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Optimization of an Economic Entity's Income under Resource Constraints: One Variant of the Analytical Solution

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Abstract:

Purpose: *The main objective of the research was to develop a mathematical model for the optimal distribution of resources between two production processes to maximize the total economic entity's income.*
Design/Methodology/Approach: *The study constructs a mathematical model assuming that production processes are described by arbitrary Cobb-Douglas production functions. An analytical expression for the general form of the contract curve is derived for the model with two production functions and two factor inputs. Additionally, the production possibility frontier (PPF) function is defined analytically, and the parameters for which the PPF curve takes different shapes are examined. The research employs an analytical approach to address the resource allocation problem. The model provides a framework for determining the optimal resource distribution between two production processes to maximize firms' total income.*
Findings: *The study defines the contract curve and production possibility frontier functions for the two-product model analytically. It identifies the parameters influencing the shape of the PPF curve. The research demonstrates that the proposed model allows for an analytical solution to the resource distribution problem, which is crucial for maximizing firms' income.*
Practical Implications: *The proposed solution enables firms with production processes described by the Cobb-Douglas production function to find optimal resource allocation options, thereby increasing their total income and market share.*
Originality/Value: *This study fills a gap by providing an analytical solution to the problem of optimal resource distribution between two production processes, which has been insufficiently addressed in existing literature.*

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1. Introduction

In the conditions of the modern economy, special attention is paid to the problem of efficient and rational use of resources. Therefore, the need for a comprehensive study of various options for using resources is becoming increasingly important.

Economic resources are one of the most important aspects that ensure the functioning of production. In this regard, today, on the one hand, the problem of providing firms with the resource-saving process is being updated, and on the other hand, the problem of determining the optimal distribution of resources in accordance with the chosen optimality criterion.

In addition, the relevance of this problem lies in the fact that during economic activity, absolutely all enterprises are faced with such a problem as limited resources, and, therefore, with the need to find solutions for their optimal use to achieve the goals set by the company (Thalassinos *et al.,* 2022).

Another factor that determines the relevance of the optimal use of resources is the increase in prices for production resources – inefficient use of resources reduces the production potential of the company and worsens its competitive position in the market. It should be noted that recently there has been an increase in prices for both raw materials and semi-finished products and components with a high level of added value (Hakim and Thalassinos, 2021).

For example, on the world market over the past two years, aluminium prices have increased by about 60%, touching $2,700/MT in December 2021. As the main reasons for the rise in prices for aluminium, V. Kumar highlights substantial increase in energy and raw material costs, growing demand, decline in China's production capacity, reduction in global inventories and the impact of the Coronavirus disease (COVID-19). The researcher predicts a further rise in prices – 10% in 2022 (Kumar, 2022).

Prices for high value-added commodities are rising on the world market. For example, Taiwanese semiconductor giant TSMC is raising prices, and PC processor prices set to rise substantially in 2022 due to higher manufacturing costs (Chen and Ke, 2022).

There are record inflationary pressures in March 2022 in the Irish manufacturing sector, with commodity and product prices rising at the fastest pace in more than two decades of research history. At the same time, demand partly reflected pre-orders from customers due to continued supply chain uncertainty and concerns about rising prices exacerbated by the Russian invasion of Ukraine (Gleeson, 2022).

Thus, in the conditions of rising prices for resources, the problem of optimal distribution of limited resources becomes more and more urgent.

In microeconomic theory, the basic model for studying resource allocation options involves the distribution of two types of resources between two production processes (two firms). The problem of optimal distribution of limited resources can be solved in accordance with the achievement of various goals. For example, the problem of finding a variant of the distribution of resources between two production processes can be solved to produce the maximum volume of products. Another task may be to find the best option for allocating resources between two production processes to achieve maximum profit.

In this study, we propose to consider the problem of optimal distribution of two limited resources between two production processes, which are described by arbitrary Cobb-Douglas production functions (with any kind of returns to scale). The target indicator in this optimization problem is proposed to consider the total income of two firms, the production processes of which are considered in the mathematical model. Such a task of determining the conditions for the optimal allocation of limited resources can be solved by firms whose medium-term goal is to increase their market share.

The paper is structured as follows. In the first section, we review the literature that deals with the problems of finding solutions for the optimal allocation and use of resources by firms. The second section consists of 3 subsections. The first subsection presents the construction of an analytical type of contract curve for a two-product production model with two limited resources. The next subsection considers the analytic view of the production possibility frontier function – for a two-product model with two limited resources.

In the third subsection, based on the analytical types of functions constructed in the previous subsections, we propose a solution to the problem of optimal resource allocation in a two-product model for optimization of an economic entity’s income. Later we provide some conclusions and future research implications.

2. Literature Review

Resource constraints are one of the main constraints in determining the optimal performance of any economic entity. In the context of rising resource prices, the next task is to determine the optimal number of resources that will be used in its activities by the economic entity to achieve its goals.

Modern research examines various aspects of efficient allocation and use of resources in various fields.

Numerous studies have focused on determining the optimal allocation of limited resources in the health care system during the COVID-19 pandemic (Parker *et al.*, 2020), for the University Bakery (Kayode *et al.*, 2020), for vegetable graft nurseries, which provide high-labour production of young plants with several stages of operation (Masoud *et al.*, 2019) for an agricultural enterprise (Zgajnar, 2016), food industry enterprises (Dziamulych *et al.*, 2021; Nestorenko *et al.*, 2020), for a multi-product manufacturing firm in conditions of uncertainty of demand in the presence of budgetary constraints (Boyabatli *et al.*, 2016), for a multi-plant firm (Liou *et al.*, 2006). The parameters of optimal distribution of limited resources in semi-continuous multi-food industries are studied on the example of yogurt production (Kopanos *et al.*, 2011).

