

# Truss-type systems exhibiting negative compressibility

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We propose novel two- and three-dimensional truss structures made from rods of different materials connected together through pin-joints to form triangular units which can exhibit anomalous compressibility properties. In particular, we show that these systems may be made to exhibit negative linear compressibility along certain directions or compressibilities that are even more positive than any of the component

materials, i.e. the end product is a system with tunable compressibility properties that can be tailor made for particular practical applications. We also show that in specific cases, these systems can exhibit an overall negative area compressibility and sometimes even negative volumetric compressibility (i.e. negative bulk modulus) thus confirming that this property can indeed exist.

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**1 Introduction** The isothermal volume compressibility  $\beta_V$ , and its inverse, the bulk modulus  $K$  are two properties which describe how the volume  $V$  of materials change when subjected to changes in the external hydrostatic pressure  $P$  and are defined by [1–4]:

$$\beta_V = \frac{1}{K} = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T.$$

In general, most common materials contract in all directions when the external pressure is increased so that compressibilities are normally positive. However, it has recently been shown that negative compressibilities are possible and in fact, in their recent work, Lakes and Wojciechowski [3] argue that the existence of negative compressibility does not violate classical thermodynamic theory which in fact only strictly forbids this property for unconstrained systems. In fact, Lakes and Wojciechowski also report that they have observed and measured negative compressibility in pre-strained foams. In this respect one should also mention the work by Grima et al. [4] and Gatt et al. [5] who have proposed examples of systems which show such properties and also the work by Baughman et al. [6] who had proposed a networked polymeric carbon based polymer which is predicted to exhibit negative linear compressibility (defined below). Also, it should be noted that it

is well known that negative linear compressibility is possible in the presence of significant anisotropy as such systems could still behave in such a way that the overall volumetric compressibility is positive.

A very basic, yet simple unit that has a potential to exhibit negative compressibility is a triangle constructed from pin-jointed rods made from conventional materials, where the rod forming the base of the triangle is made up of a material that has a different compressibility than that of the other two rods making up the sides of the triangle. More specifically, if the base rod expands to a higher extent than the sides when the structure is subjected to a decrease in pressure (i.e. the base rod material has a lower compressibility than the material used for the sides of the triangles), the height of the triangle shortens (the ‘triangular shortening’ effect) as a result of the constraint that the sum of internal angles of the triangle must remain equal to  $180^\circ$ . All this makes such triangular truss systems exhibit linear negative compressibility in the direction orthogonal to the base of the triangle. (Note that this ‘triangular shortening’ concept has already been used before for generating negative thermal expansion [7–9], although it is well known that negative thermal expansion may occur through many other mechanisms and does not require anisotropy [10].)

In this work, we analyse and discuss 2D truss structures based on this idea, which are constructed from rods made of different materials which behave differently when subjected to pressure changes as a result of differences in their mechanical properties. We derive expressions for  $\beta_L$ , a property which quantifies the change in a length  $L$  of the sample as a result of a change in the pressure  $P$  and is the 1D equivalent of the compressibility (the inverse of the bulk modulus) defined by<sup>1</sup> [6]:

$$\beta_L = -\frac{1}{L} \left( \frac{\partial L}{\partial P} \right)_T, \quad (1)$$

which at constant temperature may be re-written in terms of  $\varepsilon_L$ , the strain in the direction of  $L$ , by:

$$\beta_L = -\frac{\varepsilon_L}{dP}. \quad (2)$$

In particular, for 2D systems which when subjected to a change in pressure  $dP$  deforms by a strain defined by the 2D strain tensor  $\varepsilon_{ij}$  ( $i, j = 1, 2$ ), we derive expressions for the linear compressibility at any direction  $\zeta$  to the  $Ox_i$  direction which is given by:

$$\beta_L(\zeta) = -\left[ \frac{\varepsilon_{11}}{dP} \cos^2(\zeta) + 2 \frac{\varepsilon_{12}}{dP} \sin(\zeta) \cos(\zeta) + \frac{\varepsilon_{22}}{dP} \sin^2(\zeta) \right], \quad (3)$$

a property which is at a maximum/minimum at mutually orthogonal directions which are oriented at an angle of  $\zeta_{\max/\min}$  to the  $Ox_i$  axis where  $\zeta_{\max/\min}$  is given by:

$$\zeta_{\max/\min} = \frac{1}{2} \tan^{-1} \left( \frac{2\varepsilon_{12}}{\varepsilon_{11} - \varepsilon_{22}} \right), \quad (4)$$

<sup>1</sup> Note that for isotropic materials,  $\beta_L = 1/3\beta_V$ .

at which points the magnitude of the linear compressibility is given by:

$$\beta(\zeta)_{\max/\min} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2 dP} \pm \sqrt{\left( \frac{\varepsilon_{11} - \varepsilon_{22}}{2 dP} \right)^2 + \left( \frac{\varepsilon_{12}}{dP} \right)^2}. \quad (5)$$

We also derive expressions for  $\beta_A$ , a property which quantifies the change in an area  $A$  of the sample as a result of a change in the pressure and is the 2D equivalent of the compressibility defined by<sup>2</sup> [6]:

$$\beta_A = -\frac{1}{A} \left( \frac{\partial A}{\partial P} \right)_T, \quad (6)$$

where for 2D systems, since  $\frac{dA}{A} = \varepsilon_{11} + \varepsilon_{22}$ , the expression for  $\beta_A$  may be re-written as:

