayley graph packings, we shall construct a similar class of Cayley flip graphs associated with cyclic groups.

Let $b, r \in \mathbb{N}$ satisfy the conditions for (b, r) to be a flip sequence. Suppose that for some Abelian group Γ we can construct two Cayley graphs $G = \mathsf{Cay}\,(\Gamma; B)$ and $H = \mathsf{Cay}\,(\Gamma; R)$ satisfying the conditions in Proposition 2.3.2 (iii), $|B| = b$ and $|R| = r$. Then their packing is a (b, r)-flip graph (which in particular turns out to be another Cayley graph).

For the case when $\Gamma = \mathbb{Z}_n$, finding such sets R and B entails finding suitably large subsets of $\mathbb{Z}_n$ which are both inverse-closed and sum-free. Sum-free sets in Abelian groups have been studied vastly and are of interest in additive combinatorics and number theory, see Alon and Kleitman (1990); Green and Ruzsa (2005); Tao and Vu (2016). We therefore begin with the following useful lemma.

Lemma 3.2.5. *If A is a non-empty subset of the integer interval $\left(\frac{n}{8}, \frac{n}{4}\right)$ in $\mathbb{Z}_n$, then $A \cup A^{-1}$ is a set of order $2|A|$ which is sum-free and inverse-closed.*

Proof. Let A be as above. Firstly note that $A^{-1} \subseteq \left(\frac{3n}{4}, \frac{7n}{8}\right)$, and therefore $2A \subseteq \left(\frac{n}{4}, \frac{n}{2}\right)$ and $2A^{-1} \subseteq \left(\frac{n}{2}, \frac{3n}{4}\right)$. Moreover, $A + A^{-1} \subseteq \left(\frac{7n}{8}, n\right) \cup \left(0, \frac{n}{8}\right)$.

Therefore we have that $A \cup A^{-1}$ is sum-free, since none of the sums intersect the interval $\left(\frac{n}{8}, \frac{n}{4}\right)$ containing A and the interval $\left(\frac{3n}{4}, \frac{7n}{8}\right)$ containing A^{-1}.

More so, since $A \cap A^{-1} = \varnothing$ and A contains no involutions, it follows that $A \cup A^{-1}$ has order $2|A|$. □

Furthermore, we will also require the following lemma.

Lemma 3.2.6. *Let A_0, B_0 be non-empty disjoint integer intervals of $\left(\frac{n}{8}, \frac{n}{4}\right)$ in $\mathbb{Z}_n$, such that $\max\,(A_0) < \min\,(B_0)$. Let $B_1 \subseteq B_0$ be an integer interval, $A = A_0 \cup A_0^{-1}$ and $B = B_0 \cup B_0^{-1} \cup 2B_1 \cup 2B_1^{-1}$. Then, $(A + B) \cap A = \varnothing$. Moreover if n is even, $\left(A + \left\{\frac{n}{2}\right\}\right) \cap A = \varnothing$ and furthermore if $\min(B_1) \geq \frac{3n}{16}$ then $\left(\left\{\frac{n}{2}\right\} + B\right) \cap A = \varnothing$.*

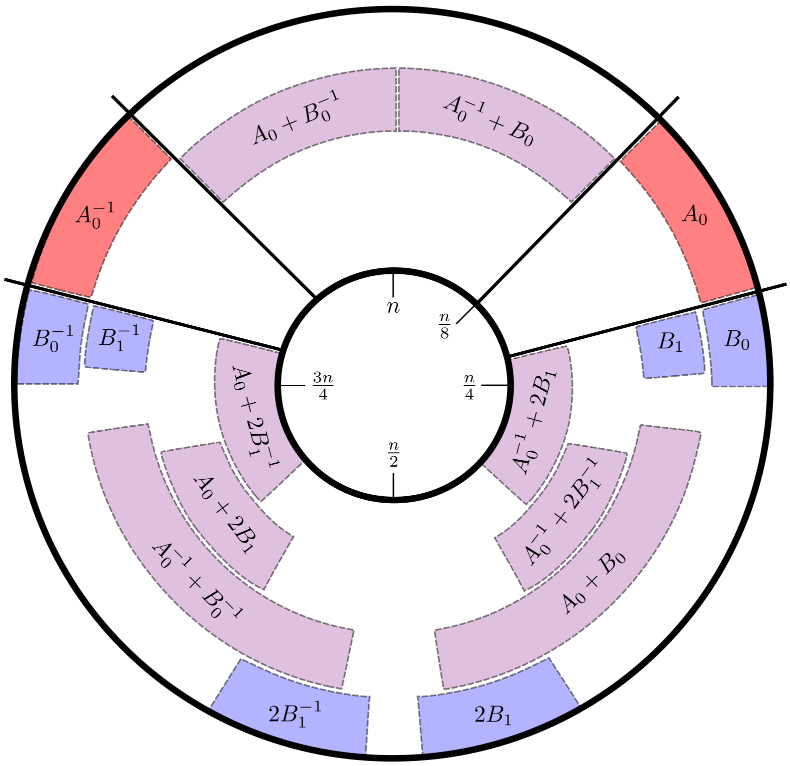

Figure 3.3: Illustration of the sets and sum-sets of interest in Lemma 3.2.6. The cyclic group $\mathbb{Z}_n$ is represented as a 'clock' with its elements written in a 'clockwise' fashion. The outermost ring illustrates the sets composing A and B, whilst the three inner rings highlight the possible sum-sets involving A and B. Observe that none of these sum-sets intersect the set $A = A_0 \cup A_0^{-1}$, and therefore $(A + B) \cap A = \emptyset$.

Proof. Since A_0 and B_0 are disjoint intervals of $\left(\frac{n}{8}, \frac{n}{4}\right)$ such that $\max(A_0) < \min(B_0)$, then there exists integers m, l, M, L such that $\frac{n}{8} < m < l < L < M < \frac{n}{4}$, $A_0 = [m, l]$ and $B_0 = [L, M]$. Observe that the sets A_0, B_0, $2B_1$ and their inverses are all disjoint and, in the case when n is even, none include the involution $\frac{n}{2}$.

We consider every possible summation across these sets, in order to show that $(A + B) \cap A = \emptyset$, as required. Figure 3.3 illustrates the sets and sum-sets of interest, highlighting their non-intersection with A.

Firstly note that since A_0 and B_0 are disjoint subsets of $\left(\frac{n}{8}, \frac{n}{4}\right)$ then any possible sum between them and their inverses does not intersect (in particular) A, as a consequence of Lemma 3.2.5.

Also, $A_0 + 2B_1$ is a subset of $\left(\frac{3n}{8}, \frac{3n}{4}\right)$ whilst $A_0^{-1} + 2B_1^{-1}$ is a subset of $\left(\frac{n}{4}, \frac{5n}{8}\right)$, and therefore they do not intersect A.

Now, $\min\left(A_0 + 2B_0^{-1}\right) = (m - 2M) \mod n > \max(A_0)$ and

$$\max\left(A_0 + 2B_0^{-1}\right) = l - 2L < -l = \min\left(A_0^{-1}\right).$$

Hence $\left(A_0 + 2B_0^{-1}\right) \cap A = \emptyset$. Since $B_1 \subseteq B_0$ we have $\left(A_0 + 2B_1^{-1}\right) \cap A = \emptyset$. Similarly we have that $\left(A_0^{-1} + 2B_1\right) \cap A = \emptyset$. This covers all possible cases and therefore we have that for our choice of A and B, $(A + B) \cap A = \emptyset$ is satisfied.

We now consider the case when n is even. Since $A + \left\{\frac{n}{2}\right\} \subseteq \left[\frac{n}{4}, \frac{3n}{8}\right] \cup \left[\frac{5n}{8}, \frac{3n}{4}\right]$ then $\left(A + \left\{\frac{n}{2}\right\}\right) \cap A = \emptyset$ as required. Also, $\left\{\frac{n}{2}\right\} + B_0 \cup B_0^{-1} \subseteq \left[\frac{n}{4}, \frac{3n}{8}\right] \cup \left[\frac{5n}{8}, \frac{3n}{4}\right]$ and $\min\left(\left\{\frac{n}{2}\right\} + 2B_1^{-1}\right) > 0$. If $\min(B_1) \geq \frac{3n}{16}$, then

$$\max\left(\left\{\frac{n}{2}\right\} + 2B_1^{-1}\right) = \frac{n}{2} - 2\min(B_1) \leq \frac{n}{8}$$

and consequently $\left(\left\{\frac{n}{2}\right\} + 2B_1^{-1}\right) \cap A = \emptyset$. By a similar argument we obtain that $\left(\left\{\frac{n}{2}\right\} + 2B_1\right) \cap A = \emptyset$ and hence $\left(\left\{\frac{n}{2}\right\} + B\right) \cap A = \emptyset$, as required. $\square$

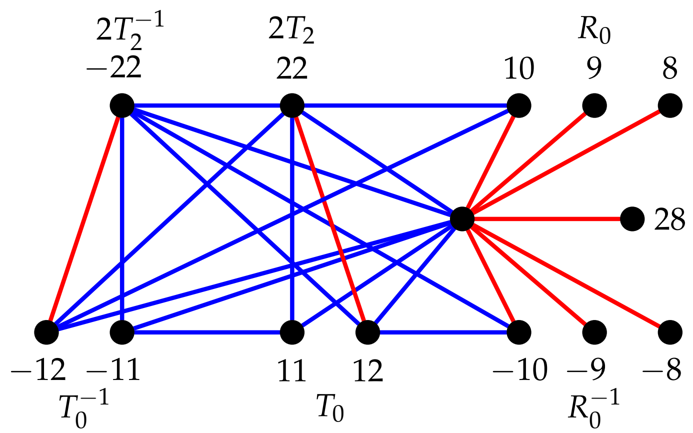

Figure 3.4: Illustration of the closed neighbourhood of the identity in the Cayley graph construction for $(b, r) = (6, 7)$ and $n = 56$ in the proof of Theorem 3.2.7, with the choice of R_0, T_0 and T_2 highlighted.

Equipped with this lemma and Proposition 2.3.2 (iii), we are able to construct small 2-flip graphs for a wide range of admissible 2-flip sequences. Figure 3.4 is an illustration of the construction presented in the following theorem.

Theorem 3.2.7. *Let $b, r \in \mathbb{N}$ such that $4 \leq b < r < b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2$. Then,*

$$h(b,r) \leq 8\lambda_{b,r}\left(2 + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor\right)$$

where $\lambda_{b,r} = \max\{1, (b \mod 2) + (r \mod 2)\}$.

Proof. Let $n \in \mathbb{N}$ such that $n = 8\left(2 + \lfloor \frac{r}{2} \rfloor + \lfloor \frac{b+2}{2} \rfloor - 2\lfloor \frac{b+2}{6} \rfloor\right)$ and consider $\mathbb{Z}_n$, the additive group modulo n. By this choice of n, the interval $\left(\frac{n}{8}, \frac{n}{4}\right)$ has two disjoint integer intervals R_0 and T_0 of sizes $\lfloor \frac{r}{2} \rfloor$ and $\lfloor \frac{b+2}{2} \rfloor - 2\lfloor \frac{b+2}{6} \rfloor$ respectively.

