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ON SUPERPARACOMPACT AND LINDELÖF GO-SPACES

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ABSTRACT. In this paper we study some compact/paracompact type properties, namely weak superparacompactness, superparacompactness and Lindelöfness. Particular attention is given to GO-spaces. It is proved that a GO-space X is weakly superparacompact if and only if every gap is a W-gap and every pseudogap is a W-pseudogap. A characterization of Lindelöf GO-spaces involving C-(pseudo)gaps is given. We also show that there is a 1–1 correspondence between superparacompact (resp. Lindelöf) GO-d-extensions and preuniversal ODF (resp. prelindelöf) GO-uniformities. Finally we give several examples corresponding to the above results.

1. PRELIMINARIES

Let X be a topological space and let $\mathcal{W} = \{W_\gamma : \gamma \in \Gamma\}$ be a collection of subsets of X. A finite sequence $W_{\gamma(i)}, i = 1, \ldots, s$ of elements of $\mathcal{W}$ is said to be a *chain* from W_γ to $W_{\gamma'}$ if $\gamma(1) = \gamma, \gamma(s) = \gamma'$ and $W_{\gamma(i)} \cap W_{\gamma(i+1)} \neq \emptyset$ for $i = 1, \ldots, s-1$. The collection $\mathcal{W}$ is said to be *connected* if for every $W_\gamma, W_{\gamma'} \in \mathcal{W}$, there exists a chain from W_γ to $W_{\gamma'}$. Maximal connected subcollections of a collection $\mathcal{W}$, that is connected subcollections of $\mathcal{W}$ which are not proper subsets of any connected subcollection of $\mathcal{W}$, are called *components* of $\mathcal{W}$.

It is known that every collection $\mathcal{W}$ of subsets of X decomposes into the union of its components and that the supports of different components are disjoint. Also, if the collection $\mathcal{W}$ is star-countable then each component is a countable subcollection of $\mathcal{W}$ [1]. By the support $\widetilde{\mathcal{W}}$ of a collection of subsets $\mathcal{W}$ we mean $\cup\mathcal{W} = \cup\{W : W \in \mathcal{W}\}$ and by $[\mathcal{W}] = [\mathcal{W}]_X$ we mean $\{[W]_X : W \in \mathcal{W}\}$, where

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$[W]_X$ is the closure of W in X. It follows that if $\mathcal{W}$ is an open cover of the space X, then the support $\widetilde{\mathcal{W}_\lambda}$ of any component $\mathcal{W}_\lambda$ of $\mathcal{W}$ is clopen (closed and open) in X.

Definition 1. A star-finite open cover of the space X is said to be a *finite component cover* if the number of elements of each component is finite.

We now turn to the definition of four classes of paracompact type topological spaces.

Definition 2. A $T_{3\frac{1}{2}}$-space X is called *weakly P-complete* (resp. *weakly superparacompact*) if for every $x \in \beta X - X$ (resp. compact $B \subset \beta X - X$), there exists a clopen cover $\mathcal{W}$ of the space X such that $x \notin \cup[\mathcal{W}]_{\beta X}$ (resp. $B \cap (\cup[\mathcal{W}]_{\beta X}) = \emptyset$), that is x is not contained in the closure in βX of any element of $\mathcal{W}$, where βX is the Stone-Čech compactification of X.

Thus every weakly superparacompact space is weakly P-complete.

Definition 3. A $T_{3\frac{1}{2}}$-space X is called *P-complete* (resp. *superparacompact*) if for every $x \in \beta X - X$ (resp. compact $B \subset \beta X - X$), there exists a finite component cover $\mathcal{W}$ of the space X such that $x \notin \cup[\mathcal{W}]_{\beta X}$ (resp. $B \cap (\cup[\mathcal{W}]_{\beta X}) = \emptyset$).

Thus every P-complete space is weakly P-complete and every superparacompact space is P-complete and weakly superparacompact (and so also weakly P-complete). The following important characterization of superparacompact spaces makes it possible to define such spaces outside the range of $T_{3\frac{1}{2}}$-spaces (cf. [8]).

Proposition 1.1. *A $T_{3\frac{1}{2}}$-space X is superparacompact if and only if for every open cover of X there exists an open finite component refinement.*

Many results concerning the above four mentioned classes of spaces can be found in [8].

It is known that the class of superparacompact spaces lies strictly between the class of compact spaces and the class of strongly paracompact spaces. There is an example ($S \times S$, where S is the Sorgenfrey line) of a P-complete (and weakly superparacompact) space which is not a superparacompact space. Also, $T(\omega_1)$ ($\equiv [0, \omega_1[$ with the standard open interval topology) is weakly superparacompact (and weakly P-complete), but it is not P-complete. Thus the class of P-complete spaces lies strictly between the class of weakly P-complete spaces and the class of superparacompact T_2-spaces.

Before we give other characterizations of the above four defined classes we need the following definition.

Definition 4. Let X be a $T_{3\frac{1}{2}}$-space. A compactification Y of X is said to be *perfect* with respect to the open (in X) set U if $Fr_Y O\langle U\rangle = [Fr_X U]_Y$, where $O\langle U\rangle$ is the biggest open set of Y such that $O\langle U\rangle \cap X = U$. The compactification Y is said to be a *perfect compactification* if it is perfect with respect to every open (in X) set U.

It is known that βX is a perfect compactification for any $T_{3\frac{1}{2}}$-space X [7]. The following results are known [8].