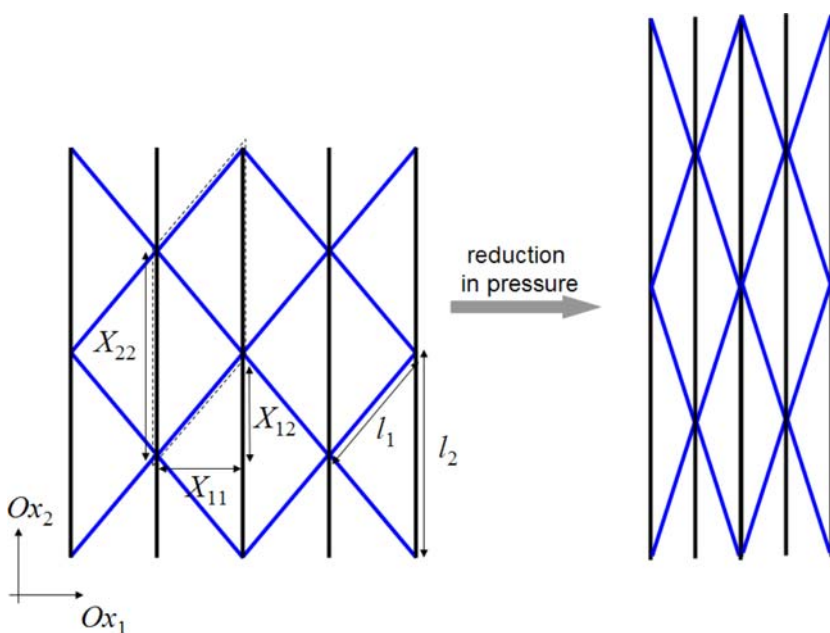
$$\beta_A = -\left[ \frac{d\varepsilon_{11}}{dP} + \frac{d\varepsilon_{22}}{dP} \right]. \quad (7)$$

Finally, we also discuss some simple 3D systems based on this idea which can exhibit negative volumetric compressibility (i.e. negative bulk modulus).

## 2 Analytical model for a 2D simplified system made from two materials

Let us first consider a simplified system, more specifically a 2D periodic truss system constructed using rods made of two different types of isotropic materials, which we shall denote by ‘1’ and ‘2’ respectively connected together as illustrated in Fig. 1, i.e. in the same way as that proposed earlier which exhibits negative thermal expansion [9]. This construct may be

<sup>2</sup> Note that for isotropic materials,  $\beta_A = 2\beta_L = 2/3\beta_V$ .



**Figure 1** (online colour at: [www.pss-b.com](http://www.pss-b.com)) A simple 2D system where the basic unit is an isosceles triangle built from two different materials. Note that if the response of material 2 to an externally applied reduction in the hydrostatic pressure is sufficiently larger than that of material 1, the system can exhibit negative compressibility.

considered as a porous solid system placed in a fluid (which may be air). Note that as illustrated in Fig. 1, if the vertical rods '2' expand much more than those of material '1' when the external pressure decreases, then the width of the system decreases (i.e. negative linear compressibility in the  $Ox_1$  direction), an effect which can even lead to an overall decrease in or overall shrinkage of the structure (the total area of the structure decreases, i.e. negative area compressibility).

To model this effect, we describe the shape and size of the system by a parallelogramic unit cell containing two triangles, as illustrated in Fig. 1 and if it is oriented such that  $l_2$  is aligned parallel to the  $Ox_2$  direction, the unit cell vectors are given by  $\mathbf{a} = (X_{11}, X_{12})$  and  $\mathbf{b} = (0, X_{22})$  where:

$$X_{11} = \frac{\sqrt{4l_1^2 - l_2^2}}{2}, \quad (8)$$

$$X_{22} = l_2, \quad (9)$$

$$X_{12} = \frac{l_2}{2}. \quad (10)$$

If the materials  $m = 1, 2$  have different Young's moduli  $E_{Sm}$ , Poisson's ratio  $\nu_{Sm}$  i.e. bulk modulus  $K_m$ , then when the structure is subjected to a change in the external pressure  $dP$ , the lengths  $l_m$  of each type of element varies by a different amount  $dl_m$  given by:

$$dl_m = -l_m s_m dP, \quad (11)$$

where for three dimensional isotropic rods (i.e. rods which can experience a pressure from all three dimensions), the term  $s_m$  which is the linear compressibility of the material  $m$  is given by:

$$s_m = \frac{1}{3K_m} = \frac{1 - 2\nu_{Sm}}{E_{Sm}}. \quad (12)$$

These changes in  $l_m$  due to a variation in the external pressure will result in changes in size and shape of the macrostructure which may be quantified through the  $2 \times 2$  strain tensor  $\varepsilon_{ij}$  ( $i, j = 1, 2$ ) where  $\varepsilon_{11}$  and  $\varepsilon_{22}$  are the axial strains in the  $Ox_1$  and  $Ox_2$  directions respectively, and  $\varepsilon_{12}$  and  $\varepsilon_{21}$  are equal to half the shear strain  $\gamma$ . These strains are given by:

$$\varepsilon_{11} = \frac{dX_{11}}{X_{11}}, \quad \varepsilon_{22} = \frac{dX_{22}}{X_{22}},$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{\gamma}{2} = \frac{1}{2X_{11}} \left[ dX_{12} - \left( \frac{X_{12}}{X_{22}} \right) dX_{22} \right]. \quad (13)$$

Thus, in analogy to the thermal model [9], since:

$$dX_{ij} = \sum_{m=1}^2 \frac{\partial X_{ij}}{\partial l_m} dl_m = - \sum_{m=1}^2 \frac{\partial X_{ij}}{\partial l_m} l_m s_m dP \quad (i, j = 1, 2) \quad (14)$$

from Eqs. (8)–(14), these elements of the strain tensor are given by:

$$\varepsilon_{11} = \frac{dX_{11}}{X_{11}} = - \frac{1}{X_{11}} \left( \frac{\partial X_{11}}{\partial l_1} l_1 s_1 + \frac{\partial X_{11}}{\partial l_2} l_2 s_2 \right) dP$$

$$= - \frac{4l_1^2 s_1 - l_2^2 s_2}{4X_{11}^2} dP = - \frac{4l_1^2 s_1 - l_2^2 s_2}{4l_1^2 - l_2^2} dP, \quad (15)$$

$$\varepsilon_{22} = \frac{dX_{22}}{X_{22}} = - \frac{1}{X_{22}} \left( \frac{\partial X_{22}}{\partial l_2} l_2 s_2 \right) dP = -s_2 dP \quad (16)$$

$$\varepsilon_{12} = \varepsilon_{21} = - \frac{1}{2X_{11}} \left[ \frac{\partial X_{12}}{\partial l_2} l_2 s_2 - \left( \frac{X_{12}}{X_{22}} \right) \frac{\partial X_{22}}{\partial l_2} l_2 s_2 \right] dP$$

$$= - \frac{1}{2X_{11}} [0] dP = 0, \quad (17)$$

and using Eq. (5), the linear compressibility at an angle  $\zeta$  to the  $Ox_1$  direction is given by:

$$\beta_L(\zeta) = \frac{4l_1^2 s_1 - l_2^2 s_2}{4l_1^2 - l_2^2} \cos^2(\zeta) + l_2 s_2 \sin^2(\zeta), \quad (18)$$

a function which is at a maximum/minimum on-axis ( $\zeta = 90^\circ$ ).