Choose these intervals such that $\max(R_0) < \min(T_0)$. We have that

$$\lfloor \tfrac{b+2}{2} \rfloor - 2\lfloor \tfrac{b+2}{6} \rfloor = \lfloor \tfrac{b+2}{6} \rfloor + \lfloor \tfrac{1}{2}(b + 2 - 6\lfloor \tfrac{b+2}{6} \rfloor) \rfloor = \lfloor \tfrac{b+2}{6} \rfloor + \lfloor \tfrac{(b+2) \mod 6}{2} \rfloor$$

and therefore T_0 has at least $\lfloor \frac{b+2}{6} \rfloor$ integers. By our choice of n, we can choose T_0 such that it has a sub-interval T_2 of size $\lfloor \frac{b+2}{6} \rfloor$, and $\min(T_2) \geq \frac{3n}{16}$ since R_0 contains at least half of the interval $\left(\frac{n}{8}, \frac{n}{4}\right)$.

Define the sets $R_1 = R_0 \dot\cup R_0^{-1}$ and $T_1 = T_0 \dot\cup T_0^{-1}$, which are inverse-closed and sum-free. Define $B_1 = T_1 \dot\cup 2T_2 \dot\cup 2T_2^{-1}$. Since T_2 is an integer interval and $2T_2$ is the sum-set of T_2 with itself, then $|2T_2| = 2|T_2| - 1$. Moreover, $|T_2| = |T_2^{-1}|$ and $|T_0| = |T_0^{-1}|$. Therefore,

$$|T_1| + |2T_2| + \left|2T_2^{-1}\right| = 2|T_0| + 4|T_2| - 2 = 6\left\lfloor \frac{b+2}{6} \right\rfloor + 2\left\lfloor \frac{(b+2) \mod 6}{2} \right\rfloor - 2$$

and hence $|B_1| = b - (b \mod 2)$.

The sets R_1 and B_1 have even size, and we may need to add some involutions to them to get the size equal to r and b respectively (which may be odd). Three cases may arise:

i. Both r and b are even and therefore $|R_1| = r$ and $|B_1| = b$. In this case let $\Gamma = \mathbb{Z}_n$, $R = R_1$ and $B = B_1$. Consequently, $\lambda_{b,r} = 1$.

ii. Either r is odd or b is odd, in which case we let $\Gamma = \mathbb{Z}_n$. If r is odd we define $R = R_1 \cup \{\frac{n}{2}\}$ and $B = B_1$. Else if b is odd we define $R = R_1$ and $B = B_1 \cup \{\frac{n}{2}\}$. Consequently R and B are inverse-closed and have size r and b respectively. Moreover, R is sum-free. In this case $\lambda_{b,r} = 1$.

iii. Both r and b are odd, in which case we let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_n$. Define $B = (\{0\} \times B_1) \cup \{(0, \frac{n}{2})\}$ and $R = (\{0\} \times R_1) \cup \{(1,0)\}$, noting that $(0, \frac{n}{2})$ and $(1,0)$ are involutions in $\mathbb{Z}_2 \times \mathbb{Z}_n$. Moreover, R is sum-free by choice of R_1 and by the properties of the direct product. In this case $\lambda_{b,r} = 2$.

Consider $H = \mathsf{Cay}(\Gamma; R)$ coloured monochromatically using colour 2. Since R is sum-free, then $e_2^H[v] = r = \deg_2^H(v)$ for any v in Γ.

Now consider $G = \mathsf{Cay}(\Gamma; B)$ coloured monochromatically using colour 1. For any v in Γ, $\deg_1^G(v) = b$. Moreover, there are at least $2\left\lfloor \frac{b+2}{6} \right\rfloor^2$ edges in the open neighbourhood of v, since the group is Abelian and therefore the $\left\lfloor \frac{b+2}{6} \right\rfloor^2$ sums from T_2 to $2T_2$, and from T_2^{-1} to $2T_2^{-1}$, all contribute an edge to the open neighbourhood. Hence,

$$e_1^G[v] \geq b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2 > r = e_2^H[v]$$

Finally observe that by construction, as a consequence of Lemma 3.2.6 and properties of the direct product, $(R + B) \cap R = \emptyset$. By Proposition 2.3.2 (iii), $\mathsf{Cay}(\Gamma; R \cup B)$ is a (b,r)-flip graph and by our choice of Γ, the bound on $h(b,r)$ follows. □

It is worth comparing this bound to the bound given in Proposition 3.2.4. We begin by noting that this bound is significantly better than the one in Proposition 3.2.4. This is illustrated in Figure 3.5, for fixed b and $1 < r - b < 2\left\lfloor \frac{b}{6} \right\rfloor^2$.

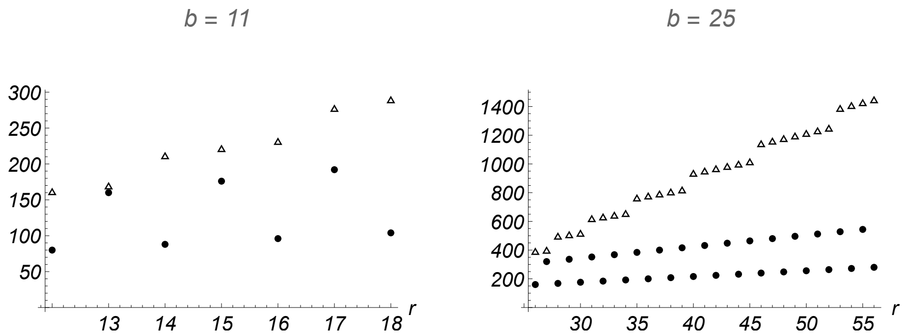

Figure 3.5: Comparison of the bounds in △ Theorem 3.2.4 and • Theorem 3.2.7 for $b = 11$ and $b = 25$ over the common range for r between the two bounds.

Observe, however, that the bound in Proposition 3.2.4 holds for a wider range of values of b, suggesting that further work is to be done towards a unified bound. Over the range for which the bound in Theorem 3.2.7 holds, we have that it is asymptotically the best possible.

Corollary 3.2.8. *Let $b, r \in \mathbb{N}$, $4 \leq b < r < b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2$. Then $h(b,r)$ is $\Theta(b+r)$.*

Proof. That $h(b,r)$ is $O(b+r)$ follows immediately from Theorem 3.2.7. Moreover, since any graph realising the flip sequence (b,r) must be $b+r$ regular, then $b + r \leq h(b,r)$ and hence $h(b,r)$ is $\Omega(b+r)$. $\square$

4

Existence and construction of t-neighbourhood flip graphs

We have so far considered an in-depth treatment of the flip colouring of graphs with regards to the immediate neighbourhood of a vertex. A natural extension of this problem, as outlined in Problem 1.3.7, is to consider not just a flip in the immediate neighbourhood of a vertex v, but in all the neighbourhoods $N_1[v], N_2[v], \ldots, N_t[v]$ for some $1 \leq t < \mathsf{diam}(G)$.

Before proceeding further, we require an adaptation of our existing notation. Let $k \in \mathbb{N}$ and let G be a graph with an edge-colouring from $\{1, \ldots, k\}$. For a vertex v and $t \in \mathbb{N}$, let $e_{j,t}(v) = \left|E_j(G) \cap E\left(N_t(v)\right)\right|$ and $e_{j,t}[v] = \left|E_j(G) \cap E\left(N_t[v]\right)\right|$.

Note that when $t = 1$ the above reconciles with our previous definitions. We now extend our general flip colouring problem to t-neighbourhoods. Given a graph $G = (V, E)$, and $k, t \in \mathbb{N}$ such that $k \geq 2$, we are interested in whether there exists an edge-colouring $f \colon E(G) \to \{1, \ldots, k\}$ such that:

i. for every $v \in V$, $\deg_j(v) > \deg_i(v)$ for $1 \leq i < j \leq k$, forcing a global majority ordering $e_j > e_i$,

ii. and for every $v \in V$ and $1 \leq l \leq t$, $e_{j,l}[v] < e_{i,l}[v]$ for $1 \leq i < j \leq k$, resulting in a local opposite majority ordering up to a distance t with respect to the global e_j and the local $\deg_j(v)$.

If such an edge-colouring exists then G is said to be a $([t], k)$-flip graph. When we do not concern ourselves with the number of colours used, we shall simply say that G is a $[t]$-flip graph.

As before, we shall consider a more restricted version of this problem, where for every $j \in \{1, \dots, k\}$, the edge-set E_j spans a regular subgraph of degree a_j, where $a_1 < a_2 < \cdots < a_k$. We term such graphs as $([t], (a_1, \dots, a_k))$-flip graphs.

In this chapter we shall consider the existence of $([t], 2)$-flip graphs, through two different perspectives, namely through Cayley graphs and packing arguments. The existence of $[t]$-flip graphs on three or more colours remains wide open at the time of writing, and is a potentially interesting line of research.

An illustrative example of a $([t], 2)$-flip graph is given in Figure 4.1. The edges between the first and second neighbourhoods are illustrated on a separate figure for clarity.

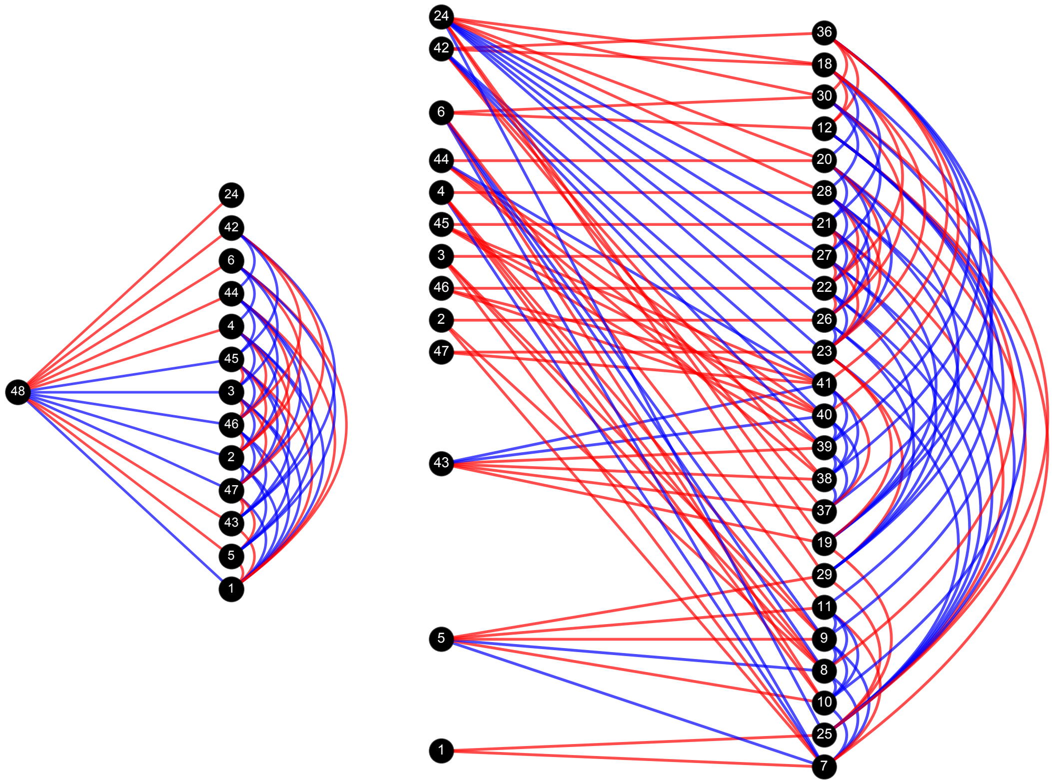

Figure 4.1: The neighbourhoods $N_0(48)$, $N_1(48)$ and $N_2(48)$ of $\mathrm{Cay}(\mathbb{Z}_{48}; R \cup B)$ drawn layered left to right, where $R = \{4, 5, 6, 24, 42, 43, 44\}$ and $B = \{1, 2, 3, 45, 46, 47\}$. The graph is a $([2], (6, 7))$-flip graph.