The task of efficient use of resources arises when determining the optimal parameters of laboratory management in colleges and universities. Improving the efficiency of laboratory equipment, ensuring the optimal allocation of experimental resources will help improve the quality of experimental teaching and provide a guarantee of "cultivation" of talent. Based on the analysis of the current state and mode of allocation of laboratory resources, Yang offers a scientific management mode in terms of optimal allocation of resources (Yang, 2021).

Efficient use of resources in the supply chain between a manufacturer and several retailers requires coordination and optimization (Yan *et al.*, 2020). For an online shopping company, Yao and Gu, based on an analysis of the supply chain resource allocation in a customized online shopping service and its performance, offers an optimization resource allocation model and builds an algorithm to solve it (Yao and Gu, 2015).

Optimal allocation of resources is crucial for their effective use in a wide range of practical applications in science and technology. In their work, Salazar and co-authors investigate the optimal distribution of resources in multidisciplinary quantum systems (Salazar *et al.*, 2021).

Effective allocation of resources in the merger of two firms depends on the establishment of true upward communication (Friebel and Raith, 2010). Researchers argue that this effect dominates the positive impact on efforts caused by competition for the firm's resources. Resource efficiency is significantly affected by the lack of spare parts and maintenance technicians (Sandeep and Bhupesh, 2016). In addition, capacity constraints on resource consumption by agents should be considered when deciding on the optimal use of resources (Kumar *et al.*, 2009).

In their work, Xianglan and Jun determine the conditions for the optimal distribution of resources in society (at the macro level) using a constructed stochastic model. The authors discuss the impact of many parameters on economic growth and social well-being (Xianglan and Jun, 2011). Some studies propose an analytical solution to the problem of optimal distribution of resources in the game of two parties based on the Cobb-Douglas production function with a constant return when changing the scale of production (Nestorenko *et al.*, 2022).

In a few research problems of optimization of logistic processes in shops are considered. Using a genetic algorithm (GA), Yang and co-authors formulate and solve a mathematical optimization model of shop planning (Yang *et al.*, 2019). The authors show that after optimization, the volume of logistics decreased by 63.5%, and capacity increased by 42.0%, which allows more efficient use of time and resources.

In their study, Nourelfath and Yalaoui present a method of integrating solutions for load balancing and tactical production planning, considering the cost of changing capacity and the cost of unused capacity. The authors investigate the parameters that will minimize the amount of capacity change costs, unused capacity costs, installation costs, storage costs, order costs, and production costs. The main limitations are to meet the demand for all products on the horizon and not to exceed the available repair resources (Nourelfath and Yalaoui, 2012).

The challenge of optimization is to define inspection and maintenance policies to minimize long-term risks and costs in deteriorating engineering environments. One of the main problems is the existence of restrictions on stochastic long-term constraints due to lack of resources and other unfeasible / undesirable reactions of the system (Andriotis and Papakonstantinou, 2020). The authors propose to solve this and several other problems within the joint structure of limited Markov decision-making processes (POMDP) and multi-agent deep reinforcement (DRL).

In a study by Yan and co-authors, optimal resource allocation strategies are obtained by introducing stochastic comparisons for different policies according to the usual stochastic and risk order. As special cases of relativity, the policy of load distribution and minimum repair is investigated. Sufficient (and necessary) conditions for various stochastic orders are established (Yan *et al.*, 2021).

Ghassemi and Scott analyzes the problems of simultaneous use of energy and water in power plants. Electricity production at power plants directly depends on water supply, and the shutdown of power systems will affect wastewater treatment processes. The authors argue that neglecting the relationship between energy and water leads to vulnerabilities that could lead to restrictions on one resource. Because the mathematical problems of energy-water communication are complex, involve many uncertain parameters, and are large-scale, the authors proposed a new multi-stage regulated fuzzy robust approach that balances the reliability of solutions with budget constraints (Ghassemi and Scott, 2020).

The literature review has shown that the problem of optimizing income with limited resources is relevant both in theoretical research and in solving practical problems. At the same time, of all the empirical production functions, the Cobb-Douglas production function is most often used. In scientific literature contract curve (CC), production possibility frontier (PPF) are presented theoretically (schematically).

The analytical form of these functions provides a tool for further theoretical research and allows us to find a practical solution to the problem of optimization of an economic entity's income under resource constraints.

3. Materials and Methods

The study utilizes mathematical modelling to describe two production processes using Cobb-Douglas production functions. It analytically defines the contract curve and production possibility frontier (PPF) for the resource allocation problem under the resources constraints. The parameters of the production functions are analysed to determine optimal resource distribution and maximize total income.

The study employs the construction of an Edgeworth box to analyse the resource allocation between two production processes described by Cobb-Douglas production functions.

Mathematical modeling methods were employed to construct the analytical form of a series of functions, including the contract curve and the production possibility frontier function. To visualize the analytical form of the constructed functions, the Excel software package was utilized. This approach facilitated the accurate representation and analysis of the optimal resource distribution between the two production processes.