Furthermore, referring to Eq. (7), the area compressibility  $\beta_A$  for this system is given by [3]:

$$\beta_A = \frac{4l_1^2 (s_1 + s_2) - 2l_2^2 s_2}{4l_1^2 - l_2^2}. \quad (19)$$

These two equations suggest that in general, the values of the linear and area compressibilities for this simplified system can be both positive or negative with the actual sign and magnitude being dependent on:

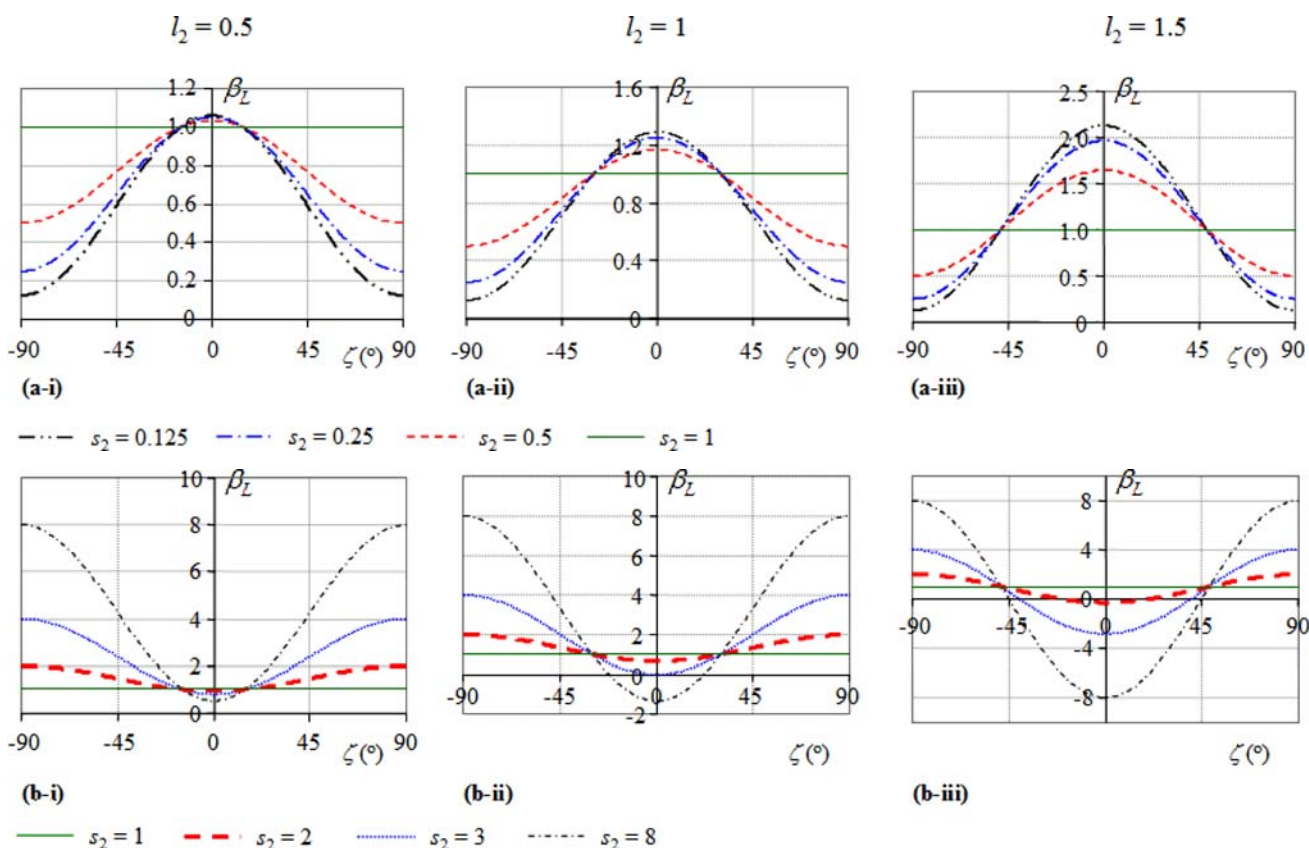
(i) the geometry of the system (i.e. the relative magnitudes of  $l_m$ );

(ii) the properties of the materials (i.e. the magnitudes of  $E_{Sm}$  and  $\nu_{Sm}$ );

and, in the case of the linear compressibility  $\beta_L$  also on the direction of measurement (i.e. the angle  $\zeta$ ) as clearly illustrated by plots of  $\beta_L$  against  $\zeta$  for various combinations of  $l_1, l_2, s_1$  and  $s_2$  (see Figs. 2, 3) which show that the compressibility can be negative for some but not all values of  $\zeta$  with maximum negative compressibility being exhibited in the  $Ox_1$  direction.

In fact, the equations and plots suggest that for this structure to exhibit negative linear compressibility, it is required that the compressibility in the  $Ox_1$  direction is negative, i.e.  $\beta_L[Ox_1] = -\varepsilon_{11}/dP < 0$ . Thus, since from geometric considerations  $l_2$  must be at least equal to  $2l_1$ , i.e.  $4l_1^2 - l_2^2 > 0$ , the requirement for negative linear compressibility in at least one direction will reduce to  $l_2^2 s_2 > 4l_1^2 s_1$  in which case, for  $\zeta \in [-90^\circ, 90^\circ]$ , the compressibility will be negative in the ranges  $\zeta \in (-\zeta_m, \zeta_m)$  where  $\zeta_m$  is defined by:

