Fibonacci Sequence and The Golden Ratio

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Fibonacci, or more correctly Leonardo da Pisa, was born in Pisa in 1175AD. He was the son of a Pisan merchant who also served as a customs officer in North Africa. He travelled widely in Barbary (Algeria) and was later sent on business trips to Egypt, Syria, Greece, Sicily and Provence. In 1200 he returned to Pisa and used the knowledge he had gained on his travels to write ***Liber abaci*** in which he introduced the Latin-speaking world to the decimal number system.

Fibonacci is perhaps best known for a simple series of numbers, introduced in ***Liber abaci*** and later named the ***Fibonacci numbers*** in his honour.

The resulting sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

(Fibonacci omitted the first term in *Liber abaci*). This sequence, in which each number is the sum of the two preceding numbers, has proved extremely fruitful and appears in many different areas of mathematics and science.

A problem in the third section of *Liber abaci* which can be solved using this sequence is the following:

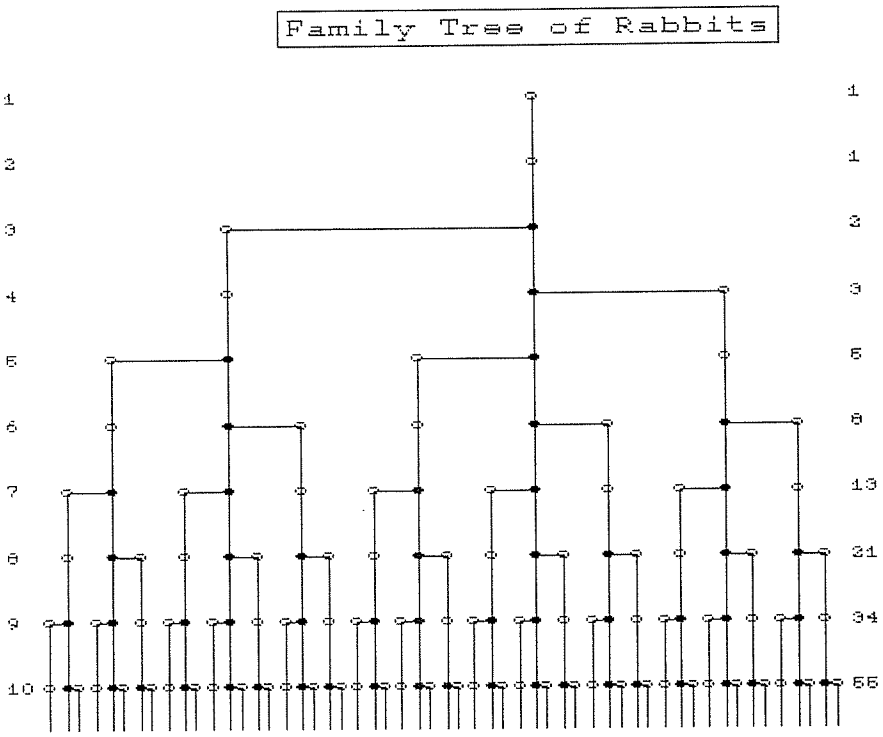
Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits ***never die*** *and that the female* ***always*** *produces one new pair (one male, one female)* ***every month*** *from the second month on. The puzzle that Fibonacci posed was...*
How many pairs will there be in one year?