4.1 | Constructing $([t], 2)$-flip graphs through Cayley graphs

This subsection shall be devoted to demonstrating the existence of $([t], 2)$-flip graphs through the perspective of algebraic graph theory, namely Cayley graphs.

By $\mathbf{0}$ we shall denote the all-zeros vector in $\mathbb{Z}_2^n$. Given any $i \in \{1, \ldots, n\}$, let $\mathbf{e}_i$ be the vector in $\mathbb{Z}_2^n$ which is 1 at the i^{th} position, and 0 everywhere else. Now, given any $1 \leq s \leq n$ and $0 \leq l \leq n - s$, we denote by $W_{s,l}$ the set of all binary vectors with the first s entries all zero and exactly l non-zero subsequent entries, namely

$$W_{s,l} = \left\{ \mathbf{w} \in \mathbb{Z}_2^n \colon \mathbf{w} \cdot \left(\sum_{i=1}^{s} \mathbf{e}_i \right) = 0 \ \wedge \ \mathbf{w} \cdot \left(\sum_{i=s+1}^{n} \mathbf{e}_i \right) = l \right\}$$

Having established our working notation, we now proceed to prove our main result for this section.

Theorem 4.1.1. *Let $s, t \in \mathbb{N}$, $t < s$. There exists a $([t], (2^s - 1, 2^s))$-flip graph.*

Proof. Let $n = 2^s + s$. Consider $\mathbf{H} = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_s\}$ and let $B = \mathbf{H} \backslash \{\mathbf{0}\}$. Since $\mathbf{H}$ is a sub-space of $\mathbb{Z}_2^n$, we have that $|B| = 2^s - 1$. Let $R = \{\mathbf{e}_{s+1}, \ldots, \mathbf{e}_n\}$; by our choice of n, we have that $|R| = 2^s$. Consider $\mathsf{Cay}\left(\mathbb{Z}_2^n; R \cup B\right)$ with the edge-colouring $f \colon E \to \{1, 2\}$ such that given $v \in \mathbb{Z}_2^n$ and $\alpha \in R \cup B$:

$$f\left(\{v, \alpha v\}\right) = \begin{cases} 1, & \alpha \in B \\ 2, & \alpha \in R \end{cases}$$

Note here that $|R \cup B| = 2^{s+1} - 1 > 2^s + s$ and therefore $R \cup B$ spans $\mathbb{Z}_2^n$. Moreover, each $\mathbf{e}_i$ is an involution and therefore $R \cup B$ is inverse-closed. Clearly also, $R \cap B = \emptyset$ and therefore the edge-colouring f is well-defined.

Since a Cayley graph is vertex-transitive, it suffices to consider a single vertex. Consider $\mathbf{0} \in \mathbb{Z}_2^n$; by the edge-colouring f, we have that

$$\deg_1(\mathbf{0}) = 2^s - 1 < 2^s = \deg_2(\mathbf{0})$$

and by vertex-transitivity the same holds for all $\mathbf{v} \in \mathbb{Z}_2^n$.

Now, $N_1(\mathbf{0}) = R \cup B$; we will add R and B to $N_1(\mathbf{0})$ so that we find the vertices in $N_2(\mathbf{0})$. Since $B \cup \{\mathbf{0}\}$ is a vector-space, then in particular $B + B = B$. On the other-hand, adding B to R results in the set $B + W_{s,1}$. Hence, $(R \cup B) + B = B + W_{s,1}$. Since neither R nor B includes $\mathbf{0}$, it follows that $(R \cup B) \cap (B + W_{s,1}) = \emptyset$ and therefore $B + W_{s,1} \subseteq N_2(\mathbf{0})$. Likewise, adding R to B gives $B + W_{s,1}$ once more and $R + R = W_{s,2}$. Therefore,

$$N_2(\mathbf{0}) = (B + W_{s,1}) \ \dot\cup \ W_{s,2}$$

and repeating the above argument for $1 \leq l < t$, we get that:

$$N_{l+1}(\mathbf{0}) = (B + W_{s,l}) \ \dot\cup \ W_{s,l+1}.$$

Note that for $\mathbf{w} \in W_{s,l}$, $B + \mathbf{w}$ is a clique isomorphic to K_{2^s-1} since $\mathbf{w}$ is not in the span of B. More so, for $\mathbf{w}_1, \mathbf{w}_2 \in W_{s,l}$ such that $\mathbf{w}_1 \neq \mathbf{w}_2$, we have that $(B + \mathbf{w}_1) \cap (B + \mathbf{w}_2) = \emptyset$. Therefore, $(2^s - 1)\binom{n-s}{l} = (2^s - 1)\binom{2^s}{l}$ edges coloured 1 arise between $W_{s,l}$ and $B + W_{s,l}$. Observe also that $(R + N_l(\mathbf{0})) \cap N_l(\mathbf{0}) = \emptyset$, and therefore the subgraph induced by $N_l(\mathbf{0})$ contains no edges coloured 2.

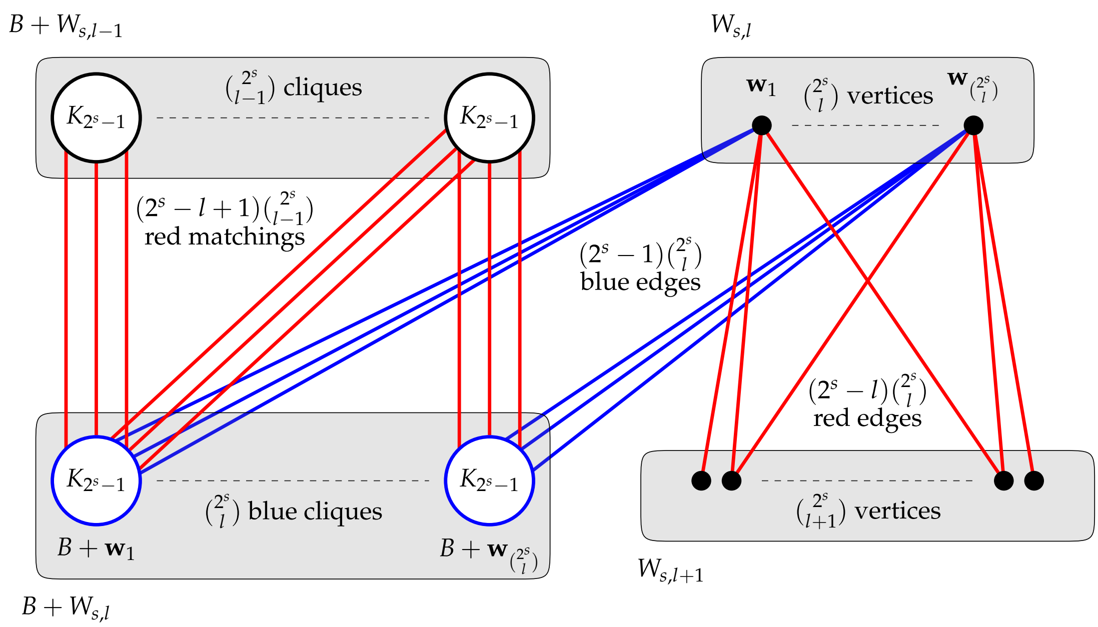

Figure 4.2: Illustration of the edges between $N_l(\mathbf{0})$ and $N_{l+1}(\mathbf{0})$, with *blue* and *red* representing the edge colours 1 and 2, respectively.

The edges coloured 1 between $N_l(\mathbf{0})$ and $N_{l+1}(\mathbf{0})$ arise by adding B to $W_{s,l}$ and therefore by our previous remark there are $(2^s-1)\binom{2^s}{l}$ such edges. Meanwhile, the edges coloured 2 between $N_l(\mathbf{0})$ and $N_{l+1}(\mathbf{0})$ arise by adding R to $N_l(\mathbf{0})$. Adding R to $W_{s,l}$ results in $(2^s-l)|W_{s,l}| = (2^s-l)\binom{2^s}{l}$ edges coloured 2 between $N_l(\mathbf{0})$ and $N_{l+1}(\mathbf{0})$. On the other-hand, adding R to $B+W_{s,l-1}$ maps each clique in $B+W_{s,l-1}$ to a total of 2^s-l+1 cliques in $B+W_{s,l}$, with a perfect-matching between every such pair of cliques. Therefore, there are an additional $(2^s-1)(2^s-l+1)\binom{2^s}{l-1}$ edges coloured 2 between $N_l(\mathbf{0})$ and $N_{l+1}(\mathbf{0})$. These edges between $N_l(\mathbf{0})$ and $N_{l+1}(\mathbf{0})$ are illustrated in Figure 4.2.

By our previous remark, the subgraph induced by $N_{l+1}(\mathbf{0})$ contains no edges coloured 2 and hence it follows that:

$$e_{2,l+1}[\mathbf{0}] = e_{2,l}[\mathbf{0}] + (2^s-1)(2^s-l+1)\binom{2^s}{l-1} + (2^s-l)\binom{2^s}{l} \tag{4.1}$$

while between the vertices in $N_{l+1}(\mathbf{0})$ there are $\binom{n-s}{l} = \binom{2^s}{l}$ cliques coloured 1 which are isomorphic to K_{2^s-1} and therefore:

$$e_{1,l+1}[\mathbf{0}] = e_{1,l}[\mathbf{0}] + \binom{2^s-1}{2}\binom{2^s}{l} + (2^s-1)\binom{2^s}{l} = e_{1,l}[\mathbf{0}] + \binom{2^s}{2}\binom{2^s}{l} \tag{4.2}$$

Now consider,

$$\begin{aligned}
&(2^s-1)(2^s-l+1)\binom{2^s}{l-1} + (2^s-l)\binom{2^s}{l} \\
&= (2^s-1)l\binom{2^s}{l} + (2^s-l)\binom{2^s}{l} \\
&\leq (2^s-1)(l+1)\binom{2^s}{l} && \because 2^s-l \leq 2^s-1 \\
&< \binom{2^s}{2}\binom{2^s}{l} && \because l+1 \leq t \leq s-1 < \frac{2^s}{2}
\end{aligned}$$

and consequently, from (4.1) and (4.2), if $e_{2,l}[\mathbf{0}] < e_{1,l}[\mathbf{0}]$ then

$$e_{2,l+1}[\mathbf{0}] < e_{1,l+1}[\mathbf{0}] \tag{4.3}$$

for $1 \leq l < t$.