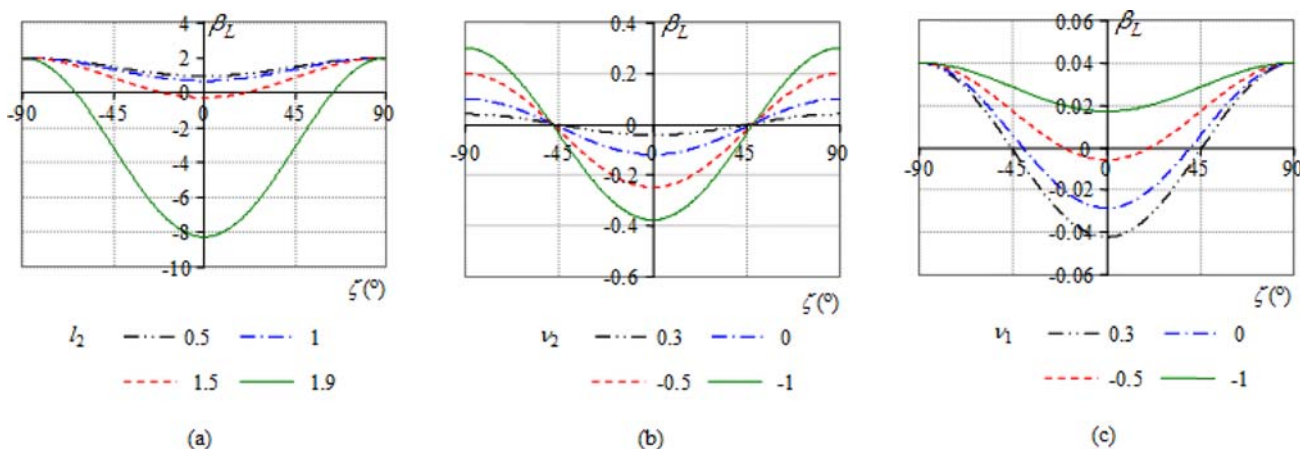
$$\zeta_m = \tan^{-1} \sqrt{\frac{l_2^2 s_2 - 4l_1^2 s_1}{l_2 s_2 (4l_1^2 - l_2^2)}}. \quad (20)$$



**Figure 2** (online colour at: [www.pss-b.com](http://www.pss-b.com)) Plots of  $\beta_L(\zeta)$ , the linear compressibility against  $\zeta$  (the direction of measurement) for simple systems where the basic unit is an isosceles triangle made from two materials with (a)  $s_1 \geq s_2$  and (b)  $s_1 \leq s_2$  where for (i)  $l_1 = 1$ ,  $l_2 = 0.5$ , (ii)  $l_1 = l_2 = 1$  and (iii)  $l_1 = 1$ ,  $l_2 = 1.5$ . Note that the values chosen are in arbitrary units.

Thus, for this simple system, the effect of negative compressibility can be maximized (i.e. increasing the magnitude of the most negative  $\beta_L$  and widening the range of values of  $\zeta$  where negative compressibility is exhibited) by increasing the magnitude of  $l_2^2 s_2 - 4l_1^2 s_1$  and minimising that of  $l_2 s_2 (4l_1^2 - l_2^2)$ . This can be achieved by:

1. maximizing  $l_2$  and/or minimizing  $l_1$ , two properties which are easy to control but are however limited by the geometric constraint that  $2l_1 > l_2$ ;
2. maximising  $s_2$  and/or minimising  $s_1$ , two properties which are limited by the availability of materials.



**Figure 3** (online colour at: [www.pss-b.com](http://www.pss-b.com)) Plots of  $\beta_L(\zeta)$ , the linear compressibility against  $\zeta$  (the direction of measurement) where the basic unit is an isosceles triangle made up from two materials where (a)  $l_1 = 1$ ,  $s_1 = 1$  and  $s_2 = 2$  (b-c)  $l_1 = 1$ ,  $l_2 = 1.5$ ,  $E_1 = 100$ ,  $E_2 = 10$  where for (b)  $\nu_1$  is held fixed at a value of 1 while for (c)  $\nu_2$  is held fixed at a value of 1. Note that the values chosen are in arbitrary units.



For normal three-dimensional rods where  $s_m = 1 - 2\nu_{sm}/E_{sm}$ , the value of  $s_2$  may be increased either by reducing the value of the Young's modulus or by reducing the value of the Poisson's ratio. In this respect it is interesting to note that for isotropic materials, the Poisson's ratio is bound to be within the range of  $-1$  to  $0.5$ , and hence auxeticity of rods '2' enhances the extent of negative compressibility while auxeticity of rods '1' enhances positive compressibility. This effect is illustrated in Fig. 3a, b which shows several plots of  $\beta_L(\zeta)$  against  $\zeta$  for various combinations of the lengths, moduli and Poisson's ratios. All this is very significant and provides another possible application where auxetic materials can result in a significant enhancement in performance when compared to conventional materials [11–25].

The plots in Figs. 2, 3 also highlight the fact that:

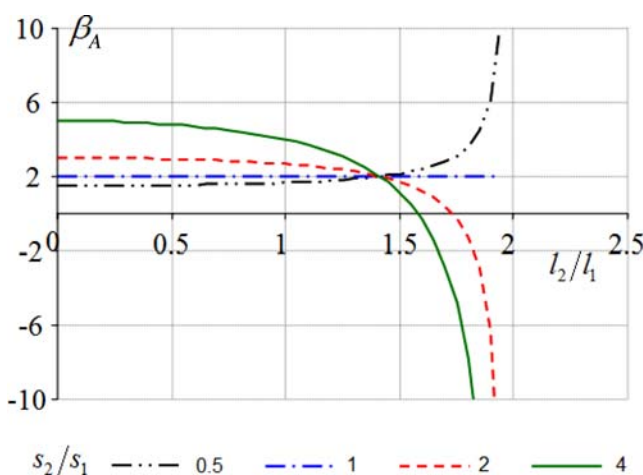
1. in the limit when  $l_2 \rightarrow 2l_1$ , the percentage of values of  $\zeta$  where  $\beta_L$  is negative will tend to 100%;
2. in some cases, the positive linear compressibilities in certain directions is greater in magnitude than the linear compressibilities of the components, i.e. the system can act as 'a linear compressibility amplifier'.