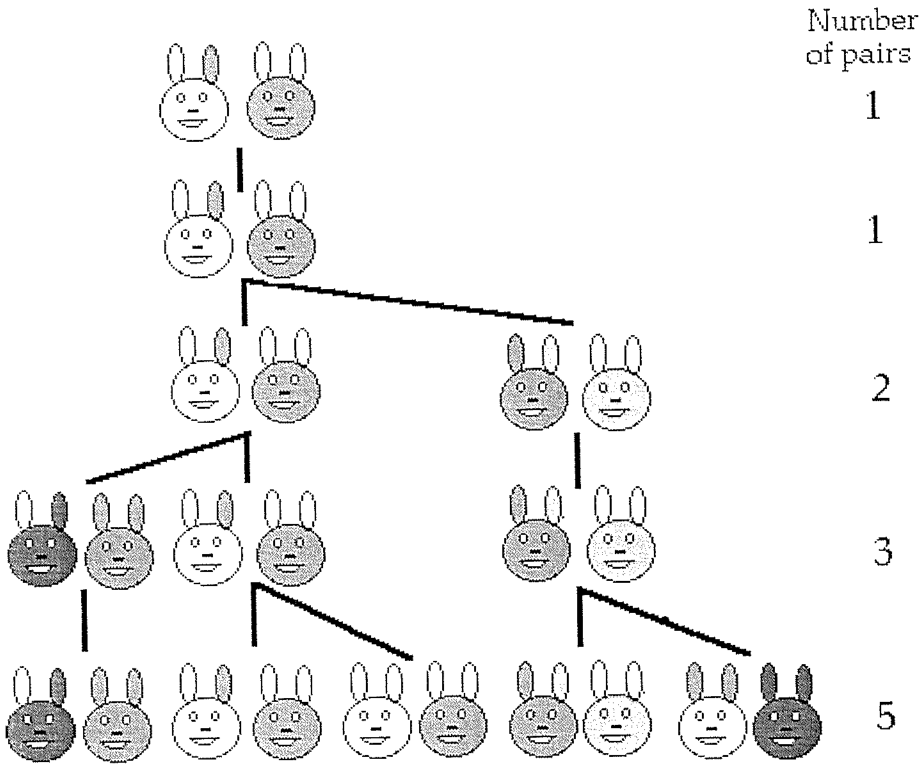
Imagine that there are x_n pairs *of rabbits after n months. The number of pairs in month n+1 will be* x_n *(in this problem, rabbits never die) plus the number of new pairs born. But new pairs are only born to pairs at least 1 month old, so there will be* x_{n-1} *new pairs.*

$$x_{n+1} = x_n + x_{n-1}$$

Which is simply the rule for generating the Fibonacci numbers!!!

The figure below is a view of the rabbit's family tree showing how the Fibonacci sequence is generate:


Family Tree of Rabbits
1
2
3
4
5
6
7
8
9
10
1
1
2
3
5
8
13
21
34
55


Number
of pairs
1
1
2
3
5

The Golden Section

Taking the ratio of successive terms in the Fibonacci series: (1, 1, 2, 3, 5, 8, 13, ..) and dividing each by the number before it, the following series of numbers is obtained:

$^{1}/_{1} = \mathbf{1}$

$^{2}/_{1} = \mathbf{2}$

$^{3}/_{2} = \mathbf{1{\cdot}5}$

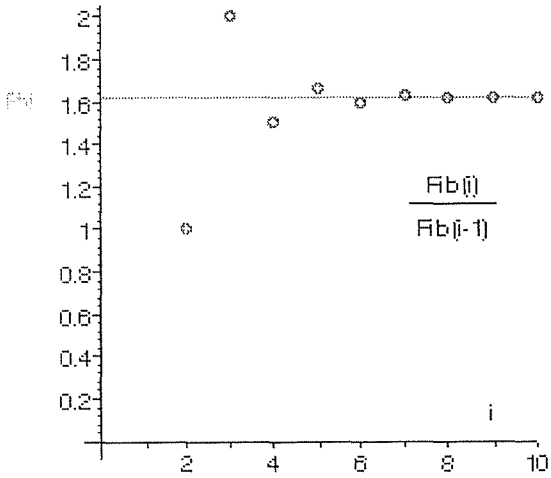
$^{5}/_{3} = \mathbf{1{\cdot}666...,}$

$^{8}/_{5} = \mathbf{1{\cdot}6}$

$^{13}/_{8} = \mathbf{1{\cdot}625}$

$^{21}/_{13} = \mathbf{1{\cdot}61538...}$

The graph shows that the values seem to be tending to a limit. This limit is actually the positive root of a quadratic equation and is called the *golden section*, *golden ratio* or sometimes the *golden mean*.



If you take two successive terms of the series, a, b, and $a+b$ then

$$\frac{b}{a} \cong \frac{a+b}{b}$$

$$\cong \frac{a}{b} + 1$$

We define the golden section, ϕ (*phi*), to be the limit of $\frac{b}{a}$, so:

$$\phi = \frac{1}{\phi} + 1$$

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Note that the golden section is denoted by the Greek letter *phi*.