4. Results

4.1 Model Construction for Practical Solution of the Income Optimization Problem with Cobb-Douglas Production Functions

Assuming that there are 2 production processes (they may be owned by the same company or by two different companies), denoted as $i = 1, 2$. In these production processes, 2 types of resources are used to produce product Q_i, denoted as X_i and Y_i.

The maximum amount of resources is defined accordingly as *L*, *K*. We assume that the scheduling time and production process time of each product are not considered.

Assuming that there are two production processes be described by Cobb-Douglas production functions:

$$Q_1(X_1,Y_1) = \gamma_1 X_1^{\alpha_1} Y_1^{\beta_1}, Q_2(X_2,Y_2) = \gamma_2 X_2^{\alpha_2} Y_2^{\beta_2}$$
$$\gamma_1, \gamma_2 > 0,\ 0 < \alpha_1, \beta_1, \alpha_2, \beta_2 < 1 \quad (1)$$

where α_i, β_i – parameters of Cobb-Douglas production function $Q_i(X_i,Y_i) = \gamma_i X_i^{\alpha_i} Y_i^{\beta_i}$, $0 < \alpha_i, \beta_i < 1$; γ_i – equalization parameter of Cobb-Douglas production function of produce the *i-th* product.

In this case, the restrictions $X_1 + X_2 \leq L$, $Y_1 + Y_2 \leq K$ are satisfied. These two products are sold at prices p_1 and p_2, accordingly.

We assume that in the problem under consideration, the Cobb-Douglas production function can be with any returns to scale:

increasing returns to scale – when $\alpha_i + \beta_i > 1$;

decreasing returns to scale – when $\alpha_i + \beta_i < 1$;

constant returns to scale – when $\alpha_i + \beta_i = 1$.

It is necessary to allocate limited resources L and K in such a way that the total income $I(Q_1, Q_2)$ from the sale of products of volume Q_1 and Q_2 is maximum.

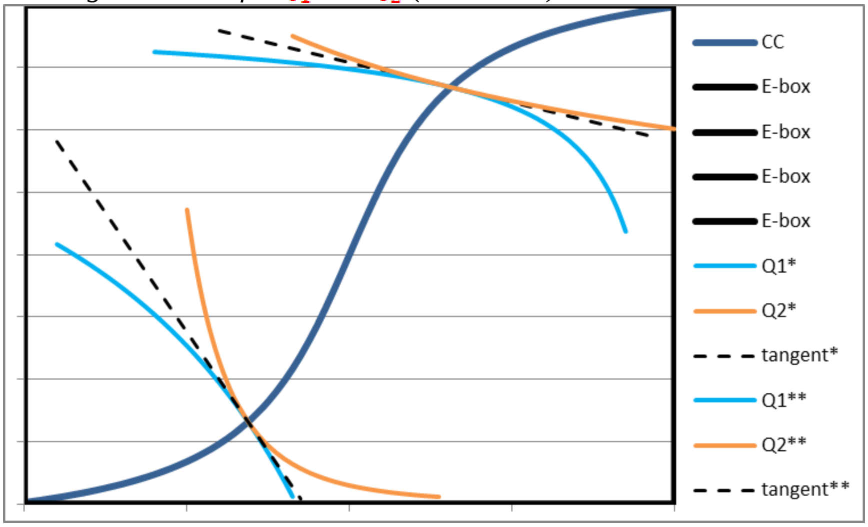
Formally, the problem can be written as follows:

$$\begin{cases} I(Q_1, Q_2) = I_1(Q_1) + I_2(Q_2) = p_1 Q_1 + p_2 Q_2 \rightarrow max \\ Q_1(X_1,Y_1) = \gamma_1 X_1^{\alpha_1} Y_1^{\beta_1},\ Q_2(X_2,Y_2) = \gamma_2 X_2^{\alpha_2} Y_2^{\beta_2} \\ 0 \leq X_1 + X_2 \leq L,\ 0 \leq Y_1 + Y_2 \leq K \end{cases} \quad (2)$$

4.1.1 Contract Curve Construction

The contract curve consists of points at which the graphs of functions (1), when following the rules for constructing an Edgeworth box, touch each other (Figure 1). The contract curve is a line on the Edgeworth box that connects all the points that illustrate Pareto-optimal distribution of two resources between two production processes.

Figure 1. *Edgeworth box for Q_1 and Q_2 (theoretical)*



Source: *Malinvaud, 1982.*

At these points, the angles of the tangents to the functions (1) are equal. Therefore, the derivatives to the functions (1) are equal at these points. Let us write down the equations of functions (1) under the conditions of construction of the Edgeworth box (we use the resource constraint conditions (2)).

$$Q_1(X_2,Y_2) = \gamma_1(L - X_2)^{\alpha_1}(K - Y_2)^{\beta_1}, \quad Q_2(X_2,Y_2) = \gamma_2 X_2^{\alpha_2} Y_2^{\beta_2} \tag{3}$$

Find and equate the derivatives of implicitly given functions (3):

$$-\frac{\alpha_1}{\beta_1}\frac{K-Y_2}{L-X_2} = -\frac{\alpha_2}{\beta_2}\frac{Y_2}{X_2} \tag{4}$$