Theorem 1.2. *For a $T_{3\frac{1}{2}}$-space X the following are equivalent:*

1. *X is weakly P-complete (resp. weakly superparacompact);*
2. *For every $x \in bX - X$ (resp. compact $B \subset bX - X$) of any perfect compactification bX of X there exists a clopen cover $\mathcal{W}$ of X such that $x \notin \cup[\mathcal{W}]_{bX}$ (resp. $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$);*
3. *There exists a perfect compactification bX of X such that for every $x \in bX - X$ (resp. compact $B \subset bX - X$) there exists a clopen cover $\mathcal{W}$ of X such that $x \notin \cup[\mathcal{W}]_{bX}$ (resp. $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$).*

Theorem 1.3. *For a $T_{3\frac{1}{2}}$-space X the following are equivalent:*

1. *X is P-complete (resp. superparacompact);*
2. *For every $x \in bX - X$ (resp. compact $B \subset bX - X$) of any perfect compactification bX of X there exists an open disjoint cover $\mathcal{W}$ (or equivalently, a finite component open cover $\mathcal{W}$) of X such that $x \notin \cup[\mathcal{W}]_{bX}$ (resp. $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$);*
3. *There exists a perfect compactification bX of X such that for every $x \in bX - X$ (resp. compact $B \subset bX - X$) there exists an open disjoint cover $\mathcal{W}$ (or equivalently, a finite component open cover $\mathcal{W}$) of X such that $x \notin \cup[\mathcal{W}]_{bX}$ (resp. $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$).*

Another result which is worth mentioning is that for $T_{3\frac{1}{2}}$-spaces, the following are equivalent: (a) X is weakly superparacompact; (b) X is weakly P-complete; (c) Every connected component of X is compact and every open neighbourhood of every connected component contains a clopen neighbourhood of the respective component. In section 3 we will show that this result can be strengthened for GO-spaces.

Two other results which we will need later are: (1) Every P-complete space is Dieudonné complete; (2) A strongly paracompact, weakly P-complete space is superparacompact.

2. Another Characterization for (Weakly) Superparacompact Spaces

Let $\mathcal{U}$ be an open cover of a space X. By $\mathcal{U}_f$ we denote the open cover of X consisting of finite unions of elements of $\mathcal{U}$.

Proposition 2.1. *A $T_{3\frac{1}{2}}$-space X is weakly superparacompact if and only if for every open cover $\mathcal{U}$ of X there exists a clopen cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}_f$.*

(Thus weak superparacompactness can be defined for any space X, without the assumption of the Tychonoff property.)

Proof. Let X be a weakly superparacompact space and let $\mathcal{U}$ be an open cover of X. Consider βX and enlarge $\mathcal{U}$ to an open (in βX) cover of X, say $\beta\mathcal{U} = \{\beta U : U \in \mathcal{U}\}$, where $\beta U \cap X = U$, for every $U \in \mathcal{U}$.

Let $B = \beta X - \cup\beta\mathcal{U}$, then B is compact and $B \subset \beta X - X$. Thus by definition, there exists a clopen cover $\mathcal{W}$ of X, such that $B \cap (\cup[\mathcal{W}]_{\beta X}) = \emptyset$. Consider an arbitrary $W \in \mathcal{W}$. Then $[W]_{\beta X}$ is compact and is a subset of $\cup\beta\mathcal{U}$. Hence there exists a finite subcover of $[W]_{\beta X}$, say $\beta U_1, \ldots, \beta U_{k(W)} \in \beta\mathcal{U}$. Consider $U(W) = \bigcup_{i=1}^{k(W)} U_i \in \mathcal{U}_f$. Then we have that $W \subset U(W)$, that is $\mathcal{W} < \mathcal{U}_f$.

Conversely, let X have the property that for every open cover $\mathcal{U}$ there exists a clopen cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}_f$. Let B be a compact subset of $\beta X - X$. Then B is closed in βX and so for every $x \in X$ there exists open (in βX) disjoint sets U_x and V_x such that $x \in U_x$ and $B \subset V_x$. Consider $\mathcal{U} = \{U_x \cap X : x \in X\}$. Then $\mathcal{U}$ is an open cover of X and so there exists a clopen (in X) cover $\mathcal{W}$ such that $\mathcal{W} < \mathcal{U}_f$. Let $W \in \mathcal{W}$, then $W \subset U_{x_1} \cup \cdots \cup U_{x_{k(W)}}$ for some $x_1, \ldots, x_{k(W)} \in X$. Then $[W]_{\beta X} \subset \left[\bigcup_{i=1}^{k(W)} U_{x_i}\right]_{\beta X} = \bigcup_{i=1}^{k(W)} [U_{x_i}]_{\beta X}$ and so $[W]_{\beta X} \cap B = \emptyset$, since $[U_{x_i}]_{\beta X} \cap B = \emptyset$ for every $i = 1, \ldots, k(W)$. Consequently we have that $B \cap (\cup[\mathcal{W}]_{\beta X}) = \emptyset$. □

Corollary 2.2. *A space X is compact if and only if it is CO-compact and weakly superparacompact, where a space is said to be CO-compact if every clopen cover has a finite subcover.*

Similarly one can prove the following.

Proposition 2.3. *A space X is superparacompact if and only if for every open cover $\mathcal{U}$ of X there exists an open disjoint cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}_f$.*

We end this section with a characterization of superparacompact spaces by uniformities. By a uniformity on a set X we understand a uniformity defined by covers of X and for a uniformity $\mathfrak{U}$, by $\tau_{\mathfrak{U}}$, we understand the topology on X generated by this uniformity.