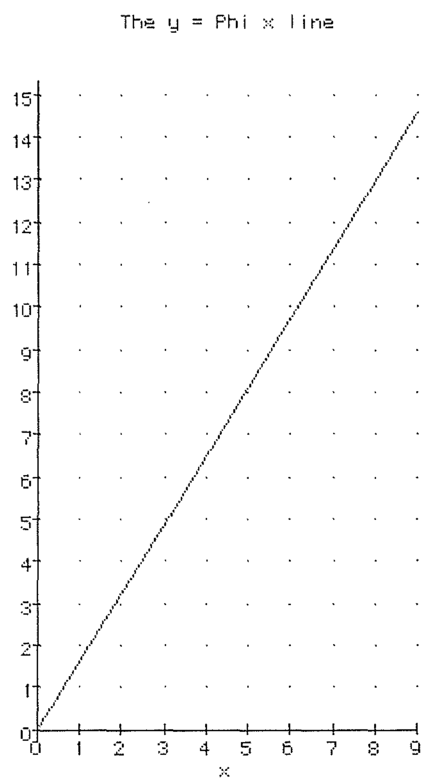
The table below shows properties of the solutions to the quadratic equation:

$$\phi^2 - \phi - 1 = 0$$

$\phi_1 . \phi_2 = 1 \quad \phi_1 - \phi_2 = 1 \quad \phi_1 + \phi_2 = \sqrt{5}$	
$\phi_1 = 1.6180339..$	$\phi_2 = 0.6180339..$
$\phi_1 = 1 + \phi_2$	$\phi_2 = \phi_1 - 1$
$\phi_1 = 1/\phi_2$	$\phi_2 = 1/\phi_1$
$\phi_1^{\ 2} = \phi_1 + 1$	$(-\phi_2)^2 = -\phi_2 + 1$ or $\phi_2^{\ 2} = 1 - \phi_2$
$\phi_1 = (\sqrt{5} + 1)/2$	$\phi_2 = (\sqrt{5} - 1)/2$

The Fibonacci numbers can also arise from the number ϕ. The graph below shows a line whose gradient is ϕ, that is the line :

$$y = \phi . x = (1{\cdot}6180339...)x$$



Since ϕ is irrational, the graph will never go through any points of the form (i,j) where are integers. The nearest integer-coordinate points to the ϕ - line are (0,1), (1,2), (2,3), (3,5)...

These coordinates are successive Fibonacci numbers. The ratio y/x for each Fibonacci point (x,y) approaches $\phi = 1.618...$

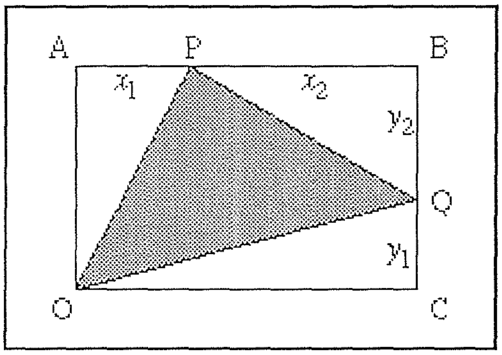
This graph also shows that the Fibonacci points are the closest points to the ϕ - line.

Problem involving the Golden Ratio:

Rectangle triangle problem

Consider a rectangle OABC from which you remove three right-angled triangles leaving a fourth triangle OPQ as shown in the diagram below.

How must you position the points P and Q so that the area of each of the three removed triangles is the same? That is, what are the ratios AP : PB and CQ : QB?



If we label the distances AP, PB, CQ and QB as shown above then we can write three equations for the areas of the triangles as follows:

Area of OAP	$=\frac{1}{2}x_1(y_1+y_2)$	(1)
Area of PBQ	$=\frac{1}{2}x_2y_2$	(2)
Area of OCQ	$=\frac{1}{2}(x_1+x_2)y_1$	(3)

From (1) and (3):

$$x_1y_1+x_1y_2=x_1y_1+x_2y_1$$
$$x_1y_2=x_2y_1$$
$$\frac{x_2}{x_1}=\frac{y_2}{y_1}$$

Notice that the two ratios are the same! If we call this ratio r then we can calculate a polynomial for r as follows:

From (1) and (2):

$$x_1y_1 + x_1y_2 = x_2y_2$$

$$y_1 + y_2 = \frac{x_2}{x_1}y_2 = ry_2$$

$$1 + \frac{y_2}{y_1} = r\frac{y_2}{y_1}$$

$$1 + r = r^2$$

$$r^2 - r - 1 = 0$$

Taking the positive root gives us the golden ratio:

$$r = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$$

Binet's Formula for the nth Fibonacci number

The n-th Fibonacci number is the sum of the (n-1)th and the (n-2)th.