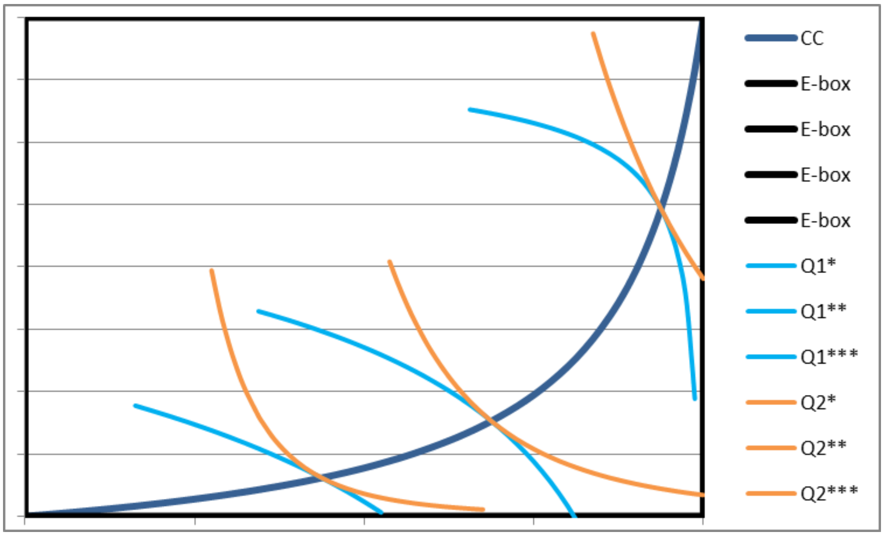
Let us denote $\delta_{12} = \frac{\beta_1\alpha_2}{\alpha_1\beta_2}$. Then from (4) we obtain the analytical form of the contract curve:

$$Y_2 = \frac{KX_2}{\delta_{12}L + (1-\delta_{12})X_2} \tag{5}$$

The plot of function (5) within the boundaries of the Edgeworth box will be part of the hyperbola branch (Figure 2).

With $\delta_{12} = 1$, the contract curve will be a straight line $Y_2 = \frac{K}{L}X_2$ (the diagonal of the Edgeworth box). Without violating the generality of the approach, we will assume that $\delta_{12} \geq 1$, and the contract curve will have the form shown in Figure 2.

Figure 2. *Type of contract curve for two arbitrary Cobb-Douglas production functions*



Source: *Built by the authors in EXCEL according to the formulas (3), (5).*

4.1.2 Production Possibility Frontier Construction

Production Possibility Frontier (PPF) is a statistical curve that visualizes the range of differences between products that require the same materials or resources to produce.

PPF has a significant role in the economy at the macro level. The PPF illustrates that the economy of any country reaches its highest level of efficiency when it produces only what is best for production and imports everything else it needs from other countries. It is also advisable to study the PPF at the micro level – at the level of firms. PPF can help firms evaluate how to distribute limited resources in production processes.

Therefore, the task of determining the PPF analytical form for the model with two production functions and with two factor inputs in production is relevance.

Substituting expressions (5) into (3), we obtain the parametric view of PPF:

$$\begin{cases} I_1(X_2) = \dfrac{p_1 \gamma_1 \delta_{12}^{\beta_1} K^{\beta_1} (L - X_2)^{\alpha_1 + \beta_1}}{(\delta_{12} L + (1 - \delta_{12}) X_2)^{\beta_1}} \\ I_2(X_2) = \dfrac{p_2 \gamma_2 K^{\beta_2} X_2^{\alpha_2 + \beta_2}}{(\delta_{12} L + (1 - \delta_{12}) X_2)^{\beta_2}} \\ 0 \le X_2 \le L \end{cases} \qquad (6)$$

Let us find the derivative of function (6):

$$\frac{dI_2(X_2)}{dI_1(X_2)} = -\frac{p_2\gamma_2 K^{\beta_2-\beta_1}\alpha_2}{p_1\gamma_1\delta_{12}^{\beta_1}\alpha_1}\frac{X_2^{\alpha_2+\beta_2-1}(L-X_2)^{1-\alpha_1-\beta_1}}{(\delta_{12}L+(1-\delta_{12})X_2)^{\beta_2-\beta_1}} < 0 \quad (7)$$

Since the derivative of function (7) is less than zero at $0 < X_2 < L$, then function (6) decreases on this interval.

An analysis of the second derivative of function (14) showed that the shape of the PPF depends on the behaviour of the function

$$h(X_2) = (\alpha_2+\beta_2-1)\delta_{12}L^2 + \Big(\big((\alpha_2+\beta_2-1)-(\beta_2-\beta_1)\big) - \big((\alpha_2+\beta_2-1)+(\alpha_2-\alpha_1)\big)\delta_{12}\Big)LX_2 - (\alpha_2-\alpha_1)(1-\delta_{12})X_2^2 \quad (8)$$

on the interval $(0, L)$.