Let us now analyse the equation for the area compressibility  $\beta_A$ . In this case, since the denominator is always positive, negative compressibility arises when  $l_2^2 s_2 > 2l_1^2 (s_1 + s_2)$  since then, the numerator of Eq. (19) would be negative. Thus, due to the geometric constraint that  $2l_1 > l_2$ , for negative area compressibility it is required that:

$$2l_1 > l_2 > l_1 \sqrt{2 \frac{s_1 + s_2}{s_2}},$$

a condition which will only be satisfied if:

$$\frac{s_1 + s_2}{s_2} < 2 \Rightarrow s_1 < s_2$$



**Figure 4** (online colour at: [www.pss-b.com](http://www.pss-b.com)) Plots of  $\beta_A$ , the area compressibility against  $\zeta$  for simple systems made from two materials where the basic unit is an isosceles triangle for different combinations of  $l_1$ ,  $l_2$ ,  $s_1$  and  $s_2$ . Note that the values chosen are in arbitrary units.

All this is graphically illustrated in Fig. 4.

At this stage it is important to emphasise that although the 'structure' may exhibit a net negative linear or area compressibility, each individual component of the system will still be exhibiting conventional positive compressibility, i.e. the total actual volume occupied by the three solid rods would have increased when the external pressure is decreased thus ensuring that there are no violations of the energy conservation law. In fact, the negative linear or area compressibilities are solely due to the fact that the rods making up the triangle would experience different strains in the free (unbound) state, and, since they are constrained to maintain a triangular shape and due to the fact that the sum of the internal angles of a triangular unit must remain equal to  $180^\circ$  the triangular construct must necessarily become shorter.

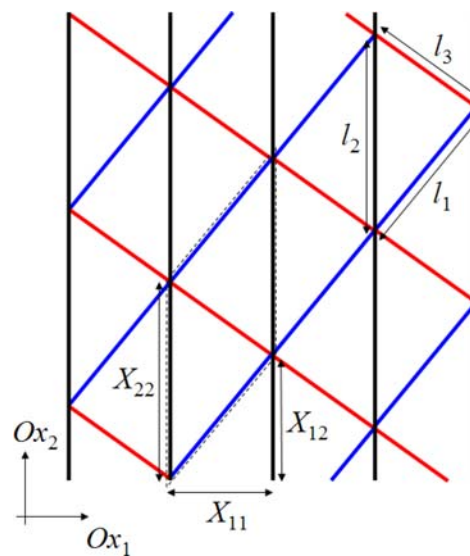
### 3 Analytical Model for generalised 2D systems made from three different materials

Having established the properties for this simplified case, let us now examine more complex cases. In particular, we consider a 2D periodic truss system constructed using rods made of three different types of isotropic materials, which we shall denote by '1', '2' and '3' (see Fig. 5). Once again, the shape and size of the system is described by a parallelogram unit cell containing two triangles where in this case:

$$X_{11} = \frac{1}{2l_2} \sqrt{(l_1 + l_2 + l_3)(-l_1 + l_2 + l_3)(l_1 - l_2 + l_3)(l_1 + l_2 - l_3)} \quad (21)$$

$$X_{22} = l_2 \quad (22)$$

$$X_{12} = \sqrt{l_1^2 - X_{11}^2} = \frac{l_1^2 + l_2^2 - l_3^2}{2l_2} \quad (23)$$



**Figure 5** (online colour at: [www.pss-b.com](http://www.pss-b.com)) A more general 2D system that can exhibit negative compressibility which can be constructed using three materials where the basic unit is a scalene triangle.

Thus, from Eqs. (8)–(14), the elements of the strain tensor are given by:

$$\begin{aligned} \varepsilon_{11} &= \frac{dX_{11}}{X_{11}} \\ &= -\frac{1}{X_{11}} \left( \frac{\partial X_{11}}{\partial l_1} l_1 s_1 + \frac{\partial X_{11}}{\partial l_2} l_2 s_2 + \frac{\partial X_{11}}{\partial l_3} l_3 s_3 \right) dP, \end{aligned} \quad (24)$$

$$\varepsilon_{22} = \frac{dX_{22}}{X_{22}} = -\frac{1}{X_{22}} \left( \frac{\partial X_{22}}{\partial l_2} l_2 s_2 \right) dP, \quad (25)$$

$$\begin{aligned} \varepsilon_{12} = \varepsilon_{21} &= -\frac{1}{2X_{11}} \\ &\left[ \frac{\partial X_{12}}{\partial l_1} l_1 s_1 + \frac{\partial X_{12}}{\partial l_2} l_2 s_2 + \frac{\partial X_{12}}{\partial l_3} l_3 s_3 - \left( \frac{X_{12}}{X_{22}} \right) \frac{\partial X_{22}}{\partial l_2} l_2 s_2 \right] dP. \end{aligned} \quad (26)$$

Thus, in this case, whilst the linear compressibility in the  $Ox_2$  direction which is given by  $\beta_L [Ox_2] = -\varepsilon_{22}/dP$  is still always positive and given by  $s_2$ , the linear compressibility in the  $Ox_1$  direction (which is equal to  $\beta_L [Ox_1] = -\varepsilon_{11}/dP$ ) will once again be able to assume both positive and negative values since all three  $\partial X_{11}/\partial l_i$  terms have both positive and negative components given by:

$$\frac{\partial X_{11}}{\partial l_1} = \frac{l_1(l_2^2 + l_3^2)}{2X_{11}l_2^2} - \frac{l_1^3}{2X_{11}l_2^2}, \quad (27)$$

$$\frac{\partial X_{11}}{\partial l_2} = \frac{l_3^4 + l_1^4}{4X_{11}l_2^2} - \frac{l_2^4 + 2l_1^2 l_3^2}{4X_{11}l_2^2}, \quad (28)$$

$$\frac{\partial X_{11}}{\partial l_3} = \frac{l_3(l_1^2 + l_2^2)}{2X_{11}l_2^2} - \frac{l_3^3}{2X_{11}l_2^2}, \quad (29)$$

i.e. the linear compressibility in the  $Ox_1$  direction  $\beta_L [Ox_1] = -\varepsilon_{11}/dP$  is given by:

$$\beta_L [Ox_1] = -\frac{\varepsilon_{11}}{dP} = \frac{dX_{11}}{X_{11}} = \frac{1}{4X_{11}^2 l_2^2} (F - G), \quad (30)$$

where:

$$F = 2l_1^2 (l_2^2 + l_3^2) s_1 + l_2 (l_3^4 + l_1^4) s_2 + 2l_3^2 (l_1^2 + l_2^2) s_3,$$