Remember that a uniform space $(X, \mathfrak{U})$ is said to be *R-paracompact* if each open cover $\mathcal{U}$ of $(X, \tau_{\mathfrak{U}})$ admits a uniformly locally finite open refinement $\mathcal{V}$ (i.e. there exists a uniform cover, each of whose elements meets at most finitely many elements of $\mathcal{V}$) [10], [4]. This is equivalent to the fact that if $\mathcal{U}$ is an open cover of $(X, \tau_{\mathfrak{U}})$, then $\mathcal{U}_f$ is a uniform cover.

We now define a new class of uniformities which we will also need in section 5. Let $\mathfrak{U}$ be a uniformity on a set X. We denote by $\mathfrak{U}_f$ the collection $\{\mathcal{U}_f : \mathcal{U} \in \mathfrak{U}\}$. Also, let $\mathfrak{B}_{OD} = \{\mathcal{U} \in \mathfrak{U} : \mathcal{U}$ is an open disjoint cover of $X\}$. Then for every $\mathcal{U}, \mathcal{V} \in \mathfrak{B}_{OD}$ we have that $\mathcal{U} \wedge \mathcal{V} \in \mathfrak{B}_{OD}$ and $\mathcal{U}$ is a star refinement of $\mathcal{U}$ for every $\mathcal{U} \in \mathfrak{B}_{OD}$. Thus $\mathfrak{B}_{OD}$ is a base for a pseudo uniformity $\mathfrak{U}_{OD} \subset \mathfrak{U}$, where $\mathcal{W} \in \mathfrak{U}_{OD}$ if there exists $\mathcal{B} \in \mathfrak{B}_{OD}$ such that $\mathcal{B} < \mathcal{W}$.

Definition 5. A uniformity $\mathfrak{U}$ on a set X is said to be a *ODF uniformity* if $\mathfrak{U}_f \subset \mathfrak{U}_{OD}$.

Proposition 2.4. *A T_2-space X is superparacompact if and only if it admits a compatible R-paracompact ODF uniformity.*

PROOF. Let X be a superparacompact T_2-space. Consider the universal uniformity $\mathfrak{U}$ on X. By Proposition 2.3 it is not difficult to see that $\mathfrak{U}$ is an R-paracompact ODF uniformity.

Conversely, say X admits a compatible R-paracompact ODF uniformity $\mathfrak{U}$ and let $\mathcal{U}$ be an open cover of X. By definition of R-paracompactness, $\mathcal{U}_f \in \mathfrak{U}$. Since $\mathfrak{U}$ is a ODF uniformity, $(\mathcal{U}_f)_f = \mathcal{U}_f$ admits an open disjoint refinement and the proof again follows from Proposition 2.3. □

3. GO-Spaces

We now turn our attention to GO-spaces. For undefined terms related with GO-spaces one can consult [5] and [9]. For a GO-space X, by X^+ we mean the Dedekind compactification (see for example [9]). As Examples 13 and 14 show, the Dedekind compactification of a GO-space is not necessarily a perfect compactification. In fact we have the following result.

Proposition 3.1. *Let X be a GO-space. If the Dedekind compactification X^+ is a perfect compactification then X has no internal gaps.*

PROOF. Let $g = (A, B) \in X^+$ be an internal gap of X. Since B is clopen in X we have that $Fr_X B = \emptyset$. Now $g \in [B]_{X^+}$ and so $g \in [O\langle B\rangle]_{X^+}$. We now show that $g \notin Int_{X^+}O\langle B\rangle = O\langle B\rangle$. If $g \in O\langle B\rangle$ then there exists a convex open set U of X^+ such that $g \in U \subset O\langle B\rangle$. By the definition of the topology of X^+ we get that $U \cap A \neq \emptyset$ and so $O\langle B\rangle \cap X \neq B$, which is a contradiction. Consequently we have that $g \in Fr_{X^+}O\langle B\rangle$, that is $Fr_{X^+}O\langle B\rangle \neq \emptyset$. □

As Example 14 shows, the converse of Proposition 3.1 is not true.

It is known [8] that, if a compactification bX of a $T_{3\frac{1}{2}}$-space X has the property:

(a) for every $x \in bX - X$ (resp. compact $B \subset bX - X$) there exists a clopen cover $\mathcal{W}$ of X such that $x \notin \cup[\mathcal{W}]_{bX}$ (resp. $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$),

(b) for every $x \in bX - X$ (resp. compact $B \subset bX - X$) there exists an open disjoint cover $\mathcal{W}$ (or equivalently, an open finite component cover $\mathcal{W}$) of X such that $x \notin \cup[\mathcal{W}]_{bX}$ (resp. $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$),

then X is respectively,

(a) weakly P-complete (resp. weakly superparacompact),

(b) P-complete (resp. superparacompact).

(Note that the above compactification bX is not necessarily perfect.)

We now prove the converse for weakly superparacompact and superparacompact spaces.