On the boundaries of the interval $h(X_2)$ takes the following values:

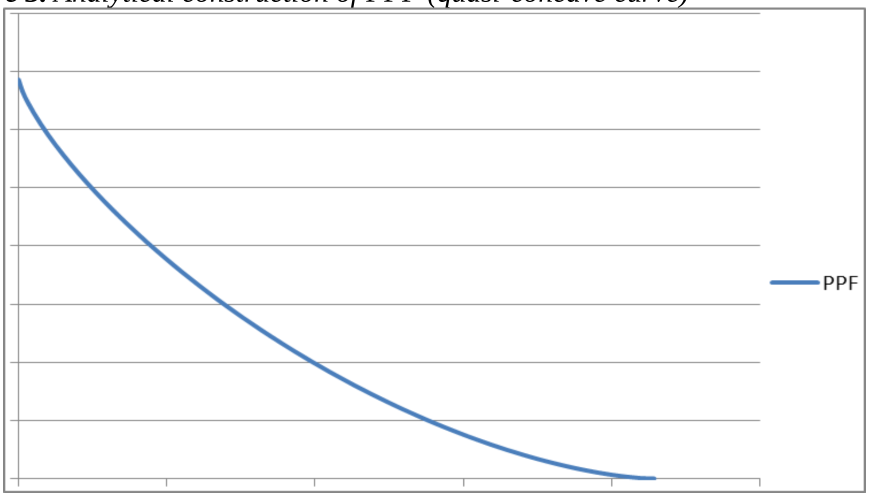
$$h(0) = (\alpha_2+\beta_2-1)\delta_{12}L^2$$
$$h(L) = (\alpha_1+\beta_1-1)L^2$$

Therefore, the following PPF behaviours are possible:

1) for $\alpha_2+\beta_2 > 1$ and $\alpha_1+\beta_1 > 1$, function (8) can be

a) $h(X_2) \geq 0$ on the interval $0 \leq X_2 \leq L$, hence PPF is quasi-concave ($\cup$) (Figure 3):

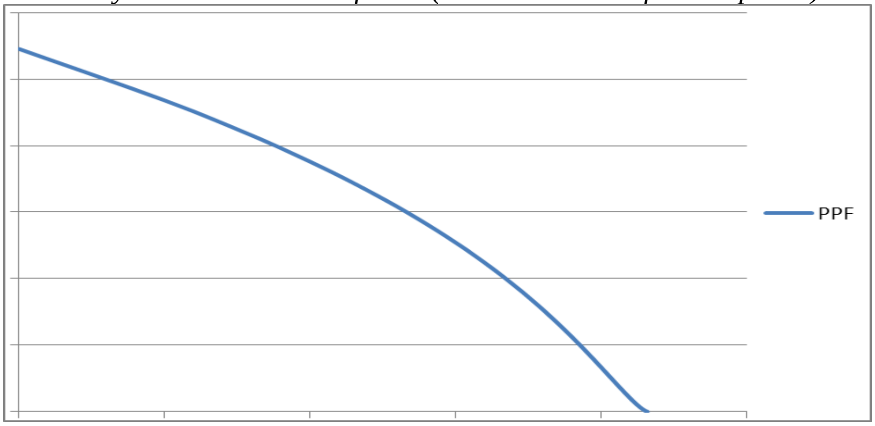
Figure 3. *Analytical construction of PPF (quasi-concave curve)*



Source: *Built by the authors in EXCEL according to the formula (6).*

b) $h(X_2) \geq 0$ under $X_2 \in (0, X_2^*) \cup (X_2^{**}, L)$, $h(X_2) < 0$ under $X_2 \in (X_2^*, X_2^{**})$, therefore, PPF has two inflection points (∪∩∪) (Figure 4).

Figure 4. *Analytical construction of PPF (curve with two inflection points)*



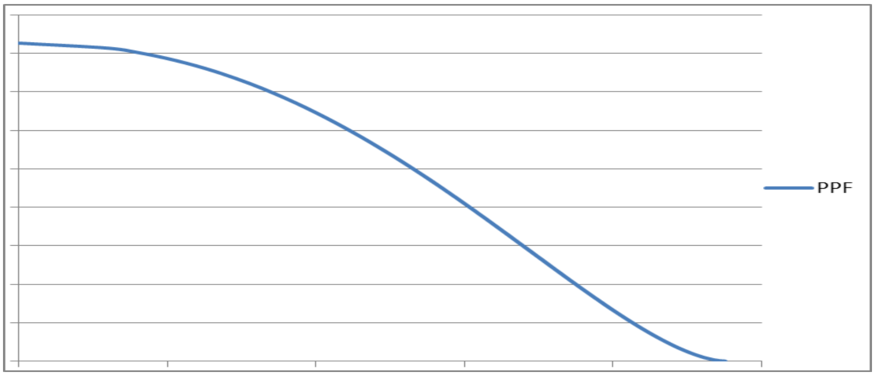
Source: *Built by the authors in EXCEL according to the formula (6).*

2) under $\alpha_2 + \beta_2 > 1$ and $\alpha_1 + \beta_1 = 1$ function (8) may be:

a) $h(X_2) \geq 0$ on the interval $0 \leq X_2 \leq L$, hence, PPF is quasi-concave (∪) (Figure 3);

b) $h(X_2) \geq 0$ under $X_2 \in (0, X_2^*)$, $h(X_2) \leq 0$ under $X_2 \in (X_2^*, L)$, hence, PPF has one inflection point (∪∩) (Figure 5).