We shall proceed by finite-induction on l. By choice of R and B, we have that the vertices of B in $\text{Cay}(\mathbb{Z}_2^n; R \cup B)$ induce the complete graph K_{2^s-1} and therefore

$$e_{1,1}[\mathbf{0}] = (2^s - 1) + \binom{2^s - 1}{2} = \binom{2^s}{2}$$

while the vertices in R are all linearly-independent and hence $e_{2,1}[\mathbf{0}] = 2^s$. Clearly the graph is $([1], (2^s - 1, 2^s))$-flipping; hence the base-case holds. We now show that the graph is $([l+1], (2^s - 1, 2^s))$-flipping for $1 \leq l < t$.

Indeed, suppose that the graph is $([l], (2^s - 1, 2^s)$-flipping. Then the result immediately follows by (4.3). □

4.2 | Constructing $([t], 2)$-flip graphs through packings

This subsection shall be devoted to demonstrating the existence of $([t], 2)$-flip graphs using two classical graph theoretic results, concerned with the existence of r regular graphs with large girth, and with graph packings.

Theorem 4.2.1 (Erdős and Sachs (1963)). *Given any $r, k \in \mathbb{N}$, there are infinitely many connected r regular graphs with girth k.*

Theorem 4.2.2 (Catlin (1976); Sauer and Spencer (1978)). *Let G and H be two graphs on n vertices, such that $2\Delta(G)\Delta(H) < n$. Then there exists a packing of G and H into an n vertex graph, with no overlapping edges.*

A tree T rooted at a vertex v and of depth $t+1$ is said to be *perfect* if every vertex in $N_l[v]$ has b neighbours in $N_{l+1}[v]$, for $0 \leq l \leq t$, and every leaf has depth $t+1$. We are now in a position to prove our main result for this subsection.

Theorem 4.2.3. *Let $t \in \mathbb{N}$. There exists infinitely many $([t], 2)$-flip graphs.*

Proof. Let $b, r \in \mathbb{N}$ such that for some $q \in \mathbb{N}$, $q \geq 2$, we have that $(q+1)b \geq r \geq 2b + 1$ and $b \geq 2(q+3)^t$. We will construct a $\big([t], (2b, r)\big)$-flip graph.

Suppose that G^* and H^* are connected graphs such that G^* is r regular with girth $g(G^*) > 2((q+3)b)^t$, and H^* is $b+1$ regular with very large girth, and hence $L(H^*)$ is $2b$ regular. The existence of such graphs G^* and H^* with prescribed degree and girth is guaranteed by Theorem 4.2.1. We shall assume subsequently that G^* and H^* are as large as necessary.

Let $p, p' \in \mathbb{N}$ such that $2\Delta(G^*)\Delta(L(H^*)) < p\,|V(G^*)| = p'\,|V(L(H^*))|$. Then let $G = pG^*$ and $H = p'L(H^*)$. Since G and H are the union of disjoint copies of G^* and $L(H^*)$ respectively, we have that $\Delta(G^*) = \Delta(G)$ and $\Delta(H) = \Delta(L(H^*))$. Furthermore, $g(G) = g(G^*)$. We will colour all the edges of H using 1 and all the edges of G using 2.

Consider the vertex $\{x, y\}$ in H. Since the girth of H^* is arbitrarily large, and in particular much larger than t, it follows that in H^*, x and y are roots of two disjoint copies of a perfect tree T of depth $l+1$, for $1 \leq l \leq t$. Joining these two trees by the edge $\{x, y\}$, the line graph of the resulting graph is two copies of some block graph with $(b+1)$-cliques, coalesced at the vertex $\{x, y\}$. The number of $(b+1)$-cliques is, by virtue of T being a perfect tree,

$$2\left(\sum_{i=0}^{l-1} b^i\right) = 2\left(\frac{b^l - 1}{b - 1}\right)$$

and consequently for any vertex $v = \{x, y\}$ in H, noting that all the edges in H are coloured 1, we have that

$$e_{1,l}^H[v] = 2\left(\frac{b^l - 1}{b - 1}\right)\binom{b+1}{2} = \frac{\left(b^{l+1} - b\right)(b+1)}{b - 1} > b^{l+1} \tag{4.4}$$

for $1 \leq l \leq t$.

Now, these two graphs G and H can be packed by Theorem 4.2.2 into a graph Q with no overlapping edges, while preserving their edge colourings. By this packing, Q is $r + 2b$ regular, where every vertex has $2b$ incident edges coloured 1 and r incident edges coloured 2. We will show that for any vertex v, $e_{1,l}[v] > e_{2,l}[v]$ in Q for $1 \leq l \leq t$, and hence Q is a $([t], (2b, r))$-flip graph.

We first compute an upper bound for the cardinality of $|N_l[v]|$ for any $v \in V(Q)$, observing that as Q is $r + 2b$ regular, and G^* and H^* are connected and can be arbitrarily large, it follows that $N_{l-1}[v]$ is strictly contained in $N_l[v]$ for

$2 \leq l \leq t$. Firstly observe that for an s regular graph we have that

$$|N_l[v]| \leq 1 + s + s(s-1) + \cdots + s(s-1)^{l-1} = 1 + s\sum_{i=0}^{l-1}(s-1)^i < 2s^l$$

recognising that the last inequality arises from a geometric sum. Since Q is $r+2b$ regular and $r \leq (q+1)b$, by the previous computation it follows that

$$|N_l[v]| < 2(r+2b)^l \leq 2((q+3)b)^l \tag{4.5}$$

for $1 \leq l \leq t$.

Due to the girth condition on G, we have that for $1 \leq l \leq t$, $g(G) > |N_l[v]|$. Hence, the subgraph in Q induced by the edges coloured 2 in $N_l[v]$ is acyclic, and therefore

$$e^Q_{2,l}[v] < |N_l[v]| < 2((q+3)b)^l$$

by (4.5). We also have, by (4.4), that

$$e^Q_{1,l}[v] \geq e^H_{1,l}[v] \geq b^{l+1}$$

Therefore, for $1 \leq l \leq t$, to flip the majority in the l^{th} neighbourhood of v we require that $2((q+3)b)^l \leq b^{l+1}$, which simplifies to $2(q+3)^l \leq b$, which is the case since $2(q+3)^t \leq b$ and $l \leq t$. Hence Q is a $([t],(2b,r))$-flip graph.

By the arbitrariness of q, it follows that for a given t there exists infinitely many such constructions. The result follows. $\square$

5

Existence and construction of k-flip graphs for $k \geq 3$

In this chapter we shall be concerned with Problems 1.3.2, 1.3.4 and 1.3.5 for the case when $k \geq 3$, as well as Problem 1.3.6 on flipping-intervals. As it shall become apparent, the case when $k \geq 3$ is astoundingly different and more difficult than the case when $k = 2$.

We first begin by considering the case when $k = 3$, giving necessary conditions for 3-flip sequences. We then introduce another technical tool, namely a new class of graphs which will be useful for constructing flipping intervals in subsequent sections.

Through the existence of flipping intervals, we then give sufficient conditions for the recognition of k-flip sequences, namely that the largest colour degree must be quadratically bound in terms of the smallest. We conclude by considering whether this condition is necessary for the recognition of a k-flip sequence when $k \geq 4$.

5.1 | Necessary conditions for 3-flip sequences

Unlike the case of two colours where (b, r)-flip sequences were completely characterised by the relation $3 \leq b < r < \binom{b+1}{2}$ in Theorem 3.1.5, for $k \geq 3$ colours we don't have a characterisation of k-flip sequences. Our first result, nonetheless, establishes a necessary condition for 3-flip sequences.

Theorem 5.1.1. *Let $a_1, a_2, a_3 \in \mathbb{N}$. If (a_1, a_2, a_3) is a 3-flip sequence, then $a_3 \leq 2(a_1)^2$.*

Proof. Suppose on the contrary that (a_1, a_2, a_3) is a flip sequence realised by some graph G, but that $2a_1^2 < a_3$. We shall prove that for some vertex v of G, $e_1[v] \leq e_2[v]$ or $e_1[v] \leq e_3[v]$. For $i \in \{1, 2, 3\}$, define the set $N^{(i)}(v)$ of neighbours u of v such that the edge $\{v, u\}$ is coloured i.

Since $|N^{(1)}(v)| = a_1$, there are at most $\binom{a_1}{2}$ edges in $N^{(1)}(v)$ coloured using 1, which leaves at least $e_1[v] - (a_1)^2$ edges of $N[v]$ coloured 1. We may assume $e_1[v] - (a_1)^2 \geq 0$, otherwise we are done. Every such 'remaining' edge $\{u, w\}$ must have both endpoints in either $N^{(2)}(v)$ or $N^{(3)}(v)$, and therefore $\{v, u, w\}$ is a triangle such that the edges $\{v, u\}$ and $\{v, w\}$ are coloured using either 2 or 3.

Hence each of the $e_1[v] - (a_1)^2$ triangles contributes two edges coloured using either 2 or 3, to some open neighbourhood, and the number of edges coloured using 2 or 3 in all open neighbourhoods is at least $\sum_{v \in V} 2\left(e_1[v] - (a_1)^2\right)$.

By a simple application of the pigeon-hole principle, there must be some vertex v with at least $2\left(e_1[v] - (a_1)^2\right)$ edges coloured using 2 or 3 in its open neighbourhood. Thus,

$$e_2[v] + e_3[v] \geq a_3 + a_2 + 2e_1[v] - 2(a_1)^2$$

but $a_3 + a_2 + 2e_1[v] - 2(a_1)^2 \geq 2e_1[v]$ since $a_3 + a_2 - 2(a_1)^2 \geq a_3 - 2(a_1)^2 \geq 0$.

Hence $e_3[v] + e_2[v] \geq 2e_1[v]$. But this means that $e_1[v] \leq e_2[v]$ or $e_1[v] \leq e_3[v]$, which is a contradiction since G is a flip graph. □

In view of Theorem 5.1.1, it is of interest to find constructions of 3-flip sequences (a_1, a_2, a_3) with as large as possible a constant c such that $a_3 = c(a_1)^2$. As we have no characterisation of 3-flip sequences, it is also of interest to outline some methods of construction, such as the following proposition.