Proposition 3.2. *Let X be a weakly superparacompact $T_{3\frac{1}{2}}$-space and bX a compactification of X. Then for every compact $B \subset bX - X$ there exists a clopen cover $\mathcal{W}$ of X such that $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$.*

PROOF. Let B be compact and $B \subset bX - X$. Let f be a continuous map from βX onto bX such that $f(x) = x$ for all $x \in X$ and $f(\beta X - X) \subseteq bX - X$. Then $f^{-1}B \subseteq \beta X - X$ and is compact. By definition there exists a clopen cover $\mathcal{W}$ of X such that $f^{-1}B \cap (\cup[\mathcal{W}]_{\beta X}) = \emptyset$. Let $W \in \mathcal{W}$, $f([W]_{\beta X})$ is a closed set of bX (and compact) and contains W. Hence, $[W]_{bX} \subseteq f([W]_{\beta X})$. Since $[W]_{\beta X} \cap f^{-1}B = \emptyset$ we have that $B \cap f([W]_{\beta X}) = \emptyset$. Consequently we have that $B \cap [W]_{bX} = \emptyset$ which implies that $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$. □

Similarly, we have the following result.

Proposition 3.3. *Let X be a superparacompact $T_{3\frac{1}{2}}$-space and bX a compactification of X. Then for every compact $B \subset bX - X$ there exists an open disjoint cover $\mathcal{W}$ (or equivalently, an open finite component cover $\mathcal{W}$) of X such that $B \cap (\cup[\mathcal{W}]_{bX}) = \emptyset$.*

Corollary 3.4. *A GO-space X is weakly superparacompact if and only if for every compact $B \subset X^+ - X$ there exists a clopen cover $\mathcal{W}$ of X such that $B \cap (\cup[\mathcal{W}]_{X^+}) = \emptyset$.*

Corollary 3.5. *A GO-space X is superparacompact if and only if for every compact $B \subset X^+ - X$ there exists an open disjoint cover $\mathcal{W}$ (or equivalently, an open finite component cover $\mathcal{W}$) of X such that $B \cap (\cup[\mathcal{W}]_{X^+}) = \emptyset$.*

From what was said at the end of section 1 and from the fact that a GO-space X is paracompact if and only if it is strongly paracompact if and only if it is Dieudonné complete, we get that for GO-spaces, P-completeness is equivalent to superparacompactness. Also, in this case, as was already noted, weak P-completeness is equivalent to weak superparacompactness. Finally we note that in this case the class of superparacompact GO-spaces is precisely the intersection of the class of paracompact GO-spaces and the class of weakly superparacompact GO-spaces. As examples will show in section 6, none of the last mentioned two classes imply the other.

We now turn to a characterization of weak superparacompactness in GO-spaces in terms of gaps and pseudogaps.

Let $(X, \tau, \leqslant)$ be a GO-space. Consider the sets $R = \{x \in X : [x, \rightarrow [\in \tau\}$, $L = \{x \in X :] \leftarrow, x] \in \tau\}$ and $G = \{g \in X^+ : g \text{ is a gap of } X\}$. Denote by $W = R \cup L \cup G$. Now let $g = (A, B)$ be an arbitrary gap of X. Consider the sets $A^+ =]\leftarrow, g[\subset X^+$ and $B^+ =]g, \rightarrow [\subset X^+$.

Definition 6. A gap (A, B) is said to be a *W-gap* if there exists a cofinal set $A' = \{a_\alpha : \alpha \in \mathcal{A}\} \subset A^+$ and a coinitial set $B' = \{b_\beta : \beta \in \mathcal{B}\} \subset B^+$ such that $A' \cup B' \subset W$.

Similarly, let $g = (A, B) = (] \leftarrow, g],]g, \rightarrow [)$ be a pseudogap. Then there is a point $g^+ \in X^+ - X$ such that $g < g^+ < b$, for every $b \in B$, and $]g, g^+[= \emptyset$. Consider the set $B^+ =]g^+, \rightarrow [\subset X^+$.

Definition 7. The pseudogap (A, B) is said to be a *W-pseudogap* if there exists a coinitial set $B' = \{b_\beta : \beta \in \mathcal{B}\} \subset B^+$ such that $B' \subset W$.

Similarly for psudogaps of the form $g = (A, B) = (] \leftarrow, g[, [g, \rightarrow [)$, where in this case there is a point $g^- \in X^+ - X$ such that $a < g^- < g$, for every $a \in A$, and $]g^-, g[= \emptyset$.

Proposition 3.6. *A GO-space X is weakly superparacompact if and only if every gap is a W-gap and every pseudogap is a W-pseudogap.*

PROOF. Let $g = (A, B)$ be a gap of a weakly superparacompact GO-space X. Then $g \in X^+ - X$ and by definition, there exists a clopen cover $\mathcal{W}$ of X such that $g \notin \cup[\mathcal{W}]_{X^+}$. One can assume that $\mathcal{W}$ is a cover consisting of convex sets. Then each $V \in \mathcal{W}$ lies either in A or in B. Let $\mathcal{W}_A = \{V \in \mathcal{W} : V \subset A\}$. For every $V \in \mathcal{W}_A$, consider $[V]_{X^+}$. Since this is a compact LOTS, it has a maximal element $x_V < g$ in X^+.

If $x_V \in X^+ - X$ then $x_V \in G$ or $x_V = y_V^-$ for some $y_V \in R$, and if $x_V \in X$ then $x_V \in L$. Now let $z_V = x_V$ if $x_V \in G \cup L$ and $z_V = y_V$ if $x_V = y_V^-$ for $y_V \in R$. It is not difficult to see that $A' = \{z_V : V \in \mathcal{W}_A\}$ is cofinal in A^+. Similarly one can find a coinitial set B' of B^+ such that $B' \subset W$.