Figure 5. *Analytical construction of PPF (curve with one inflection point)*



Source: *Built by the authors in EXCEL according to the formula (6).*

3) under $\alpha_2 + \beta_2 = 1$ and $\alpha_1 + \beta_1 > 1$ function (8) may be:

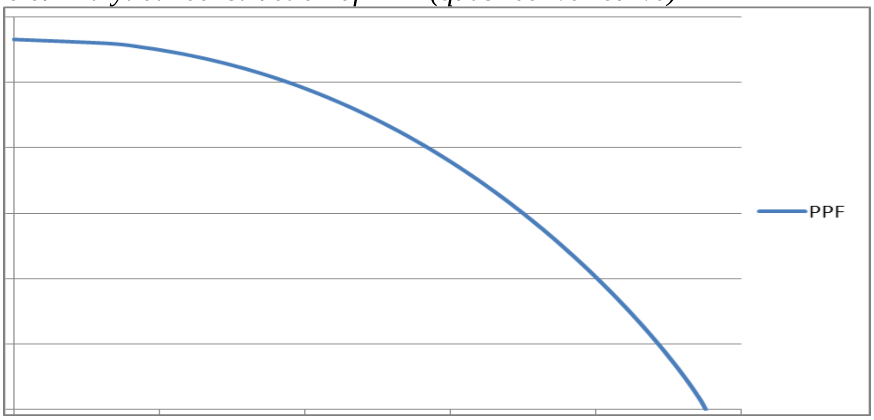
a) $h(X_2) \geq 0$ on the interval $0 \leq X_2 \leq L$, therefore, PPF is quasi-concave (∪) (Figure 3);

b) $h(X_2) \leq 0$ under $X_2 \in (0, X_2^*)$, $h(X_2) \geq 0$ under $X_2 \in (X_2^*, L)$, consequently, PPF has one inflection point (∩∪) (Figure 5).

4) under $\alpha_2 + \beta_2 = 1$ and $\alpha_1 + \beta_1 = 1$ function (8) may be:

$h(X_2) \leq 0$ on the interval $0 \leq X_2 \leq L$, therefore, PPF is quasi-convex (∩) (Figure 6);

***Figure 6.** Analytical construction of PPF (quasi-convex curve)*



***Source:** Built by the authors in EXCEL according to the formula (6).*

5) under $\alpha_2 + \beta_2 > 1$ and $\alpha_1 + \beta_1 < 1$ function (8) may be:

$h(X_2) \geq 0$ under $X_2 \in (0, X_2^*)$, $h(X_2) \leq 0$ under $X_2 \in (X_2^*, L)$, consequently, PPF has one inflection point (∪∩) (Figure 5).

6) under $\alpha_2 + \beta_2 < 1$ and $\alpha_1 + \beta_1 > 1$ function (8) may be:

$h(X_2) \leq 0$ under $X_2 \in (0, X_2^*)$, $h(X_2) \geq 0$ under $X_2 \in (X_2^*, L)$, hence, PPF has one inflection point (∩∪) (Figure 5).

7) under $\alpha_2 + \beta_2 < 1$ and $\alpha_1 + \beta_1 = 1$ function (8) may be:
a) $h(X_2) \leq 0$ on the interval $0 \leq X_2 \leq L$, consequently, PPF is quasi-convex (∩) (Figure 6);

b) $h(X_2) \leq 0$ under $X_2 \in (0, X_2^*)$, $h(X_2) \geq 0$ under $X_2 \in (X_2^*, L)$, therefore, PPF has one inflection point (∩∪) (Figure 5).

8) under $\alpha_2 + \beta_2 = 1$ and $\alpha_1 + \beta_1 < 1$ function (8) may be:

a) $h(X_2) \leq 0$ on the interval $0 \leq X_2 \leq L$, consequently, PPF is quasi-convex (∩) (Figure 6);

b) $h(X_2) \geq 0$ under $X_2 \in (0, X_2^*)$, $h(X_2) \leq 0$ under $X_2 \in (X_2^*, L)$, consequently, PPF has one inflection point (∪∩) (Figure 5).

9) under $\alpha_2 + \beta_2 < 1$ and $\alpha_1 + \beta_1 < 1$ function (8) may be:

$h(X_2) \leq 0$ on the interval $0 \leq X_2 \leq L$, hence, PPF is quasi-convex (∩) (Figure 6).

The analytical form of the production possibility frontier function proposed in this section will be used to determine the optimal parameters for the distribution of scarce resources to maximize the total income of firms in a two-product model.

4.2 Solving the Problem of Optimal Resource Allocation in a Two-Product Model

In this section, we extend the results developed in the previous sections.

Substituting expression (6) into the objective function (2), we obtain the problem of finding the largest value on the segment:

$$\begin{cases} I(X_2) = p_1\gamma_1\delta_{12}^{\beta_1}K^{\beta_1}\frac{(L-X_2)^{\alpha_1+\beta_1}}{(\delta_{12}L+(1-\delta_{12})X_2)^{\beta_1}} + \\ \quad p_2\gamma_2K^{\beta_2}\frac{X_2^{\alpha_2+\beta_2}}{(\delta_{12}L+(1-\delta_{12})X_2)^{\beta_2}} \to max \\ 0 \leq X_2 \leq L \end{cases} \tag{9}$$

The solution to problem (9) can be located either at the ends of the segment (angular solution) or inside the segment (internal solution). The internal solution can be in the current, where the derivative of the function (9) is equal to zero.

$$\frac{dI(X_2)}{dX_2} = \left((\alpha_1\delta_{12} + \beta_1)L + \alpha_1(1-\delta_{12})X_2\right)\left(-\frac{p_1\gamma_1\delta_{12}^{\beta_1}K^{\beta_1}(L-X_2)^{\alpha_1+\beta_1-1}}{(\delta_{12}L+(1-\delta_{12})X_2)^{\beta_1+1}} + \frac{p_2\gamma_2\alpha_2K^{\beta_2}X_2^{\alpha_2+\beta_2-1}}{\alpha_1(\delta_{12}L+(1-\delta_{12})X_2)^{\beta_2+1}}\right) = 0 \tag{10}$$

