# CALCULUS — A POWERFUL MATHEMATICAL TOOL

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A basic understanding of physics, and science in general, is necessary to lead a well-informed life. Most of us see the need for physics: it gives a basic background both for the physical sciences and the biological sciences. There are many topics in the life sciences which require a background knowledge of physics for a proper understanding, for example photosynthesis, osmosis, pigments or X-rays. To be able to formulate the laws governing natural phenomena we need an unambiguous language — Mathematics.

Man has always been fascinated by the wealth of nature around him, and intrigued by the complexity of the heavens. His never-ending study of the universe has led him to improve techniques and open up new fields of study. Until the seventeenth century the predominant school of thought laid down the foundations for a static and immutable universe. This theory did not explain exactly the facts, collected over the centuries, which advocated the idea of a dynamic and developing world. This necessitated a review of the mathematics available at the time, to enable it to deal with problems of change and growth. The chief credit for the development of such a powerful mathematical tool — the Calculus — is due to Sir Isaac Newton and Baron von Leibniz, two of the scientists who were working on the same problem, and who independently and simultaneously contributed to the solution.

Calculus — Latin for *pebble*, since calculating two thousand years ago was largely a matter of counting pebbles) — was invented in the first place to deal with slopes of curves and the areas 'under' them. It has since proved an invaluable aid to science and engineering. The theory of calculus first appeared in England in 1687 when Newton published his *Principia Mathematica*, a 250,000 word treatise on various topics of physics and mathematics. Simultaneously with Newton's publication, Leibniz's idea of the calculus appeared on the continent. Immediately a race for priorities was initiated, with scientists in England supporting Newton's theory, and European scientists supporting Leibniz. One basic difference finally settled the issue — Newton's 'fluxion' notation in calculus was somewhat clumsy, whereas Leibniz offered the 'd' notation, a more elegant symbolism. Some fifty years later Bishop Berkeley, a philosopher, attacked differential calculus (enabling the slope of a curve at a point to be found by taking a small sector and reducing it to a point mathematically), maintaining that a point could not have a slope. This attack opened up enquiries into the logical foundations of the calculus, with Colin Maclaurin, a professor of mathematics, finding a reasonably sound basis for the calculus in 1742. Leonard Euler, Marquis de l-Hopital, and the Bernoulli brothers (Daniel and John) further developed the calculus, with notation only slightly different from that which we use today, treating the most varied problems of differential calculus, integral calculus, and the calculus of variations. They applied it in astronomical analysis, examining the expected motions of earth, sun, moon under the mutual influence of each other's gravitational fields.

But what does 'calculus' mean to the Physics student? Quite often one gets the mechanical answer: 'dy/dx'. However for the really interested physics student, 'calculus' should imply a concurrent course dealing with the derivative and integral, and the related physical concepts of slope and area under a curve.

Perhaps the most obvious section in physics where the student comes across calculus is the 'Mechanics' part, involving the all-too-often 'rate of change'. Thus, velocity is defined as the rate of change of position with time; acceleration as the rate of change of velocity with time Both these definitions imply limiting values, leading to the concept of the derivative. The derivative we are speaking of is interpreted as the instantaneous value of the quantity under observation, and is defined as the limit of the average value, as both elements of the average value approach zero. In the case of velocity the two elements are distance and time.

## DIFFERENTIAL EQUATIONS

Many natural laws, especially those concerned with rates of change, can be phrased as equations involving derivatives or differentials. Take for instance Newton's second law. Taken for granted, though not obvious, it took two thousand years of thinking by the most capacious minds before it was achieved — "a uniform force produces a uniform *change* of motion". Aristotle thought that a uniform force produced a uniform motion (we can associate his term with our 'velocity'). Using our symbolism, he would have written his law as F = Rv, where R includes all resisting properties. But Aristotle did not have a concept of acceleration. For the case of a stone falling through the medium of air the time of flight would be too short for any refined observations to reveal a continuous change of velocity. His equation would apply, though, to a body falling through a very resistive medium, as this represents Stokes' law, R being the viscosity. And as soon as we handle differentials in such equations, we are then dealing with differential equations.

Differential equations are prominent in the study of various physical phenomena, such as the slipping of a belt on a pulley, radioactive decay, and the bending of a cantilever beam under the action of vertical loads. Differential equations find their way also into electricity, the most common case occurring in the simple LRC circuit. Here the current 'i' amperes and the charge 'q' coulombs for a capacitance C are related by: i = dq/dt; the voltage drop across a coil of inductance L henries is L (di/dt).