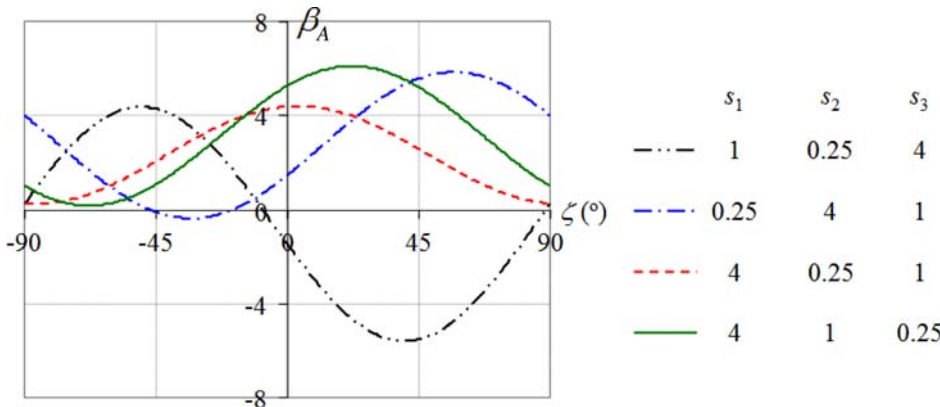
$$G = 2l_1^4 s_1 + l_2 (l_2^4 + 2l_1^2 l_3^2) s_2 + 2l_3^4 s_3,$$

thus suggesting that  $\beta_L [Ox_1] = -\varepsilon_{11}/dP$  is positive when  $F > G$  and negative when  $G > F$ .

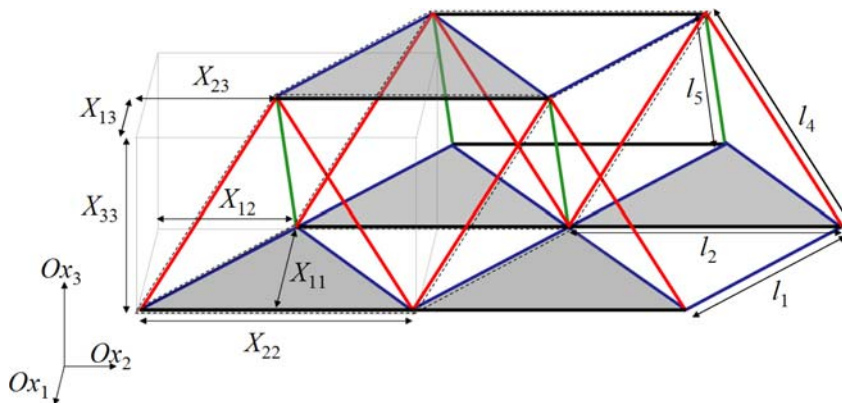
Furthermore, it is interesting to note that in general,  $\varepsilon_{12} = \varepsilon_{21}$  are not zero and instead these assume values given by:

$$\begin{aligned} \varepsilon_{12} = \varepsilon_{21} &= \frac{\gamma}{2} \\ &= -\frac{1}{2l_2 X_{11}} \left[ l_1^2 s_1 + (-l_1^2 + l_2^2 + l_3^2) s_2 - l_3^2 s_3 - \left( \frac{X_{12}}{X_{22}} \right) l_2^2 s_2 \right] dP \\ &= -\frac{l_1^2 (s_1 - s_2) - l_3^2 (s_3 - s_2)}{2X_{11}l_2} dP. \end{aligned} \quad (31)$$

This suggests that in general, provided that the system is anisotropic, the macrostructure shears when subjected to a change in pressure, unless  $l_1^2 (s_1 - s_2) = l_3^2 (s_3 - s_2)$ , a condition which may be satisfied in a number of special cases including the situation when  $s_1 = s_2 = s_3$  (the trivial solution corresponding to a structure made from one material in which case the compressibility is equal to the intrinsic compressibility of the materials), or the situation described in the previous section. Furthermore, we note that when  $\varepsilon_{12} = \varepsilon_{21} \neq 0$ , then the directions of maximum/minimum linear compressibilities will not be on-axis, as in fact illustrated in Fig. 6. Thus, in this case, maximum negative compressibility is not given by  $\beta_L [Ox_1] = -\varepsilon_{11}/dP$ , but by Eq. (5). In fact, it is important to note that such systems may be able to exhibit negative linear compressibilities in off-axis directions even if  $-\varepsilon_{11}/dP$  is positive. Nevertheless, for negative  $\beta_A$ , the term  $\beta_L [Ox_1] = -\varepsilon_{11}/dP$  must not only be negative, but its magnitude must be greater than  $\beta_L [Ox_2] = -\varepsilon_{22}/dP = s_2$ .



**Figure 6** (online colour at: [www.pss-b.com](http://www.pss-b.com)) Variation of linear compressibility  $\beta_L(\zeta)$  with the angle of loading ( $\zeta$ ) for a 2D system where the basic triangular unit has  $l_1 = 1$ ,  $l_2 = 1.5$  and  $l_3 = 2$ . Note that the values chosen are in arbitrary units.



**Figure 7** (online colour at: www.pss-b.com) A 3D truss system which can exhibit negative volume compressibility, constructed of layers of the 2D structure shown in Fig. 1 using 4 different materials.

**4 Three dimensional systems** Let us now consider a three dimensional truss system which in general may be described by parallel layers of the 2D structure in Fig. 1 aligned in the  $Ox_1 - Ox_2$  plane where half of the triangles (the ones shaded in Fig. 7a) form the base of two tetrahedra, one on either side of the 2D structure. Whilst in general, the six edges of the tetrahedra may have different lengths  $l_1, l_2, \dots, l_6$  provided that they satisfy some geometric criteria) and can be made from different materials '1', '2', ... '6', here we only consider the simple case when the base is an isosceles triangle made from two materials with

side lengths  $l_1, l_2$  and  $l_3 = l_1$  (i.e. the structure as in Fig. 8) where the two edges opposite to those of length  $l_1$  also have an equal length  $l_4$  and are made from the same material (material '4') whilst the edge opposite to that of length  $l_2$  has a length  $l_5$  and is made from material '5'.