Since $(\alpha_1\delta_{12} + \beta_1)L + \alpha_1(1-\delta_{12})X_2 > 0$ under $0 \leq X_2 \leq L$, from (10) we obtain a nonlinear equation, the solution of which can give an internal solution to problem (9):

$$(\delta_{12}L + (1-\delta_{12})X_2)^{\beta_2-\beta_1} = \frac{p_2\gamma_2\alpha_2K^{\beta_2-\beta_1}}{p_1\gamma_1\alpha_1\delta_{12}^{\beta_1}}X_2^{\alpha_2+\beta_2-1}(L-X_2)^{1-\alpha_1-\beta_1} \tag{11}$$

Let's denote $\varepsilon_{12} = \frac{p_2 \gamma_2 \alpha_2}{p_1 \gamma_1 \alpha_1 \delta_{12}^{\beta_1}}$, then (11) will look like:

$$(\delta_{12}L + (1 - \delta_{12})X_2)^{\beta_2 - \beta_1} = \varepsilon_{12} K^{\beta_2 - \beta_1} X_2^{\alpha_2 + \beta_2 - 1} (L - X_2)^{1 - \alpha_1 - \beta_1} \quad (12)$$

Since equation (12) cannot be solved explicitly (it is possible only in special cases, for example, for Cobb-Douglas production function with a constant return), let us consider possible solutions to problem (9).

To do this, consider the mutual behaviour of the functions:

$$f(X_2) = (\delta_{12}L + (1 - \delta_{12})X_2)^{\beta_2 - \beta_1}$$
$$g(X_2) = \varepsilon_{12} K^{\beta_2 - \beta_1} X_2^{\alpha_2 + \beta_2 - 1} (L - X_2)^{1 - \alpha_1 - \beta_1} \quad (13)$$

Let $\beta_2 \geq \beta_1$ (for case $\beta_2 < \beta_1$ analysis will be similar).

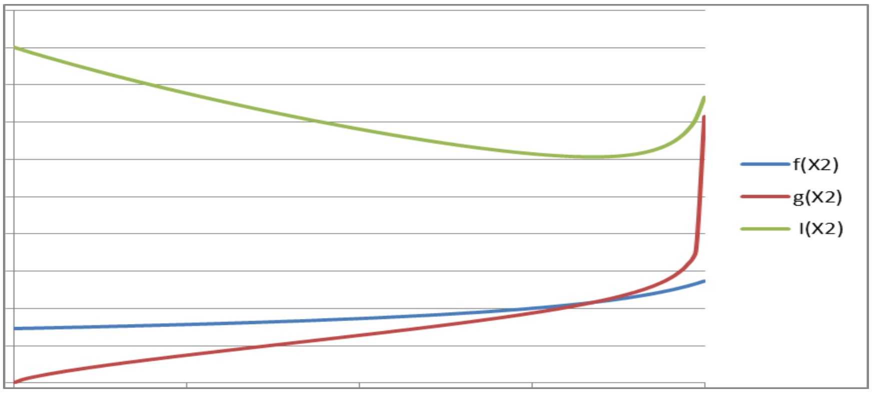
On the boundaries of the interval $g(X_2)$ takes the values 0 or $+\infty$. Moreover, where the inequality $g(X_2) < f(X_2)$, is satisfied, the function $I(X_2)$ decreases.

Therefore, the following behaviors of these functions are possible:

1) under $\alpha_2 + \beta_2 > 1$ and $\alpha_1 + \beta_1 > 1$ may be:

a) intersect at one point, therefore, the solution of problem (9) will be angular (Figure 7a):

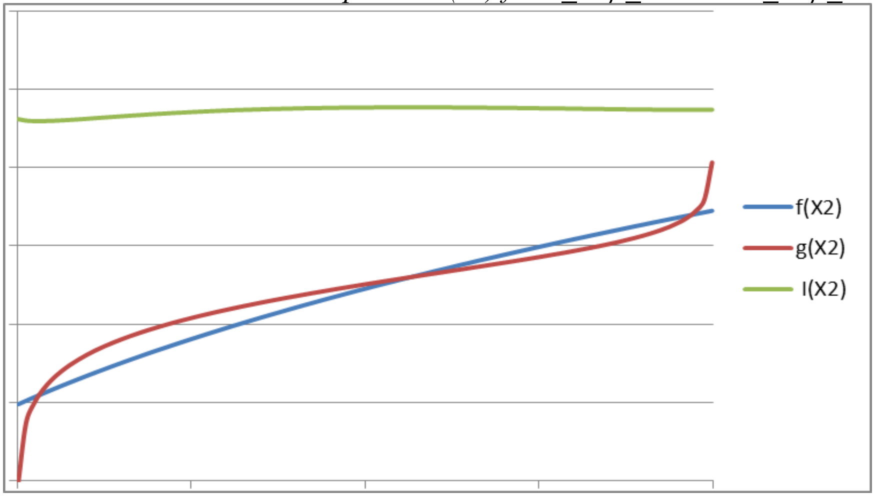
Figure 7a. *The first solution to problem (9) for α_2+β_2>1 and α_1+β_1>1*



Source: *Built by the authors in EXCEL according to the formulas (9), (13).*

b) intersect at two points, therefore, the solution of problem (9) can be both internal and angular (Figure 7b).