In the same way one can prove that every pseudogap is a W-pseudogap.

Conversely, let X be a GO-space and $x \in X^+ - X$. Then x is either a gap or a pseudogap. Say x is a gap, $x = (A, B)$. By definition, there exists a cofinal set $A' = \{a_\alpha : \alpha \in \mathcal{A}\} \subset A^+$ with $A' \subset W$. There can be three cases: (a) a_α is a gap, that is $a_\alpha \in G$, then $a_\alpha = (A_\alpha, B_\alpha)$, $A_\alpha \subset A$ and $x \notin [A_\alpha]_{X^+}$; (b) $a_\alpha \in L$, then let $A_\alpha =]\leftarrow, a_\alpha]$, and we have that $x \notin [A_\alpha]_{X^+}$; (c) $a_\alpha \in R$, then let $A_\alpha =]\leftarrow, a_\alpha[$, and again we have that $x \notin [A_\alpha]_{X^+}$. Thus $\{A_\alpha : \alpha \in \mathcal{A}\}$ is a clopen (in X) cover of A and $x \notin \cup[A_\alpha]_{X^+}$. In the same way one can construct a clopen cover $\{B_\beta : \beta \in \mathcal{B}\}$ of B such that $x \notin \cup[B_\beta]_{X^+}$.

A similar argument applies for the case that the point x is a pseudogap, since if, say, $x = (]\leftarrow, a],]a, \rightarrow[)$, then $x \notin [\,]\leftarrow, a]]_{X^+}$. □

Corollary 3.7. *A GO-space X is superparacompact if and only if every gap is a QW-gap and every pseudogap is a QW-pseudogap, where by a QW-(pseudo)gap we mean a (pseudo)gap which is a Q- and W- (pseudo)gap.*

PROOF. This follows from the following two facts: (i) for GO-spaces, paracompactness plus weak superparacompactness imply superparacompactness, and (ii) a GO-space is paracompact if and only if every gap is a Q-gap and every pseudogap is a Q-pseudogap. □

As mentioned in the last part of section 1 we have the following result for GO-spaces.

Proposition 3.8. *A GO-space X is weakly superparacompact if and only if the connected components of X are compact.*

PROOF. We need only to prove the sufficiency. Say the connected components $C_\alpha, \alpha \in \mathcal{A}$, are all compact and X is not weakly superparacompact. Then there is at least one gap or pseudogap which is not a W-gap or W-pseudogap respectively.

Say $g = (A, B)$ is a gap which is not a W-gap. Without loss of generality one can assume that there is no cofinal (in A^+) subset of W. Then there exists some $a \in A$ such that $[a, \rightarrow [\cap A$ has no gaps and no elements in $R \cup L$, and so is connected. Thus the connected component containing $[a, \rightarrow [\cap A$ is not compact, which is a contradiction. A similar argument holds for the case that (A, B) is a pseudogap which is not a W-pseudogap. □

4. Some Results On Lindelöf Spaces

Definition 8. We call a space X CO-countable, if every disjoint collection of clopen sets covering X is not more than countable.

Examples of CO-countable spaces are CO-finite spaces (in particular, pseudocompact spaces and so countably compact and compact spaces, and connected spaces [8]). Also, Lindelöf spaces are CO-countable, where a space X is said to be *Lindelöf* if every open cover of X has a countable subcover.

Definition 9. A space X is called *CO-Lindelöf*, if every clopen cover of X has a countable subcover.

Obviously, every CO-Lindelöf space is CO-countable. The converse is not true as Example 15 shows. Remember that a space X is called *countably strongly paracompact* if every open cover of X has an open star countable refinement.

Proposition 4.1. *Let X be a countably strongly paracompact space. Then the following are equivalent:*

1. *X is Lindelöf;*
2. *X is CO-Lindelöf;*
3. *X is CO-countable.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. We now prove that (3) $\Rightarrow$ (1) for countably strongly paracompact spaces.

Let $\mathcal{U}$ be an open cover of a countably strongly paracompact space X. Then there exists a star countable open refinement $\mathcal{V}$ of $\mathcal{U}$. The components $\mathcal{V}_\alpha, \alpha \in \mathcal{A}$, of $\mathcal{V}$ are countable and disjoint, that is $\mathcal{V} = \bigcup_\alpha \mathcal{V}_\alpha$, where each $\mathcal{V}_\alpha$ has a countable number of elements and each $U(\mathcal{V}_\alpha) = \cup\{V : V \in \mathcal{V}_\alpha\}$ is clopen in X. Since X is CO-countable, there are countably many $U(\mathcal{V}_\alpha)$'s, that is $X = \bigcup_{i=1}^\infty U(\mathcal{V}_i)$, and each $U(\mathcal{V}_i)$ is covered by countably many elements of $\mathcal{U}$. Hence, X is covered by countably many elements of $\mathcal{U}$ and so is Lindelöf. □

Corollary 4.2. *Let X be a paracompact GO-space, then the following are equivalent:*

1. *X is Lindelöf;*
2. *X is CO-Lindelöf;*
3. *X is CO-countable.*

From the results of section 2 one can also add (see Proposition 2.3);

Proposition 4.3. *A space X is Lindelöf and superparacompact if and only if for every open cover $\mathcal{U}$ there exists an open disjoint countable cover $\mathcal{V} < \mathcal{U}_f$.*

We now give a characterization of Lindelöf spaces similar to that of (weakly) superparacompact spaces, the proof of which runs on the same lines as that of Proposition 2.1 and so we omit it.