Proposition 5.1.2. *Let $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 < a_2 < a_3$. If H is an (a_2, a_3)-flip graph with $e_3[v] < e_2[v] < \binom{a_1+1}{2}$, then (a_1, a_2, a_3) is a 3-flip sequence.*

Proof. Consider the coloured Cartesian product $G = H \,\square\, K_{a_1+1}$ where H is coloured using 2 and 3, and K_{a_1+1} is coloured using 1. Then G is a graph with colour-degrees a_i for colour $i \in \{1, 2, 3\}$. Moreover, $e_3[v] < e_2[v] < \binom{a_1+1}{2} = e_1[v]$. Hence G is an (a_1, a_2, a_3)-flip graph and we are done. □

Constructing 3-flip sequences (a_1, a_2, a_3) using Proposition 5.1.2 is feasible when a_2 and a_3 are not too far apart. For example, through Theorem 3.2.1 and the minimising value in Lemma 3.2.3, we have seen the existence of $(b, b+1)$-flip graphs with $e_1[v] = b+3$ and $e_2[v] = b+1$ for any $b \geq 3$.

Hence with $a_2 = b, a_3 = b+1$ and $a_1 < a_2$ such that $b+3 < \binom{a_1+1}{2}$, it follows that (a_1, a_2, a_3) is a 3-flip sequence. Indeed, $b+3 < \binom{b}{2}$ whenever $b > 4$.

Corollary 5.1.3. *Let $b \in \mathbb{N}, b > 4$. Then $(b-1, b, b+1)$ is a 3-flip sequence.*

5.2 | (r, c)-constant graphs

Two very important and well-studied classes of graphs are graphs with constant link and (r, b)-regular graphs. A graph G is said to be a constant link graph if there exists a graph H such that for every vertex $v \in V$, the subgraph induced by $N(v)$ is isomorphic to H. The graph H is called a *link graph*. The existence of such graphs and the problem concerning which graphs H can be link graphs in constant link graphs is an old and mostly unsolved problem, stated first by Zykov (1964). This problem received much attention over the years, some of which we highlight in chronological order: Blass et al. (1980), Hall (1985), Larrión et al. (2011), and Conder et al. (2021).

On the other-hand, a graph G is said to be an (r, b)-regular graph if it is an r regular graph such that for every vertex $v \in V$, the subgraph induced by $N(v)$ is a b regular graph. Hence for such graphs, $e[v] = r + \frac{br}{2}$. For a recent article and further references, see Conder et al. (2021).

Generalising these two families of graphs, we introduce the family of (r, c)-constant graphs, which are r regular graphs such that every vertex $v \in V$ has c edges in its open neighbourhood, *i.e.* $e(v) = c$ and consequently $e[v] = r + c$.

Observe that every constant link (respectively (r, b)-regular) graph is an (r, c)-constant graph, but an (r, c)-constant graph need not be a constant link (respectively (r, b)-regular) graph. Figure 5.1 illustrates the hierarchy of many important families of graphs in relation to (r, c)-constant graphs.

Similar to flip graphs, we can construct 'new' (r, c)-constant graphs from 'old', using the Cartesian product. This is summarised in Proposition 5.2.1.

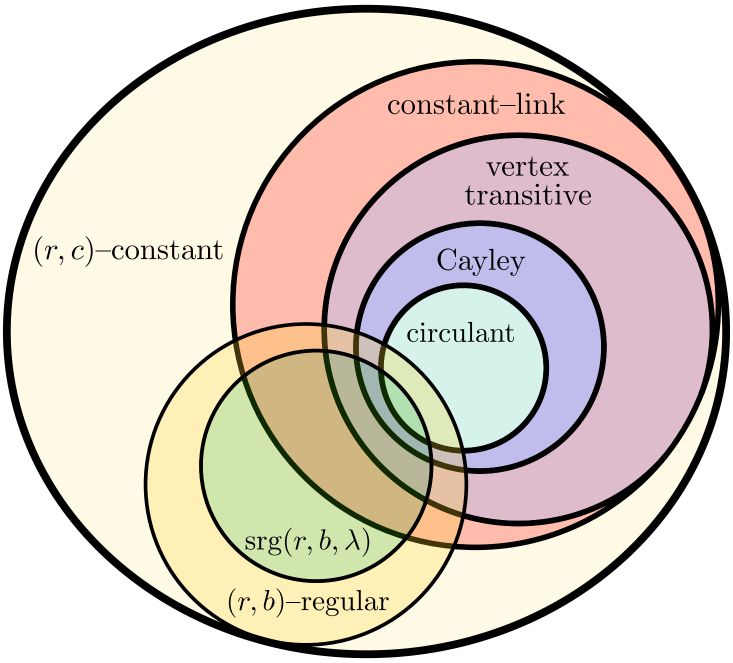

Figure 5.1: Hierarchy of (r, c)-constant graphs and their sub-families of interest.

Proposition 5.2.1. *If G and H are, respectively, (r_1, c_1)-constant and (r_2, c_2)-constant graphs, then their Cartesian product $G \square H$ is an $(r_1 + r_2, c_1 + c_2)$-graph.*

One can also define the *spectrum* of c, denoted by $spec(c)$, which is the set of all integers r such that an (r, c)-graph exists. We note that $spec(c)$ is in fact determined by the smallest r for which an (r, c)-graph G exists, as by the proposition above we have that $G \square K_2$ is an $(r + 1, c)$-graph.

Our treatment of (r, c)-constant graphs shall be limited in scope, namely with regards to questions on their existence, however a more complete treatment is given in Caro and Mifsud (2024a). A public database of several (r, c)-constant graphs is also maintained by Caro and Mifsud (2024b).

Before proceeding any further, however, we provide some motivation for the introduction of such graphs. Suppose that for some $r_1, r_2, c_1, c_2 \in \mathbb{N}$ such that $r_1 < r_2$ but $r_1 + c_1 > r_2 + c_2$, (r_1, c_1)-constant and (r_2, c_2)-constant graphs exist. Let these graphs be G and H respectively. Then not only is $G \square H$ an (r, c)-constant graph by virtue of Proposition 5.2.1, but if we monochromatically colour the edges of G and H distinctly then by the CCP Lemma I (Lemma 2.1.3), we have that $G \square H$ is an (r_1, r_2)-flip graph.

5.2.1 | Existence of (r,c)-constant graphs

A natural problem regarding (r,c)-constant graphs, given $r \in \mathbb{N}$, is that of finding for which integers c, $0 \leq c \leq \binom{r}{2}$, does an (r,c)-constant graph exist.

The *spectrum* of r, denoted by $\mathsf{spec}(r)$, is the set of all such integers c such that an (r,c)-constant graph exists:

$$\mathsf{spec}(r) = \left\{ c \in \mathbb{Z} : 0 \leq c \leq \binom{r}{2}, \text{ an } (r,c)\text{-constant graph exists} \right\}$$

The following theorem demonstrates that, in fact, $\mathsf{spec}(r)$ contains nearly all of the integral interval $\left[0, \binom{r}{2}\right]$.

Theorem 5.2.2 (Existence of (r,c)-constant graphs)**.** *Let* $r \in \mathbb{N}$.

i. *For every integer* c *such that* $0 \leq c \leq \dfrac{r^2}{2} - 5r^{\frac{3}{2}}$, $c \in \mathsf{spec}(r)$. *Hence there exists an* (r,c)*-constant graph.*

ii. *Suppose* $k \in \mathbb{N}$ *and* $r \geq 3k$. *Then* $\binom{r}{2} - k \notin \mathsf{spec}(r)$. *Hence there does not exist an* $\left(r, \binom{r}{2} - k\right)$*-constant graph.*

Before proceeding with the proof of Theorem 5.2.2, we recall a seemingly unrelated theorem concerning the feasibility problem for line graphs. Let N, M be integers such that $N \geq 1$ and $0 \leq M \leq \binom{N}{2}$. Given a family of graphs $\mathcal{F}$, we say that (N, M) is a *feasible* pair for $\mathcal{F}$ if there exists a graph $G \in \mathcal{F}$ with N vertices and M edges. Otherwise we say (N, M) is *non-feasible* pair for $\mathcal{F}$.

Theorem 5.2.3 (Caro et al. (2023b), Lemma 3.4)**.** *The smallest value of* M *for which* (N, M) *is a non-feasible pair for the line graphs of star-forests satisfies* $M > \frac{N^2}{2} - 5N^{\frac{3}{2}}$.

We now proceed to prove Theorem 5.2.2.

Proof. Let $N \in \mathbb{N}$. If M is an integer such that $0 \leq M \leq \frac{N^2}{2} - 5N^{\frac{3}{2}}$, as a consequence of Theorem 5.2.3, there exists a star-forest $F = \bigcup_{j=1}^{s} K_{1,a_j}$ such that its line graph $H = L(F)$ has N vertices and M edges.

Observe that, since for a star $K_{1,n}$ we have $L(K_{1,n}) = K_n$, then $N = \sum_{j=1}^{s} a_j$ and $M = \sum_{j=1}^{s} \binom{a_j}{2}$.

Consider the modified star-forest $F^* = \bigcup_{j=1}^{s} K_{1,a_j+1}$. The line graph $H^* = L(F^*)$ has components K_{a_j+1} for $1 \leq j \leq s$. In every component K_{a_j+1}, for $1 \leq j \leq s$, every vertex u in the component has $\deg(u) = a_j$ and $e(u) = \binom{a_j}{2}$.

Consider the Cartesian product $G = \square_{j=1}^{s} K_{a_j+1}$, which is well-defined since for $1 \leq j \leq s$, $a_j + 1 \geq 2$. Then G is a regular graph such that every vertex v has degree $N = \sum_{j=1}^{s} a_j$, and $e(v) = \sum_{j=1}^{s} \binom{a_j}{2} = M$. Setting $N = r$ and $M = c$, (i) follows immediately.

We now proceed to prove (ii). Suppose the contrary, and let H be an r regular graph where each vertex v has $e(v) = \binom{r}{2} - k$. Then every vertex in $N(v)$ has at least $r - k$ neighbours in $N[v]$.

Let x be a vertex in $N(v)$ with at most $r - 1$ neighbours in $N[v]$. Observe that for k edges not in the subgraph induced by $N[v]$, they must span at most $2k$ vertices. Hence there are $r + 1 - 2k \geq k + 1$ vertices in $N[v]$, including v, whose r neighbours are all in $N[v]$ and therefore they are all neighbours of x.

Now consider $N[x]$, which contains some vertex w not in $N[v]$, since x has at most $r - 1$ neighbours in $N[v]$. Then this vertex w is not adjacent to at least $k + 1$ vertices in $N[x] \cap N[v]$ (those having degree r in $N[v]$). Therefore w has at most $r - k - 1$ neighbours in $N[x]$. Consequently, $e(x) \leq \binom{r}{2} - k - 1$, which is a contradiction. The result follows. $\square$

5.3 | Flipping intervals and sufficient conditions for k-flip sequences

Through Theorem 3.1.5 and Corollary 5.1.3, we have seen the existence of small flipping intervals of length 2 and 3. This naturally begs the question of how long can an integral interval be such that it is flipping. This serves as the motivation behind Problem 1.3.6.