It can be easily shown that this system can form a periodic system with a unit cell (which is not the smallest unit cell), where if we assume that  $\mathbf{b}$  is always aligned in the  $Ox_2$  direction and  $\mathbf{a}$  is always in the  $Ox_1 - Ox_2$  plane, then the unit cell vectors will be  $\mathbf{a} = (X_{11}, X_{12}, 0)$ ,  $\mathbf{b} = (0, X_{22}, 0)$  and  $\mathbf{c} = (X_{13}, X_{23}, X_{33})$  where, referring to Figs. 7, 8,  $\mathbf{c}$  refers to the vector  $\mathbf{AD}$  with  $X_{33}$  being the height  $h$  of the tetrahedron ABCD which is given by:

$$h = |\mathbf{DF}| = \frac{\sqrt{(|\mathbf{CD}| + |\mathbf{CE}| + |\mathbf{DE}|)(-\mathbf{CD} + |\mathbf{CE}| + |\mathbf{DE}|)(|\mathbf{CD}| - |\mathbf{CE}| + |\mathbf{DE}|)(|\mathbf{CD}| + |\mathbf{CE}| - |\mathbf{DE}|)}}{2|\mathbf{CE}|} \quad (32)$$

where:

$$|\mathbf{CD}| = l_5 \quad |\mathbf{CE}| = \sqrt{l_1^2 - \frac{l_2^2}{4}} \quad |\mathbf{DE}| = \sqrt{l_4^2 - \frac{l_2^2}{4}}, \quad (33)$$

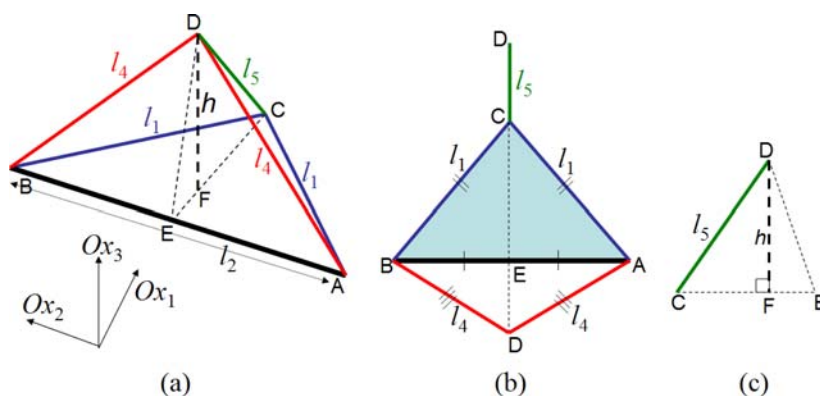
whilst:

$$X_{13} = |\mathbf{EF}| = \sqrt{|\mathbf{DE}|^2 - h^2}, \quad (34)$$

$$X_{23} = |\mathbf{AE}| = \frac{l_2}{2}. \quad (35)$$

The changes in shape and size when the system is subjected to a pressure change can be defined by the  $3 \times 3$  strain tensor  $\varepsilon_{ij}$  ( $i, j = 1, 2, 3$ ) where  $\varepsilon_{11}$  and  $\varepsilon_{22}$  are given by Eq. (15) and (16) respectively (i.e. the discussion in section 2 also applies for the  $Ox_1 - Ox_2$  plane of this structure) whilst the linear compressibility in the  $Ox_3$  direction which is equal to  $\varepsilon_{33}/dP$  is given by:

$$\beta_L [Ox_3] = -\frac{\varepsilon_{33}}{dP} = -\frac{1}{X_{33}} \frac{dX_{33}}{dP} = -\frac{1}{h} \frac{dh}{dP}, \quad (36)$$



**Figure 8** (online colour at: www.pss-b.com) Basic 3D unit (which is analogous to the 2D triangular unit) that can be used to construct 3D truss system similar to that shown in Fig. 7.

where since  $h = h(l_1, l_2, l_4, l_5)$ , the linear compressibility in the  $Ox_3$  direction

$$\beta_L [Ox_3] = -\frac{1}{h} \frac{dh}{dP} = \left[ \frac{\partial h}{\partial l_1} s_1 l_1 + \frac{\partial h}{\partial l_2} s_2 l_2 + \frac{\partial h}{\partial l_4} s_4 l_4 + \frac{\partial h}{\partial l_5} s_5 l_5 \right]. \quad (37)$$

As in the case of the 2D systems, for this system to exhibit linear compressibility in the  $Ox_3$  direction, it is required (but not sufficient) that at least one of the four partial derivatives  $\partial h / \partial l_i$  ( $i = 1, 2, 4, 5$ ) in the expression for  $dh/h$  must be negative. It may be shown that for particular combinations of  $l_i$  ( $i = 1, 2, 4, 5$ ), some of these partial derivatives are indeed negative and in fact, it may be shown that for  $l_1 = 1, l_2 = 1.8, l_4 = 1.1$  and  $l_5 = 0.5$ , the term  $\beta_L [Ox_3]$  is given by:

$$\beta_L [Ox_3] = -\frac{1}{h} \frac{dh}{dP} \approx -(0.380s_1 + 0.036s_2) + (0.514s_4 + 0.902s_5),$$

i.e. the linear compressibility in the  $Ox_3$  direction is negative if the terms  $s_i$  ( $i = 1, 2, 4, 5$ ) are such that  $(0.380s_1 + 0.036s_2) > (0.514s_4 + 0.902s_5)$ . Note that as discussed in Section 2, these length combination may also result in negative linear and area compressibility in the  $Ox_1 - Ox_2$  for certain combinations of  $s_i$  ( $i = 1, 2$ ).