Proposition 4.4. *For a $T_{3\frac{1}{2}}$-space X the following are equivalent:*

1. *The space X is Lindelöf;*
2. *For every compactification bX of X and every compact $B \subset bX - X$ there exists a countable open cover $\mathcal{U}$ of X such that $B \cap (\cup[\mathcal{U}]_{bX}) = \emptyset$;*
3. *For every compact $B \subset \beta X - X$ there exists a countable open cover $\mathcal{U}$ of X such that $B \cap (\cup[\mathcal{U}]_{\beta X}) = \emptyset$;*
4. *There exists a compactification bX of X such that for every compact $B \subset bX - X$ there exists a countable open cover $\mathcal{U}$ of X such that $B \cap (\cup[\mathcal{U}]_{bX}) = \emptyset$.*

We end this section with a result concerning GO-spaces.

Remember that if X is a GO-space and U is a subset of X, then a (pseudo)gap (A, B) is said to be *covered* by U if there is a convex set V such that $V \subset U$, $V \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. A cover $\mathcal{U}$ of X is said to *cover* the (pseudo)gap (A, B) if $\mathcal{U}$ has an element which covers (A, B).

The following lemma is known [9]:

Lemma 4.5. *An open cover $\mathcal{U}$ of a GO-space X has a finite subcover if every gap and pseudogap of X is covered by $\mathcal{U}$.*

Definition 10. A (pseudo)gap (A, B) of a GO-space X is said to be a *C-(pseudo)gap* if A has a countable cofinal subset and B has a countable coinitial subset.

Note that every C-(pseudo)gap is a Q-(pseudo)gap.

Now let $(X, \tau, \leqslant)$ be a GO-space and $\mathcal{U}$ an open cover of X. Denote by $F_{\mathcal{U}}$, the set of all gaps and pseudogaps of X which are not covered by $\mathcal{U}$. It can be easily seen that $F_{\mathcal{U}}$ is closed in X^+ and so is compact.

Lemma 4.6. *Let $(X, \tau, \leqslant)$ be such that every gap is a C-gap and every pseudogap is a C-pseudogap. Let $\mathcal{U}$ be an open cover of X. If $X^+ - F_{\mathcal{U}}$ is decomposed into a countable number of convex components, then $\mathcal{U}$ has a countable subcover.*

PROOF. Let $G_i, i < \omega$, be the convex components of $X^+ - F_{\mathcal{U}}$. Since $F_{\mathcal{U}}$ is a closed set, the convex components G_i are open in X^+. Let $H_i = G_i \cap X$, then $\{H_i : i < \omega\}$ is a disjoint open cover of X by convex sets. Regard H_i as a GO-space covered by the open cover $\mathcal{U}$. Then $\mathcal{U}$ covers every gap and pseudogap of H_i except possibly its endgaps. Select an arbitrary point a of H_i. If H_i has a maximal point, then by Lemma 4.5, $H_i' = \{x \in H_i : x \geqslant a\}$ is covered by finitely many elements of $\mathcal{U}$. If $(H_i, \emptyset)$ is an endgap of H_i, then it determines a C-gap or C-pseudogap. In either case there is a countable cofinal set $a_1, a_2, \ldots$ in H_i, which one can take to be monotonically increasing and $a_1 \geqslant a$. By Lemma 4.5 we get that $[a, a_j]$ is covered by finitely many elements of $\mathcal{U}$ for every $j < \omega$ and so H_i' is covered by countably many elements of $\mathcal{U}$. We apply the same argument to the left half of H_i to conclude that H_i is covered by countably many elements of the open cover $\mathcal{U}$. □

Proposition 4.7. *The GO-space $(X, \tau, \leqslant)$ is Lindelöf if and only if*

1. *Every gap is a C-gap and every pseudogap is a C-pseudogap;*
2. *For every compact set $F \subset X^+ - X$, $X^+ - F$ is decomposed into a countable number of convex components.*

PROOF. If (1) and (2) hold then X is Lindelöf by Lemma 4.6. Conversely, if X is Lindelöf, then for every compact set $F \subset X^+ - X$, the convex components of $X^+ - F$ gives rise to an open disjoint cover of X and so they are not more than countable. The fact that every (pseudo)gap is a C-(pseudo)gap is not difficult to see. □ □

Remark. One might ask about which GO-spaces have property (1) of Proposition 4.7. It can be proved that a GO-space X has property (1) if and only if for every $x \in X^+ - X$ there exists a countable open cover $\mathcal{U}$ of X such that $x \notin \cup[\mathcal{U}]_{X^+}$. This characterization is a characterization of realcompact spaces if one changes X^+ to βX. Thus every GO-space satisfying property (1) is realcompact. Unfortunately, property (1) is not a topological property as Example 19 shows.

Remark. With respect to property (2) of Proposition 4.7 we have that if a compact set $F \subset X^+ - X$ is countable then $X^+ - F$ is decomposed into a countable number of convex components but as Example 20 shows, the converse is not true.