The full strength and importance of (r, c)-constant graphs shall now become apparent, as together with the coloured Cartesian product, they serve as the building blocks for such long flipping intervals. More so, by the nature of these constructions, we shall obtain a sufficient condition for k-flip sequences.

Theorem 5.3.1. *Let $b \in \mathbb{N}$.*

i. If $b \geq 101$ then the interval $\left[b, b + \left\lfloor \frac{1}{4}(b^2 - 10b^{\frac{3}{2}}) \right\rfloor\right]$ is a flipping-interval.

ii. If $b \geq 3$ then the interval $[b, 2b-2]$ is also a flipping-interval.

Proof. Consider the interval $[b, b+k]$ where $k = \left\lfloor \frac{1}{4}(b^2 - 10b^{\frac{3}{2}}) \right\rfloor$. Since $b \geq 101$ it follows that $k \geq 12$. Set $M_1 = \left\lfloor \frac{b^2}{2} - 5b^{\frac{3}{2}} \right\rfloor$. For $1 \leq j \leq k$, set H_j to be a $(b+j-1, M_1 - 2(j-1))$-constant graph which exists by Theorem 5.2.2 and observe that $M_1 \geq 2k \geq 2(j-1)$ for $1 \leq j \leq k$.

Consider the coloured Cartesian product $G = \square_{j=1}^{k} H_j$, where H_j is coloured using colour j. By CCP Lemma I (Lemma 2.1.3), G is a $(b, b+1, \ldots, b+k)$-flip graph and therefore (i) follows.

We now prove (ii). Consider the triple $(b+j, b+1-j, 2j)$ where $0 \leq j \leq b-2$. Consider a regular bipartite graph H of degree $\sum_{j=1}^{b-2} 2j = (b-1)(b-2)$, where a $2j$-factor of H is coloured j for $1 \leq j \leq b-2$.

We may take H to be $K_{n,n}$ where $n = (b-1)(b-2)$. Now consider the coloured Cartesian product $G = H \,\square \left(\square_{j=1}^{b-2} K_{b+1-j}\right)$ where K_{b+1-j} is coloured j for $1 \leq j \leq b-2$.

In every vertex v of G, for $1 \leq j \leq b-2$, $\deg_j(v) = b+j$ which is increasing and $e_j[v] = \binom{b+1-j}{2} + 2j$, which is decreasing. Hence G is a $(b, b+1, \ldots, 2b-2)$-flip graph. This concludes the proof. $\square$

Let $k(b) = \left\lfloor \frac{1}{4}(b^2 - 10b^{\frac{3}{2}}) \right\rfloor$. We require $b \geq 101$ in (i) since for smaller values of b, we have that $k(b) \leq 0$. Moreover, for $b \geq 108$ we have that $k(b) \geq b-2$. This justifies the simple interval $[b, 2b-2]$ in (ii), as for $3 \leq b \leq 107$, it allows for longer intervals than (i).

We are now able to deduce a sufficient condition for k-flip sequences.

Corollary 5.3.2 (Sufficient condition for k-flip sequences)**.** *Suppose that $k \geq 2$. Let $3 \leq a_1 < a_2 < \cdots < a_k$ be a sequence of k integers such that either $a_k \leq 2a_1 - 2$ or $a_k \leq a_1 + \left\lfloor \frac{1}{4}\left((a_1)^2 - 10(a_1)^{\frac{3}{2}}\right) \right\rfloor$, then $(a_1, \ldots, a_k)$ is a k-flip sequence.*

Proof. In both cases, the sequence $(a_1, \ldots, a_k)$ is a subsequence of a flipping interval in Theorem 5.3.1. The result follows by CCP Lemma III (Lemma 2.1.5), noting that all the factors in the construction of Theorem 5.3.1 are either monochromatic or triangle-free. $\square$

5.4 | Constructing small k-flip graphs

Our constructions of k-flip graphs realising a sequence $(a_1, \ldots, a_k)$ in Corollary 5.3.2 involve the product of k graphs $H_1, \ldots H_k$ which are $a_1, \ldots, a_k$ regular respectively. Thus such constructions are very large, involving $\Omega(k!)$ vertices.

We introduce a special class of flip graphs: *flip graphs with constant gaps*. We say that a flip graph has *constant gaps* with parameters (λ, Λ) if there exists $\lambda, \Lambda \in \mathbb{N}$ such that

$$\deg_{j+1}(v) - \deg_j(v) = \lambda \quad \text{and} \quad e_j[v] - e_{j+1}[v] = \Lambda$$

for $1 \leq j < k$ and $v \in V$.

We outline here an infinite sub-family of these flip graphs with constant gaps which can be constructed on a small number of vertices, and therefore are an illustration of relatively small k-flip graphs.

Remark 5.4.1. Observe from the construction in Theorem 5.3.1 (i) that the graphs constructed are flip graphs with constant gap parameters $(\lambda, \Lambda) = (1, 1)$, where $\lambda = 1$ follows since the graph realise a flipping interval, and $\Lambda = 1$ follows from the choice of (r, c)-constant graphs.

Proposition 5.4.2. *Let $k, \zeta \in \mathbb{N}$, $k \geq 2$. For any $t_0 \in \mathbb{N}$ such that $t_0 > k + 2$, there exists a flip graph G with constant gaps having parameters (λ, Λ) where*

$$\lambda = 1 + \zeta\left(\frac{k(k+1)}{2} - 1\right) \quad \text{and} \quad \Lambda = \zeta - 1 + \zeta k(t_0 - k - 2)$$

and if ζ is even then $|V(G)|$ is $O(\zeta t_0 k^3)$.

Proof. Let Γ be a sufficiently large finite group such that Γ has k disjoint and inverse-closed subsets S_j where $|S_j| = \zeta(k - j + 1)$ for $1 \leq j \leq k$, and such that $S = \bigcup_{j=1}^{k} S_j$ is sum-free in Γ. Let $\mu = \dfrac{\zeta k(k+1)}{2}$ denote the degree of each vertex in $\mathsf{Cay}(\Gamma; S)$. Consider $\mathsf{Cay}(\Gamma; S)$ with an edge-colouring such that the edges labelled in S_j are coloured with j.

Hence the number of edges coloured j incident to a vertex is $\zeta(k-j+1)$; more so by the sum-free condition, this is also equal to the number of edges coloured j in the closed neighbourhood of a vertex of $\mathsf{Cay}(\Gamma; S)$.

Such a group Γ exists, as it suffices to consider a sufficiently large cyclic group by Lemma 3.2.5 with, if necessary, a number of direct products of $\mathbb{Z}_2$ to add a number of involutions matching the number of odd colour degrees in $\mathsf{Cay}(\Gamma; S)$.

In particular we can choose $\mathbb{Z}_n$ such that the integral interval $\left(\frac{n}{8}, \frac{n}{4}\right)$ has at least $\frac{\mu}{2}$ integers. In the case that ζ is even then there are no odd colour degrees in $\mathsf{Cay}(\Gamma; S)$ and therefore no involutions are necessary in Γ. Hence we can choose Γ to be of order $O(\zeta k^2)$ in this case.

Let $\rho = kt_0 + \binom{k}{2}$ and consider $K_{\rho,\rho}$ such that for $1 \leq j \leq k$ we have $t_0 + j - 1$ matchings coloured j. Let G be the coloured strong product $K_{\rho,\rho} \boxtimes \mathsf{Cay}(\Gamma; S)$.

By Lemma 2.2.2 and the properties of $\mathsf{Cay}(\Gamma; S)$, we have the following in G:

$$\deg_j(v) = \zeta(k-j+1) + (t_0+j-1)(1+\mu)$$

and

$$e_j[v] = \zeta(k-j+1)(\rho+1) + (t_0+j-1)(1+3\mu)$$

for $v \in V(G)$ and $1 \leq j \leq h$. Substituting for μ and ρ we get that:

$$\deg_{j+1}(v) - \deg_j(v) = 1 + \mu - \zeta = 1 + \zeta\left(\frac{k(k+1)}{2} - 1\right) = \lambda$$

and

$$e_j[v] - e_{j+1}[v] = \zeta(\rho+1) - 3\mu - 1 = \zeta - 1 + \zeta k(t_0 - k - 2) = \Lambda$$

as required.

Hence we have 'constant gaps'. To satisfy the requirement that the graph is a flip graph, all that remains is to show that $\Lambda > 0$. Re-arranging, we require that

$$t_0 > (k+2) - \frac{\zeta - 1}{\zeta k} = \frac{\zeta(k^2 + 2k - 1) + 1}{\zeta k}$$

which is satisfied from our theorem statement.

Recall that if ζ is even then we can choose Γ to be of order $O(\zeta k^2)$. Moreover, $K_{\rho,\rho}$ has $O(kt_0)$ vertices, recalling that $t_0 > k+2$. Consequently the flip graph G can be constructed on $O(\zeta t_0 k^3)$ vertices whenever ζ is even. □

The construction above depends on 3 parameters, namely the number of colours k, a parameter ζ which controls λ *i.e.* the gaps in the colour-degree sequence, and a parameter t_0 which controls Λ *i.e.* the gaps in the number of coloured edges in a closed neighbourhood.

In the case when ζ is even, choosing t_0 and ζ to be $O(k)$, the k-flip graphs constructed above are on $O(k^5)$ vertices, which is significantly smaller than the $\Omega(k!)$ constructions obtained previously.

5.5 | Unbounded gaps in k-flip sequences

We have seen that for $k = 2$ and $k = 3$ we have a necessary condition for flip sequences, namely that the largest colour-degree a_k is quadratically bound in terms of the smallest colour-degree a_1. We have also seen a sufficient condition for $k \geq 2$ colours, where a_k was also quadratically bound in terms of a_1.

A natural question to ask is whether such a necessary condition, where a_k is quadratically bound in a_1, exists for $k \geq 4$ colours. The intuition for $k = 2$ and $k = 3$ was that for a flip to be possible, the differences in the colour-degrees must not be too large as otherwise it would be impossible to have a flip.

Rather surprisingly however, this intuition completely breaks down for four or more colours. The following proposition establishes that for $k \geq 4$ colours, there exists some $m(k) \in \mathbb{N}$ such that given any $N \in \mathbb{N}$, there is a k-flip sequence $(a_1, \ldots, a_k)$ where $a_1 = m(k)$ and $a_k > N$. In particular then, the largest colour-degree is *not* bound in terms of the smallest colour-degree!