Having determined the strains in the  $Ox_i$  directions, one should note that these may be used to calculate the volume compressibility  $\beta_V$  (the inverse of the bulk modulus,  $K$ ) which at constant temperature is given by:

$$\beta_V = \frac{1}{K} = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = -\left[ \frac{d\varepsilon_{11}}{dP} + \frac{d\varepsilon_{22}}{dP} + \frac{d\varepsilon_{33}}{dP} \right] \quad (38)$$

i.e.:

$$\beta_V = \beta_A + \beta_L [Ox_3] \quad (39)$$

All this suggests that for certain combinations of  $l_i$  and  $s_i$  ( $i = 1, 2, 4, 5$ ),  $\beta_V$  (and hence the bulk modulus) for this simple truss system can indeed be negative, for example if for the length combination that we have just considered  $s_1 = 3, s_2 = 6, s_4 = 1$  and  $s_5 = 0.5$  negative volume compressibility is observed. Once again it is important to emphasise that although this ‘structure’ exhibits a net negative volume compressibility, each individual component of the system will still be exhibiting conventional positive compressibility, i.e. the total actual volume occupied by the solid rods would have increased when the external pressure is decreased thus ensuring that there are no violations of the energy conservation law.

**5 Final considerations** Before we conclude this discussion, it is important to highlight some important aspects of the model presented here.

First of all, it is important to note that the systems described here are of interest not only due to the fact that they can exhibit negative compressibility/bulk modulus but also because these models allow for the possibility to construct them with a pre-desired set of compressibility characteristics, i.e. they can be tailor made for particular practical applications.

In this respect it is important to highlight that the values of the compressibility of the structure depend on the relative instantaneous lengths  $l_m$  of the trusses in the system which in turn depend on the external pressure. In fact, it is important to note that the compressibility properties discussed above are only valid for small pressure changes. This pressure dependence of the compressibility can have some very interesting consequences as may be illustrated, for example, by considering a special case of the simple model where the triangles are equilateral at a temperature  $T_0$ , with  $s_2 = 4s_1 = 4s_3$ . From Eq. (18), this system exhibits zero compressibility in the  $Ox_1$  direction at  $P = P_0$ . However, as the pressure is decreased, the lengths  $l_m$  will increase in length in such a way that  $l_2$  will always be longer than  $l_1$  for pressures  $P < P_0$ , and conversely, if the pressure is increased, the lengths  $l_m$  decrease in length in such a way that  $l_2$  is always shorter than  $l_1$  and  $l_3$  for pressured  $P > P_0$  with the result that although the system exhibits zero compressibility in the  $Ox_1$  direction at  $P = P_0$ , it will exhibit negative compressibility in the  $Ox_1$  direction when  $P < P_0$ , which becomes more negative as  $P$  is increased and positive compressibility in the  $Ox_1$  direction when  $P > P_0$ .

Furthermore, in this discussion the behaviour of this system has been discussed at constant temperature conditions. In reality, when this system is constructed using different materials, these materials will not only have different mechanical properties, but probably also different thermal expansion properties. In this respect, it is important to note that the thermal expansion properties of such systems have already been discussed before [7–9], where, for example, it was shown that the 2D systems in Figs. 1 and 5 can also be constructed to exhibit pre-desired thermal expansion properties (which could include negative thermal expansion). Since in practical applications it is possible for, such systems to experience a simultaneous change in both the external hydrostatic pressure and the temperature, it is important that the model presented here for the compressibility is considered together with the model for thermal expansion properties presented elsewhere [9] so as to obtain a more complete picture of the behaviour of these systems. In particular it would be interesting to investigate systems where changes in shape/size due to changes in temperature could be reduced or even nullified by counter changes caused by a change in pressure.

Also, it is important to highlight that this system can be considered as a ‘multifunctional negative system’ which can exhibit more than one negative property at the same time, in particular negative compressibility and negative thermal expansion. Furthermore, the concept presented he-



re for generating negative compressibility due to triangle shortening can easily be applied in the construction of systems similar to the one proposed to exhibit simultaneous negative thermal expansion (due to triangular shortening) and negative Poisson's ratio (due to rotating triangles) so as to achieve systems which can exhibit all these three negative properties simultaneously [25].

In this respect it is important to note that to maximise the effects of negative compressibility and negative thermal expansion, it may be very useful if the rods which are meant to respond mostly to changes in temperature/pressures could be replaced by piston-like elements filled with a gas (see Fig. 9). In such cases, assuming that the gas obeys ideal  $PV = nRT$  behaviour where  $V = Al$  with  $A$  being the cross-sectional area and  $l$  being the length of the piston containing the gas, then the changes in length as a result to a change in pressure or temperature (with the other component being kept constant) is given by:

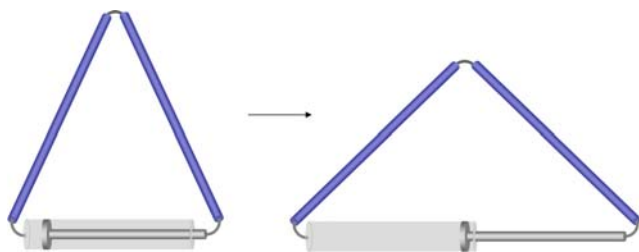
$$dl = -\frac{nRT}{AP^2} dP, \quad dl = \frac{nR}{AP} dT,$$

i.e.:

$$s_i = \frac{1}{P}, \quad \alpha_i = \frac{1}{T} dT,$$

which in practice results in values of  $s_i$  and  $\alpha_i$  which may have much greater values than those afforded by conventional commonly available solid materials.

Finally, it is important to highlight that the properties described here can be exhibited at any length scale, i.e. the systems presented here can be constructed at the micro and/or nano level so that the resultant product can be considered as a solid 'material' rather than a 'structure' which exhibits negative compressibility (i.e. negative bulk modulus). This would result in the first solid state material that exhibits negative bulk modulus, thus confirming the necessity that scientists and engineers must allow for the possibility that the bulk modulus is negative in their investigations.



**Figure 9** (online colour at: [www.pss-b.com](http://www.pss-b.com)) A triangular unit, which exhibits negative linear compressibility, constructed using a gas-filled piston which can be used to maximise the negative compressibility effect.

**6 Conclusion** In this study we have shown that it is possible to engineer 2D and 3D systems, that can be constructed at any scale, with adjustable linear, area and/or volume compressibility, which can also be negative, i.e. systems which shrink in size (at least in one dimension) when the external pressure decreases and that the linear compressibility of these systems is highly anisotropic and pressure dependent. We have also shown that these systems may also exhibit other negative properties, including negative thermal expansion. Given the simplicity of the construction, its adjustability and its structural rigidity (since the construct under analysis consists of triangles which confer substantial structural rigidity), we envisage that the proposed construct or variations of it should find extensive use in many practical applications.

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