5. Lindelöf And Superparacompact GO-d-Extensions

It is well known that a topological space (X, τ) is a GO-space together with some ordering $\leqslant_X$ on X if and only if (X, τ) is a topological subspace of some LOTS $(Y, \lambda(\leqslant_Y), \leqslant_Y)$ with $\leqslant_X = \leqslant_Y|_X$, where the symbol $\leqslant_Y|_X$ is the restriction of the order $\leqslant_Y$ to X, so any GO-space has a linearly ordered extension. Note that a LOTS $(Y, \lambda(\leqslant_Y), \leqslant_Y)$ is called a *linearly ordered extension* of a GO-space $(X, \tau, \leqslant_X)$ if $X \subset Y, \tau = \lambda(\leqslant_Y)|_X$ and $\leqslant_X = \leqslant_Y|_X$ [6]. Any GO-space X has a linearly ordered extension Y such that X is dense in Y (such an extension is called a linearly ordered d-extension). A *GO-extension* of the GO-space $(X, \tau_X, \leqslant_X)$ is a GO-space $(Y, \tau_Y, \leqslant_Y)$ such that $X \subset Y, \tau_X = \tau_Y|_X$ and $\leqslant_X = \leqslant_Y|_X$. If X is dense in Y then the GO-extension is called a *GO-d-extension* [3]. The extensions that we will consider are all GO-d-extensions, so by an extension we always mean a GO-d-extension.

Let X be a set, $\mathfrak{U}$ a uniformity on X, τ a topology on X and $\leqslant$ a linear order on X. If a cover $\mathcal{U}$ of X consists of convex (w.r.t. $\leqslant$) sets, then it is called a *convex cover.*

Definition 11. The triple $(X, \mathfrak{U}, \leqslant)$ is called a *GO-uniform space* if the uniformity $\mathfrak{U}$ has a base $\mathfrak{B}$, consisting of convex covers. In this case $\mathfrak{U}$ is called a *GO-uniformity* on $(X, \leqslant)$ [3].

Every GO-uniformity induces a GO-topology on $(X, \leqslant)$. This follows from the fact that if $\mathfrak{U}$ is a GO-uniformity on $(X, \leqslant)$, then $\tau_{\mathfrak{U}}$ (the topology on X generated by $\mathfrak{U}$) is T_1 and has a base consisting of convex sets [3]. We say that a GO-uniformity $\mathfrak{U}$ is a *GO-uniformity of a GO-space* $(X, \tau, \leqslant)$ if $\tau_{\mathfrak{U}} = \tau$. The universal uniformity of a GO-space $(X, \tau, \leqslant)$ is always a GO-uniformity.

Let $U(X, \tau, \leqslant)$ be the set of all GO-uniformities of a GO-space $(X, \tau, \leqslant)$. It is partially ordered by inclusion. If $\mathfrak{U} \in U(X, \tau, \leqslant)$, then by $\Phi(\mathfrak{U})$ we denote the set of all minimal Cauchy filters of the uniform space $(X, \mathfrak{U})$. For the set $U(X, \tau, \leqslant)$ an equivalence relation is defined in the following manner: $\mathfrak{U}_1 \sim \mathfrak{U}_2$ if and only if $\Phi(\mathfrak{U}_1) = \Phi(\mathfrak{U}_2)$. By $E(\mathfrak{U})$ we denote the equivalence class containing the uniformity $\mathfrak{U}$ and let $\mathfrak{U}_E = \sup\{\mathfrak{U}' : \mathfrak{U}' \in E(\mathfrak{U})\}$.

Let $(X, \mathfrak{U}, \leqslant)$ be a GO-uniform space. The GO-uniformity $\mathfrak{U}_E$ is called *E-leader* of the GO-uniformity $\mathfrak{U}$. The GO-uniformity $\mathfrak{U}$ is called a *preuniversal* GO-uniformity if the equality $\mathfrak{U} = \mathfrak{U}_E$ holds [2].

In [3] it was proved that if $(X, \mathfrak{U}, \leqslant)$ is a GO-uniform space and $(\widetilde{X}, \widetilde{\mathfrak{U}})$ is the completion of the uniform space $(X, \mathfrak{U})$, then there exists a linear order $\widetilde{\leqslant}$ on $\widetilde{X}$ such that $(\widetilde{X}, \tau_{\widetilde{\mathfrak{U}}}, \widetilde{\leqslant})$ is a GO-d-extension of the GO-space $(X, \tau_{\mathfrak{U}}, \leqslant)$.

Also, it was proved that for a GO-space $(X, \tau, \leqslant)$ there is a 1–1 correspondence between GO-paracompactifications (that is, paracompact GO-d-extensions) and GO-uniformity classes (and so preuniversal GO-uniformities). We now prove similar theorems concerning Lindelöf and superparacompact GO-d-extensions.

Definition 12. A GO-uniformity $\mathfrak{U}$ on $(X, \leqslant)$ is said to be *prelindelöf*, if it is preuniversal and has a base consisting of convex countable covers.

Let $(X, \tau, \leqslant)$ be a GO-space. If $\mathfrak{U}$ is a prelindelöf uniformity compatible with τ , then the completion $\widetilde{\mathfrak{U}}$ is the universal uniformity on $(\widetilde{X}, \tau_{\widetilde{\mathfrak{U}}}, \widetilde{\leqslant})$ [2], having a base consisting of convex (with respect to $\widetilde{\leqslant}$) countable covers. Since $(\widetilde{X}, \tau_{\widetilde{\mathfrak{U}}})$ is paracompact and so every open cover is a uniform cover with respect to $\widetilde{\mathfrak{U}}$, we get that $(\widetilde{X}, \tau_{\widetilde{\mathfrak{U}}})$ is Lindelöf.