Substituting these quantities into an equation for the total voltage drop gives a linear differential equation (a linear differential equation of the first order implies that the dependent variable y, and its derivative with respect to x, dy/dx, occur in the first degree).

Of the two types of differential equations — 'ordinary' and 'partial' — partial differential equations tend to be more complex. (O.D.E.'s involve only one variable and derivatives with respect to it; P.D.E.'s deal with more than one independent variable). The two partial differential equations that one comes across most often in physics are the wave equation and the heat conduction equation.

Most general one-dimensional wave motion with velocity 'a' obeys the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This partial differential equation applies to the vertical vibrations of a flexible, elastic string stretched between two supports on the x-axis. By analogy to the 'standing-wave equation', the motion can be regarded as a superposition of two waves moving with velocity 'a' in opposite direction. As a result the free vibrations are periodic, no matter what the initial conditions may be. Since periodic vibration is associated with musical effects, perhaps one can realise the importance of this fact in the development of musical intruments. Deviating from ideal conditions as encountered above, namely, free vibrations with constant amplitude, one can then consider damped oscillation of the string vibrating in air, or perhaps forced oscillations, in which the force function is independent of the vertical displacement (for instance, the gravitational force on a horizontal vibrating string).

Suppose it is required to find the distribution of temperature in an infinite rod at an arbitrary moment. If the x-axis is directed along the rod, then the temperature T at a point x of the rod at moment t satisfies the HEAT CONDUCTIVITY EQUATION

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2}$$

where 'a' is a characteristic of the rod. This equation assumes that the initial temperature is a prescribed function of the distance 'x', and that the ends of the bar have the temperature zero. If an insulated rod of length 1 has one end maintained at temperature zero while the other end radiates into a medium of temperature zero, then this agrees with Newton's law of cooling. If, on the other hand, the ends of the bar are insulated, then the rate of flow across the ends (but not the temperature itself) is zero, which is an essentially different problem.

An extension of the above problem leads to the three-dimensional form of the equation (in Cartesian coordinates) for the flow of substance, such as heat and water, and known as the Laplace Equation of Continuity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

This assumes that there are no sources or sinks within the region occupied by the fluid, and that the flow of heat is steady, so that T is independent of t. The temperature at an arbitrary point (x, y, z) of the body at moment t is represented by T. The same equation arises also in electrostatics and gravitation, since the mathematical structures of Newton's and Coulomb's laws are identical. Thus there is an equilibrium distribution of electric charge over the surface of a conductor, a stationary fluid flow in a closed region, and the like.

The student will, however, come across simpler differential equations, such as Newton's second law, simple harmonic motion, and the catenary (the equation for a flexible chain in equilibrium under gravity).

More important is the fact that such results indicate a certain parallelism between various physical phenomena.

Thus, for instance, all oscillations whether mechanical or electromagnetic, are described by the same basic mathematical equations. The three obvious types of vibration consist of free vibration, vib ation with viscous damping, and forced vibration leading to resonance. The analogue of a freely vibrating mechanical system is a closed LC circuit wherein a capacitance C discharges through an inductance L, the latter assumed to be of negligible resistance. The resistance is the direct equivalent of damping effects. In the case of viscous damping, the electrical parallel is just an LCR circuit, this time C discharging through L whose resistance R is not negligible.

At the other extreme, if C is placed in series with a source of electromotive force and allowed to discharge through a coil containing L and R, resonance occurs, which is exactly identical to resonance in mechanical systems.

No matter how simple derivatives and differential equations might be, for anything of value to be extracted in the form of results one has to go through the reverse process of differentiation: integration.

Thus, whereas velocity and acceleration led to the concept of the derivative, work leads to the concept of the integral. Defined as the product of force and distance, an estimate of the work done can be obtained from a 'force-displacement', graph simply by finding the area enclosed between the curve and the displacement axis. (A formal derivation of the integral using summations and limits is too detailed to go into at this stage, and as such the student is referred to any relevant publication regarding the subject.)

The mechanics section will call to mind such applications of the integral calculus as areas, volumes, centre of mass, centre of gravity, and moments of inertia, wheras in the electricity section, to quote one instance, one comes across Ampere's law, giving the relation between the current 'i' and the magnetic field 'B' using integral notation.

"In the space of almost precisely one century infinitesimal calculus or, as we now call it in English, The Calculus, the calculating tool par excellence, had been forged; and nearly three centuries of constant use have not completely dulled this incomparable instrument." Some of the more common applications of this 'instrument', so adequately described by N. Bourbaki, have been touched upon in this article with the hope that some of the results previously 'believed' will someday be 'understood', when one knows how to handle this tool with dexterity, if not perfection.

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