Proposition 5.5.1. *Let $k \in \mathbb{N}$, $k \geq 4$. Then there is some constant $m = m(k) \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, there exists a k-flip sequence $(m, a_2, \ldots, a_k)$ such that $a_k > N$.*

Proof. Let K be the complete graph K_{2n} where $n > \dfrac{k(k^2-2k+1)}{4(k-3)}$. Since K is a complete graph on an even number of vertices, K has a 1-factorisation. For

$1 < j < k$, let $k-j$ 1-factors be coloured using colour j and let the remaining edges be coloured 1. It follows then that every vertex v in K_{2n} has

$$\deg_1^K(v) = 2n - 1 - \binom{k-1}{2}$$

incident edges coloured 1 and $\deg_j^K(v) = k - j$ incident edges coloured j for $1 < j < k$. For convenience, define $\deg_k^K(v) = 0$. Observe that the sequence $\deg_1^K(v), \ldots, \deg_k^K(b)$ is strictly decreasing, noting that

$$\deg_1^K(v) - \deg_2^K(v) > \frac{k(k-1)}{k-3} > 0$$

since $n > \dfrac{k(k^2-2k+1)}{4(k-3)}$. Since K is a complete graph and each vertex v has the same number of incident edges coloured j, then $e_j^K[v] = n \deg_j^K(v)$ for $1 \leq j \leq k$.

We now show that for every vertex v in K, $(k-1)(e_1^K[v] - e_2^K[v]) > 4n^2$. Rearranging and substituting for $e_1^K[v]$ and $e_2^K[v]$ in terms of n and k, we must show that $n > \dfrac{k(k^2-2k+1)}{4(k-3)}$. This follows immediately by our choice of n.

Consider $t \in \mathbb{N}$ such that

$$t \geq \frac{4n^2}{(k-1) \min\limits_{1<j<k} \{e_j^K[v] - e_{j+1}^K[v]\}}$$

and let H be a $\rho = \dfrac{(k-1)(2t+k-2)}{2}$ regular bipartite graph. For $0 \leq j \leq k-2$, let $t+j$ matchings of H be coloured using colour $2+j$.

Let G be the graph $H \boxtimes K$, inheriting the edge-colourings of H and K respectively. By Corollary 2.2.3, since G is the strong product of a triangle-free graph H and a complete graph K, each vertex (u,v) in G has colour-degree

$$\deg_j^G((u,v)) = \begin{cases} 2n - 1 - \binom{k-1}{2} & j = 1 \\ (k-j) + 2n(t+j-2) & 2 \leq j \leq k \end{cases}$$

which is strictly increasing, and associated coloured closed neighbourhood sizes

$$e_j^G[(u,v)] = \begin{cases} (\rho+1)e_1^K[v] & j = 1 \\ (\rho+1)e_j^K[v] + 4n^2(t+j-2) & 2 \leq j \leq k \end{cases}$$

which we now show to be strictly decreasing.

Firstly note that since $\rho + 1 = (k-1)t + \binom{k-1}{2} + 1$, there exists $\kappa \in \mathbb{R}$ such that $\kappa > 1$ and $\rho + 1 = (k-1)t\kappa$. Now, recall that K has the property that $(k-1)(e_1^K[v] - e_2^K[v]) > 4n^2$. Since $\kappa > 1$, it follows that

$$(\rho+1)\left(e_1^K[v] - e_2^K[v]\right) = (k-1)\left(e_1^K[v] - e_2^K[v]\right)(t\kappa) > 4n^2 t$$

and therefore since $e_1^K[v] > e_2^K[v]$ we have that $(\rho+1)e_1^K[v] > (\rho+1)e_2^K[v] + 4n^2 t$. Consequently, $e_1^G[(u,v)] > e_2^G[(u,v)]$ as required.

Now consider $2 \leq j \leq k-1$. By the choice of t and $\kappa > 1$, we have that

$$\begin{aligned}(\rho+1)\left(e_j^K[v] - e_{j+1}^K[v]\right) &= (k-1)\left(e_j^K[v] - e_{j+1}^K[v]\right)(t\kappa) \\ &> 4n^2 \\ &= 4n^2(t+j-1) - 4n^2(t+j-2)\end{aligned}$$

and therefore $e_j^G[(u,v)] > e_{j+1}^G[(u,v)]$.

It follows that G is a flip graph on $k > 3$ colours, such that for any vertex (u,v), $\deg_1((u,v))$ is only dependent on k and $\deg_k((u,v))$ increases with t. Since t is not bounded above in the construction, then given any $N \in \mathbb{N}$, a sufficiently large t can be found such that $\deg_k((u,v)) > N$. The result follows. □

In the context of Problem 1.3.3 for $k \geq 4$, it follows that $h(a_1, \ldots, a_k)$ is not bound in a_1 either. It is therefore of interest to generalise these notions and find, given $k \geq 4$, the longest subsequence $a_1, \ldots, a_{q(k)}$ possible such that a_k and hence $h(a_1, \ldots, a_k)$ is not bound in the terms of this subsequence.

More formally, let $q(k)$ be an integer, $q(k) < k$, such that there exists some $m_1(k), \ldots, m_{q(k)}(k) \in \mathbb{N}$ and for all $N \in \mathbb{N}$, there is a k-flip sequence $(a_1, \ldots, a_k)$ where $a_i = m_i(k)$ for $1 \leq i \leq q(k)$ and $a_k > N$.

5.5.1 | Bounding the parameter $q(k)$

It turns out that $q(k)$ can be substantially large, at least $\max\left\{1, \left\lceil \frac{k}{4} \right\rceil - 1\right\}$, but not *too* large – indeed no more than $\left\lceil \frac{k}{2} \right\rceil$. We conclude by proving the following theorem which summarises the aforementioned, however it is worth noting that the relation between the colour-degrees past $q(k)$ is still not well understood.

Theorem 5.5.2. *Let $k \in \mathbb{N}$ such that $k > 3$. Then,*

$$\max\left\{1, \left\lceil \frac{k}{4} \right\rceil - 1\right\} \leq q(k) < \begin{cases} \frac{k}{3} & \text{if } k \pmod 3 = 0 \\ \left\lceil \frac{k}{2} \right\rceil & \text{otherwise} \end{cases}$$

Our strategy to prove the lower bound in Theorem 5.5.2 will be as follows: If q-flip graphs satisfying particular properties (as outlined in the following lemma) exist, then $(a_1, \ldots, a_k)$-flip graphs exist for $k > 4q$, where a_k can be arbitrarily large whilst a_i is constant for $1 \leq i \leq q$.

Observe how this strategy differs from that employed in the proof of Proposition 5.5.1, namely that here we are considering a flip graph as our starting point whereas before we considered a coloured complete graph (which is not a flip graph).

Lemma 5.5.3. *Let $q, k \in \mathbb{N}$ such that $1 < q < \frac{k}{4}$. Let $D_1, \ldots, D_q \in \mathbb{N}$ such that $D_q(k - 4q) > 1 + \xi q(q-1) + 5\binom{k-q}{2}$ where $\xi = \max_{1 \leq j < q}\{D_j - D_{j+1}\}$.*

If there exists a $(a_1, \ldots, a_q)$-flip graph F such that for every $v \in V(F)$ and $1 \leq j \leq q$, $e_j^F[v] = D_j$, then given any $N \in \mathbb{N}$ there exists a $(a_1, \ldots, a_k)$-flip graph for some $a_{q+1}, \ldots, a_k \in \mathbb{N}$ where $a_k > N$.

Proof. Let Γ be a sufficiently large finite group such that Γ has $k - q - 1$ disjoint and inverse-closed subsets S_j where $|S_j| = k - q - j$ for $1 \leq j < k - q$, and such that $S = \bigcup_{j=1}^{k-q-1} S_j$ is sum-free in Γ.

Consider $K = \mathrm{Cay}(\Gamma; S)$ with an edge-colouring such that the edges labelled in S_j are coloured using $q + j$. Hence for any $v \in \Gamma$, $\deg^K_{q+j}(v) = k - q - j$ and by the sum-free condition on S, $e^K_{q+j}[v] = k - q - j$.

Such a group Γ exists, as it suffices to consider a sufficiently large cyclic group by Lemma 3.2.5 with, if necessary, a number of direct products of $\mathbb{Z}_2$ to add a number of involutions matching the number of odd colour degrees in K.

Now consider the coloured Cartesian product $F \,\square\, K$, which is $k-1$ coloured since F is coloured using $1, \ldots, q$ and K is coloured using $q+1, \ldots, k-1$.

By Lemma 2.1.2, for a given colour j the graph has colour-degree a_j inherited from F for $1 \leq j \leq q$, and $a_j = k - j$ inherited from K when $q < j < k$. Likewise, the number of edges coloured j in a closed neighbourhood is D_j inherited from F for $1 \leq j \leq q$ and $D_j = k - j$ inherited from K for $q < j < k$.

Finally, note that $F \square K$ is $\mu = \binom{k-q}{2} + \sum_{i=1}^{q} a_i$ regular. Let $t \in \mathbb{N}$ such that

$$t \geq \frac{1 + \mu + 2\sum_{i=1}^{k} D_i}{(k-q) \min_{q<i<k} \{D_i - D_{i+1}\}}$$

and let H be a $\rho = (k-q)t + \binom{k-q}{2}$ regular bipartite graph. For $1 \leq j \leq k - q$, colour $t + j - 1$ matchings of H using $q + j$.

Let G be the coloured strong product $H \boxtimes (F \square K)$. By Lemma 2.2.2, for $v \in V(G)$ and $1 \leq j \leq k$, the edge-colouring in G is such that

$$\deg_j(v) = \begin{cases} a_j & 1 \leq j \leq q \\ a_j + (t + i - q - 1)(1 + \mu) & q < j \leq k \end{cases}$$

and

$$e_j[v] = \begin{cases} D_j(\rho + 1) & 1 \leq j \leq q \\ D_j(\rho + 1) + (t + j - q - 1)\left(1 + \mu + 2\sum_{i=1}^{k} D_i\right) & q < j \leq k \end{cases}$$

We will show that G as constructed and edge-coloured is a k-flip graph. Observe that for $1 \leq j \leq k$, $a_j < 1 + \mu$. Hence for $j = q$ we have that

$$\deg_q(v) = a_q < 1 + \mu < a_{q+1} + t(1 + \mu) = \deg_{q+1}(v)$$

and for $j > q$ we have that $\deg_{j+1}(v) - \deg_j(v) = \mu > 0$. Consequently the colour-degree sequence in G is strictly increasing.

Since for $1 \leq j < q$ we have $D_j > D_{j+1}$ then in G we have $e_j[v] > e_{j+1}[v]$. Next note that $\rho + 1 = (k-q)t\kappa$ for some $\kappa > 1$. Hence $(D_q - D_{q+1})(\rho + 1) > (D_q - D_{q+1})(k-q)t$. Therefore to show that $e_q[v] > e_{q+1}[v]$ it suffices to show

$$(D_q - D_{q+1})(k - q) > 1 + \mu + 2\sum_{i=1}^{k} D_i$$

From the lower bound on D_q in the theorem statement we have that

$$\begin{aligned}
&D_q(k-q) \\
&> 1 + (3q)D_q + \xi q(q-1) + 5\binom{k-q}{2} \\
&> 1 + \sum_{i=1}^{q} a_i + (2q)D_q + \xi q(q-1) + 5\binom{k-q}{2} && \because a_1 < \cdots < a_q \le D_q \\
&= 1 + \mu + (2q)D_q + \xi q(q-1) + 4\binom{k-q}{2} && \because \mu = \sum_{i=1}^{q} a_i + \binom{k-q}{2} \\
&\ge 1 + \mu + 2\sum_{i=1}^{q} D_i + 4\binom{k-q}{2} && \because D_{q-i} \le D_q + i\xi \\
&= 1 + \mu + 2\sum_{i=1}^{k} D_i + 2\binom{k-q}{2} && \because \binom{k-q}{2} = \sum_{i=q+1}^{k} D_i \\
&= D_{q+1}(k-q) + 1 + \mu + 2\sum_{i=1}^{k} D_i && \because D_{q+1} = k-q-1
\end{aligned}$$

as required and therefore the colours q and $q+1$ flip in G.