Now let $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$ be a Lindelöf GO-d-extension of $(X, \tau, \leqslant)$. Being Lindelöf, it is paracompact. Let $\widetilde{\mathfrak{U}}$ be the universal uniformity on $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$, that is the uniformity that consists of the set of all open (convex) covers. It has a base of countable convex covers and it is clear that the uniformity $\mathfrak{U}$ induced on X by $\widetilde{\mathfrak{U}}$ is a preuniversal uniformity having a base of convex countable covers, that is, it is prelindelöf.

We have thus proved the following theorem.

Theorem 5.1. *In a GO-space $(X, \tau, \leqslant)$, there is a 1–1 correspondence between Lindelöf GO-d-extensions and prelindelöf GO-uniformities.*

We now turn to superparacompact GO-d-extensions.

Let $(X, \tau, \leqslant)$ be a GO-space. If $\mathfrak{U}$ is a preuniversal ODF GO-uniformity compatible with τ, then the completion $\widetilde{\mathfrak{U}}$ is the universal uniformity on $(\widetilde{X}, \tau_{\widetilde{\mathfrak{U}}}, \widetilde{\leqslant})$. From the construction of the completion (see for example [2] or [9]), it is not difficult to see that $\widetilde{\mathfrak{U}}$ is also an ODF uniformity. Hence by Proposition 2.4, $(\widetilde{X}, \tau_{\widetilde{\mathfrak{U}}})$ is a superparacompact space.

Now let $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$ be a superparacompact GO-d-extension of $(X, \tau, \leqslant)$. Being superparacompact, it is paracompact. Let $\widetilde{\mathfrak{U}}$ be the universal uniformity on $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$, that is the uniformity that consists of the set of all open (convex) covers. As shown in Proposition 2.4, this uniformity is an ODF uniformity and thus the uniformity $\mathfrak{U}$ induced on X by $\widetilde{\mathfrak{U}}$ is a preuniversal ODF uniformity.

We have thus proved the following theorem.

Theorem 5.2. *In a GO-space $(X, \tau, \leqslant)$, there is a 1–1 correspondence between superparacompact GO-d-extensions and preuniversal ODF GO-uniformities.*

6. Examples

Example 13. Let $X = \mathbb{Q}$, the set of rational numbers with standard order and topology (that is, as a subset of $\mathbb{R}$ with standard order and topology). Then $X^+ = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. Let $U =]\leftarrow, p[\cap \mathbb{Q}$, where p is an irrational number. One can easily see that $O\langle U \rangle =]\leftarrow, p[$ and so we get that, $Fr_X U = \emptyset$, $[Fr_X U]_{X^+} = \emptyset$ and $Fr_{X^+} O\langle U \rangle = \{p\}$. Hence X^+ is not a perfect compactification.

Example 14. Consider the subspace $]0,1] \subseteq \mathbb{R}$ and let $X \subset]0,1]$ be the subspace $\cup \{ [\frac{1}{2^{i+1}}, \frac{1}{2^i}] : i = 0,2,4,\dots \}$. Then X is a LOTS. Now take the subset U of the space X to be $\cup \{ [\frac{1}{2^{i+1}}, \frac{1}{2^i}] : i = 0,4,8,\dots \}$. Then U is clopen in X and so $Fr_X U = \emptyset$, which in turn implies that $[Fr_X U]_{X^+} = \emptyset$. But since $O\langle U \rangle = U$ and $0 \in [O\langle U \rangle]_{X^+}$, we have that $0 \in Fr_{X^+} O\langle U \rangle$. Thus X^+ is not a perfect compactification and X has no internal gaps (cf. Proposition 3.1).

Example 15. Let $X = T(\omega_1) = [0, \omega_1[$. Then X is CO-finite (and so also CO-countable), but the cover $\mathcal{U} = \{O_\alpha = [0, \alpha] : \alpha < \omega_1\}$ of X is clopen in X and does not have a countable subcover. Thus, X is not CO-Lindelöf (and not Lindelöf). Note that this space is not paracompact (and so not superparacompact), while it is weakly superparacompact.

Example 16. Let S be the Sorgenfrey line. The GO-space S is weakly superparacompact and paracompact, and so is superparacompact. The space S is also Lindelöf, but not compact.

Example 17. Let M be the Michael line in which the subspace of all irrational numbers is discrete. The GO-space M is weakly superparacompact and paracompact, and so is superparacompact, but M is not Lindelöf.

Example 18. Let $\mathbb{R}$ be the real line with standard order and topology. The space $\mathbb{R}$ is Lindelöf (and so is paracompact), but is not weakly superparacompact, and so is not superparacompact.

Example 19. Let $X = [0, \omega_1[$ with standard order and discrete topology. Then we have that X is homeomorphic to $D(\aleph_1)$ (i.e., the discrete space of cardinality $\aleph_1$) and as a GO-space does not have property (1) of Proposition 4.7. On the other hand, let $X' = \mathbb{R}$ with standard order and discrete topology. Then, if one assumes CH to hold, we also have that X' is homeomorphic to $D(\aleph_1)$ (and so is homeomorphic to X) and as a GO-space has property (1) of Proposition 4.7.

Example 20. Let $\Re$ be the set of rational numbers with the standard order and discrete topology. It is not difficult to see that the uncountable set $\Re^+ - \Re$ is closed in $\Re^+$, while $\Re \subset \Re^+$ is decomposed into countably many convex (with respect to the order in $\Re^+$) components.

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