Tipping the Scales . . . Functionally!

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Although one might be tempted to attribute the choice of notes in musical scales and chords to purely aesthetic considerations, a growing body of work seeks to establish correspondences with mathematical notions. Intriguingly, the most important musical scales turn out to be solutions to optimisation problems.

In this abstract, we review mathematical characterisations which yield some of the most important musical scales. We also describe an implementation of these notions using the functional programming language Haskell, which has allowed us to conduct interactive experiments and to visualise various approaches from the literature.

1 Pitches, Pitch Classes and Scales

The *pitch* of a musical note is quantified by its frequency. Pitches are partitioned into *pitch classes* with frequencies $p \times 2^n, -\infty < n < \infty$ (doubling frequencies) under the *octave equivalence* relation. Pitches within the same pitch class, although having different frequencies, are perceived as being of the same quality [6].

The continuum of pitch classes can be discretised by choosing a finite subset of pitch classes as a palette. This is the *chromatic* pitch-class set[1], the raw material at a composer's disposition. Frequencies are typically chosen at (or near) equidistant points on a logarithmic scale, so that the brain perceives the distance between successive pairs of pitches as being the same. Though by no means universal, the use of twelve pitch classes is predominant in western music [3] and is widely represented in other cultures. Western music also exhibits a preference for scales containing seven note pitch-class sets chosen from the twelve-note chromatic set. The scales are characterised not by the presence of specific pitch classes but by the distances between them.

The *clock diagram* [2] places chromatic pitch classes at equidistant points on the circumference of a circle. This representation is useful because it eliminates bias towards particular pitch classes or distance patterns (unlike staff notation or a piano keyboard), while embodying the cyclic nature of pitches with respect to octave equivalence.

2 Formalising Musical Scales using Euclidean Geometry

The chromatic pitch classes are represented as c equidistant points on the circumference of a circle. We write $pos_c(i)$ (where $0 \leq i < c$) to represent the euclidean coordinates of the ith point (of c equidistantly distributed points) on the circumference of a circle centred at the origin and with unit radius such that the 0th position lies at $(0, 1)$. A pitch-class set ψ is a sequence of d (with $d \leq c$) distinct pitch classes $\psi = \langle p_0, p_1 \ldots p_{d-1} \rangle$ from a chromatic set of c (i.e. $0 \leq p_i < c$). We write $\Psi_{d/c}$ to denote the set of all possible pitch-class sets of size d from c chromatic pitch classes, and Ψ_c to denote all possible pitch-class sets of any size from c chromatic pitch classes. We write $poly(\psi)$ to denote the d-sided polygon which is formed by joining nearest pairs of chosen points from pitch-class set ψ with straight lines. Given two distinct pitch classes p and p' in a pitch-class set $\psi \in \Psi_c^d$, we denote the *euclidean distance* between the points as $\delta_c^{euc}(p, p') = |pos_c(p) - pos_c(p')|$ and the *diatonic distance* for a pitch-class set ps between the points as the number of lines between them: $\delta_{c,ps}^{dia}(p, p') = \#([p, p') \cap ps)$.

[1] Contrary to the mathematical definition, in musical set theory the pitch classes in a pitch-class set are ordered.

3 Scales as Solutions to Optimisation Problems

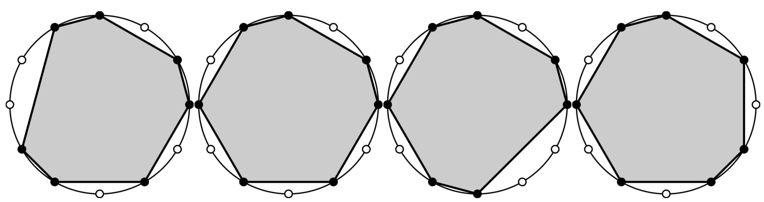
Many important scales can be identified as optimal solutions to geometric measures [1, 5, 4]. Given a metric $f \in \Psi_c \to \mathbb{R}$, we identify the set of its maximal (or minimal) musical scales of length d from c chromatic pitch classes as: $\mathit{maximise}_{d/c}(f) = \{\psi \in \Psi_{d/c} \mid f(\psi) = M\}$ where $M = \max\{f(\psi) \mid \psi \in \Psi_{d/c}\}$. In the Haskell implementation, we use higher-order functions to choose the maximal and minimal solutions of a given metric. Maximisation is defined as:

```
maximise f (c,d) =
  maximumBy
    (\x y -> compare (f (c,d) x) (f (c,d) y))
      (scalesOfSize (c,d))
```

Achieving *maximal evenness* [1] is analogous to seating a group of, for instance, seven people at a round table with twelve chairs in such a way that they are spread as evenly as possible. A set of d pitch classes chosen from c chromatic pitch classes is maximally even if the pitch classes are arranged as evenly as possible on the circle when their positions are restricted to the positions of the c chromatic 'slots'. Going further, we may optimise metrics over a scale for pitch classes with a particular diatonic distance between them, rather than limit ourselves to ones which are a diatonic distance of one apart [5]. We can, for instance, implement the maximal evenness property for a generalised distance as:

```
unevennessNApart n (c,d) s = maximum distances - minimum distances
  where
    distances = map (diatonicDistance (c,d)) (pairsNApart n s)
maximalEvennessNApart n = minimise (unevennessNApart n)
```

With `maximalEvennessNApart 2 (12,7)` we identify the seven-note scales which maximise evenness between alternate notes in the scale (also known as diatonic thirds), and our JavaScript visualiser illustrates the result using clock diagrams. The harmonic minor, the melodic minor, the harmonic major and the major scale appear on the screen:



4 Future Work

We observed that the abstraction and extensibility provided by Haskell lends itself to free-wheeling exploration. As the framework is augmented to handle higher degrees of freedom such as harmony and time we expect it to unlock substantial exploratory and experimental research potential.

References

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