Figure 7b. *The second solution to problem (17) for α_2+β_2>1 and α_1+β_1>1*

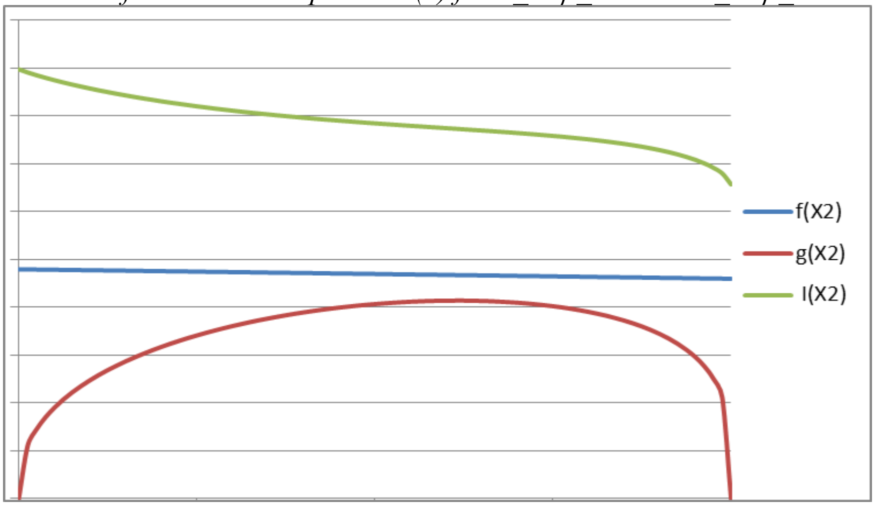


Source: *Built by the authors in EXCEL according to the formulas (9), (13).*

2) under α_2+β_2>1 and α_1+β_1<1 may be:

a) do not intersect, therefore, the solution of problem (9) will be angular (Figure 8a):

Figure 8a. *The first solution to problem (9) for α_2+β_2>1 and α_1+β_1<1*

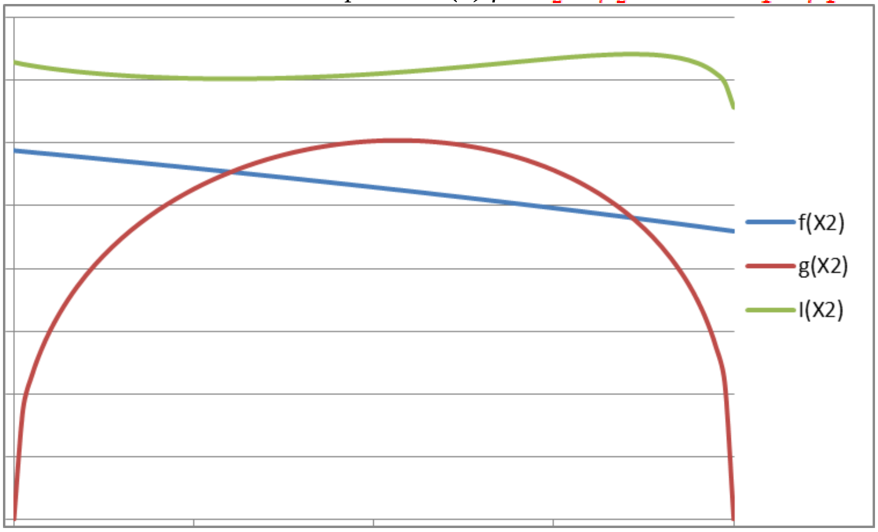


Source: *Built by the authors in EXCEL according to the formulas (9), (13).*

b) intersect at two points, therefore, the solution of problem (9) can be both internal and angular (Figure 8b).

In case $\alpha_2+\beta_2<1$ and $\alpha_1+\beta_1>1$ or $\alpha_2+\beta_2<1$ and $\alpha_1+\beta_1<1$ the definition of the solution to problem (9) will be similar.

Figure 8b. *The second solution to problem (9) for* $\alpha_2 + \beta_2 > 1$ *and* $\alpha_1 + \beta_1 < 1$



Source: *Built by the authors in EXCEL according to the formulas (9), (13).*

5. Discussion and Conclusion

In modern conditions, resource constraints, the constant rise in prices for most production resources require firms to implement an effective mechanism for their distribution and use.

The literature review has shown that the problem of maximizing the firms' income with the optimal resources' distribution for various production function types is in most cases considered in the economic and theoretical aspect. Emphasis is predominantly placed on the economic interpretation of the results. At the same time, insufficient attention was paid to the analytical solution of this class of problems. This study was aimed at filling this gap. The study considers an analytical approach to finding the optimal resource allocation variant in the case when the production processes of firms are described by an arbitrary Cobb-Douglas production function.

The paper considers one of the approaches to solving the actual problem of developing a mathematical model for the distribution of two limited resources between two production processes to maximize the total income of firms.

The study proposes an analytical form of a series of functions (contract curve, production possibility frontier function), the use of which allows you to find the optimal distribution of resources between two production processes (or firms – for example, duopoly) to maximize the total income from the sale of manufactured products.

The use of the presented solution allows firms, whose production processes can be described by the Cobb-Douglas production function, to determine the optimal options for allocating resources to maximize total income and, thereby, increase their market share.

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