Consider the final case when $q < j < k$. By the choice of t and $\kappa > 1$, we have

$$(D_j - D_{j+1})(\rho + 1) = (D_j - D_{j+1})(k-q)(t\kappa) > 1 + \mu + 2\sum_{i=1}^{k} D_i$$

which we can re-arrange to get $e_j[v] > e_{j+1}[v]$. Hence the sequence of closed neighbourhood sizes is strictly decreasing as required.

It follows that G is a flip graph on k colours, such that for any vertex v, the difference between $\deg_q(v)$ and $\deg_k(v)$ grows in t as $t \to \infty$. □

We are finally in a position to prove the lower bound in Theorem 5.5.2.

Proposition 5.5.4. *Let $k \in \mathbb{N}$ such that $k > 3$. Then for any $q \in \mathbb{N}$ such that $q = 1$ or $q < \frac{k}{4}$, there exists $a_1, \ldots, a_q \in \mathbb{N}$ such that given any $N \in \mathbb{N}$ there exists a $(a_1, \ldots, a_k)$-flip graph for some $a_{q+1}, \ldots, a_k \in \mathbb{N}$ where $a_k > N$.*

Proof. The case $q = 1$ follows immediately from Proposition 5.5.1. Hence consider the case when $1 < q < \frac{k}{4}$.

Let $b \in \mathbb{N}$ be sufficiently large such that $b \geq \max\left\{101, \dfrac{1+q(q-1)+5\binom{k-q}{2}}{k-4q}\right\}$ and $\left\lfloor \frac{1}{4}(b^2 - 10b^{\frac{3}{2}}) \right\rfloor \geq q-1$. By Corollary 5.3.2 and the choice of b, there exists a $(b, \ldots, b+q-1)$ flip graph F where for every vertex $v \in V(F)$,

$$e_q[v](k-4q) \geq (b+q-1)(k-4q) > b(k-4q) \geq 1 + q(q-1) + 5\binom{k-q}{2}$$

and therefore the result follows as an immediate consequence of Lemma 5.5.3, noting that $\xi = 1$ by Remark 5.4.1 for our choice of flip graph. $\square$

The proof of the upper bound in Theorem 5.5.2 is much less involved, and is given next. It still remains open whether the two cases can be reconciled.

Proposition 5.5.5. *Let $k \in \mathbb{N}, k \geq 2$. Then* $q(k) < \begin{cases} \frac{k}{3} & \textit{if } k \pmod 3 = 0 \\ \left\lceil \frac{k}{2} \right\rceil & \textit{otherwise} \end{cases}$.

Proof. The cases $k = 2$ and $k = 3$ immediately follows from Theorems 3.1.5 and 5.1.1. Hence consider $k \geq 4$. Let G be an $(a_1, \ldots, a_k)$-flip graph. Consider the case when $k \pmod 3 = 0$ and let $p = \frac{k}{3}$. Re-colour the edges of G such that the p colours $p(j-1)+1, \ldots, pj$ are coloured j for $j \in \{1,2,3\}$. For $j \in \{1,2,3\}$, define $b_j = \sum_{i=1}^{s} a_{(i-1)s+j}$. Note that $b_1 < b_2 < b_3$ by the monotonicity of the k-flip sequence and the fact that each b_j is a sum of s terms.

Applying a similar argument to the coloured closed neighbourhood sizes, one observes that G is a (b_1, b_2, b_3)-flip graph.

By Theorem 5.1.1, it follows that $b_3 \leq 2(b_1)^2$ and hence $a_k \leq 2k^2(a_s)^2$. Therefore a_k is bound in $a_{\frac{k}{3}}$ and hence $q(k) < \frac{k}{3}$ when $k \pmod 3 = 0$.

Otherwise, consider $t = \left\lceil \frac{k}{2} \right\rceil$. Let $c_1 = \sum_{i=1}^{t} a_i$ and $c_2 = \sum_{i=1}^{k-t} a_{t+i}$. Two cases are possible: either $2t = k$ or $2t - 1 = k$. In the case that $2t - 1 = k$, it need not necessarily be that case that $c_1 < c_2$. Indeed, suppose that $c_1 \geq c_2$. Then $ta_t \geq a_k$ and hence a_k is bounded in $a_{\lceil \frac{k}{2} \rceil}$.

Consider the cases when $2t = k$ or $2t - 1 = k$ but $c_1 < c_2$. In both these cases, by a similar argument to the case when $k \pmod 3 = 0$, we have that G is a (c_1, c_2)-flip graph. By Theorem 3.1.5 it follows that $c_2 < \binom{c_1+1}{2}$ and hence a_k is quadratically bound in $a_{\lceil \frac{k}{2} \rceil}$. It follows that $q(k) < \left\lceil \frac{k}{2} \right\rceil$. $\square$

6

Conclusion and open problems

We began this thesis with a list of several enticing problems on flip graphs, and have provided an in-depth treatment of a number of them, most notably Problems 1.3.2 – 1.3.6. Nonetheless, many open problems remain.

We have seen a complete treatment of the two-colour case in Chapter 3 concerning Problems 1.3.2, 1.3.4, and 1.3.5. In Chapters 6 and 7 we also explored these problems for the case when $k \geq 3$, along with Problem 1.3.6. In particular for three colours, in Theorem 5.1.1 we have seen a necessary condition for a sequence (a_1, a_2, a_3) to be a 3-flip sequence. In view of this theorem, it is of interest to find constructions of 3-flip sequences (a_1, a_2, a_3) with as large as possible a constant c such that $a_3 = c(a_1)^2$, where it is known that $c \leq 2$.

Problem 6.0.1. Determine the supremum of c such that there exists infinitely many 3-flip sequences (a_1, a_2, a_3) satisfying $a_3 \geq c(a_1)^2$.

With regards to Problem 1.3.3, albeit a comprehensive treatment was given in Chapter 3, it still remains open whether an explicit form of $h(b, r)$ can be given, or at a minimum better upper and lower bounds.

Problem 6.0.2. Determine the exact value of $h(b, r)$ or at least expand the range of admissible values (b, r) for which the best possible asymptotic bound in Corollary 3.2.8 holds, and obtain a non-trivial lower bound.

Finding large subsets of Abelian groups which are both sum-free and inverse-closed is of interest with regards to expanding the range of admissible values r

for fixed b such that $h(b,r)$ is $\Theta(b+r)$. As mentioned in Section 3.2.2, this problem has been studied greatly in the absence of the inverse-closed condition. We therefore pose the following problem.

Problem 6.0.3. Given a finite Abelian group Γ, what is size of the largest subset which is both sum-free and inverse-closed?

We next turn our attention to Problem 1.3.3 for three or more colours. Given $k \geq 4$, as a consequence of Theorem 5.5.4, we have that there exists k-flip sequences were $a_1, \ldots, a_{q(k)}$ are fixed but $a_k \to \infty$. In particular since $h(a_1, \ldots, a_k) > a_k$, one observes that $h(a_1, \ldots, a_k)$ is not bounded above by a polynomial in $a_1, \ldots, a_{q(k)}$.

This contrasts the cases when $k = 2$ and $k = 3$, where $h(a_1, a_2)$ and $h(a_1, a_2, a_3)$ are polynomially bound in a_1, and in particular we have proven that $h(a_1, a_2)$ is quadratically bound in a_1 whenever $a_2 < a_1 + 2\left\lfloor \frac{a_1+2}{6} \right\rfloor^2$. In light of this, we pose the following problem.

Problem 6.0.4. For $k \geq 4$, is there a smallest integer $p(k)$, $\frac{k}{4} \leq p(k) \leq k$, such that $h(a_1, \ldots, a_k)$ is polynomially bound in $a_{p(k)}$?

In Chapter 5 we introduced the family of (r,c)-constant graphs, which served as a crucial tool for establishing a sufficient condition in Corollary 5.3.2 for the recognition of a k-flip sequence, especially in the case when $k \geq 3$.

Given the demonstrated usefulness of (r,c)-constant graphs, it is therefore important to advance our knowledge concerning $\mathsf{spec}(r)$.

Problem 6.0.5. Determine $\mathsf{spec}(r)$, or at least improve upon the lower bound given in Theorem 5.2.2 (i) for membership into $\mathsf{spec}(r)$ and the non-membership in $\mathsf{spec}(r)$ given in Theorem 5.2.2 (ii).

In light of Problem 1.3.3, small constructions of (r,c)-constant graphs are of interest, as they may lead to small constructions of flip graphs.

Problem 6.0.6. Find lower and upper bounds to

$$g(r,c) = \min\{|V(G)| \colon G \text{ is an } (r,c)\text{−constant graph}\}$$

Chapter 4 dealt with Problem 1.3.7 and the extension of the flipping problem to the closed t-neighbourhoods. In particular we illustrated two distinct constructions for the case of two colours. Problems 1.3.2 – 1.3.6 naturally extend to this generalisation of the flipping problem and remain to be studied extensively.

For $t \in \mathbb{N}$, define $b(t) = \min\{b\colon ([t],(b,r))\text{−flip graph exists}\}$, in other words the smallest colour degree such that a $[t]$-flip graph exists.

Observe that for $t \geq 2$, using $q = 2$ in the proof of Theorem 4.2.3, we get the existence of $\left([t], \left(4\left(5^t\right), 4\left(5^t\right)+1\right)\right)$-flip graphs and therefore $b(t) \leq 4(5^t)$. Furthermore, by Theorem 4.1.1 we improve this to $b(t) \leq 2^{t+1} - 1$. This suggests the following problem.

Problem 6.0.7. For $t \in \mathbb{N}$, $t \geq 2$ determine $b(t)$ or get a lower bound and improve the upper bound $2^{t+1} - 1$.

Another perspective which one can consider is with regards to the computational complexity of recognising flip sequences and flip graphs, summarised in the two problems below.

Problem 6.0.8 (Complexity of deciding that G is a k-flip graph)**.** Given a graph G and an integer $k \geq 2$, what is the complexity class of deciding whether G is a k-flip graph or not? It is also of interest to determine the complexity class of this decision problem if G belongs to a restricted class of graphs.

Problem 6.0.9 (Complexity of deciding that $(a_1, \ldots, a_k)$ is a k-flip sequence)**.** Given an integer $k \geq 2$ and an integer sequence $(a_1, \ldots, a_k)$, what is the complexity class of deciding whether the sequence is a k-flip sequence or not?

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