Theorem: A formula for the nth Fibonacci number Fib(n), which contains only n and does not need any other (earlier) Fibonacci values involves the golden section number ϕ_1 and its reciprocal ϕ_2;

$$\text{Fib}(n) = \frac{\phi_1^{\,n} - (-\phi_1)^{-n}}{\sqrt{5}} = \frac{\phi_1^{\,n} - (-\phi_2)^{n}}{\sqrt{5}}$$

Proof:

Since ϕ_1 and $-\phi_2$ are the two roots of $x^2 = x+1$, we get the following:

If $x^2 = x+1$ then,

$$\mathbf{x^n = fib(n)\, x + fib(n-1)}$$

...... (1) for n>0.

Proving (1) by induction:

RTP *True for n = 1*

$x^1 = (\text{fib}(1))x + \text{fib}(0)$

$x = 1 . x + 0$

$x = x$

Suppose it is true for n = k

i.e. $x^k = fib(k) + fib(k-1)$

RTP *True for n = k + 1*

i.e. $x^{k+1} = (fib(k+1))x + fib(k)$

$x^{k+1} = x^k . x$

$= (fib(k) . x + fib(k-1)) . x$

$= fib(k) . x^2 + fib(k-1) . x$

But $x^2 = x + 1$,

$x^{k+1} = fib(k) . (x+1) + fib(k-1) . x$

$= fib(k) . x + fib(k) + fib(k-1) . x$

$= (fib(k) + fib(k-1)) . x + fib(k)$

But $fib(k) + fib(k-1) = fib(k+1)$

Therefore,

$x^{k+1} = fib(k+1) . x + fib(k)$

Now the two roots of $x^2 = x+1$ are $\phi_1 = (1+\sqrt{5})/2 = 1{\cdot}6180339...$ and $-\phi_2 = (1-\sqrt{5})/2 = -0{\cdot}6180339....$ and thus that these are the only two values for which their powers can be expressed as Fibonacci multiples of themselves, as given in the formula.

So, from the formula above, we have:

$$\phi_1^n = fib(n)\,\phi_1 + fib(n-1) \qquad \textbf{(A)}$$

and also

$$(-\phi_2)^n = fib(n)\,(-\phi_2) + fib(n-1) \qquad \textbf{(B)}$$

Subtracting (B) from (A) gives:

$$\phi_1^n - (-\phi_2)^n = fib(n)(\,\phi_1 - (-\phi_2)) \qquad \textbf{(C)}$$

and from this we derive an initial formula for fib(n):

$$fib\,(n) = \frac{\phi_1^n - (-\phi_2)^n}{(\,\phi_1 - (-\phi_2))}$$

But $\phi_1 - (-\phi_2) = \sqrt{5}$, so we can write this as:

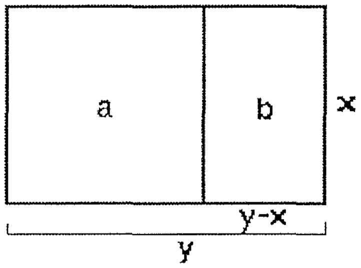
$$fib\,(n) = \frac{\phi_1^n - (-\phi_2)^n}{\sqrt{5}}$$

To get the form of the formula which involves only ϕ_1, we replace ϕ_2 by $1/\phi_1$ so that:

$$fib\,(n) = \frac{\phi_1^n - (-(1\,/\,\phi_1))^n}{\sqrt{5}}$$

$$fib\,(n) = \frac{\phi_1^n - (-\phi_1)^{-n}}{\sqrt{5}}$$

The Golden Rectangle

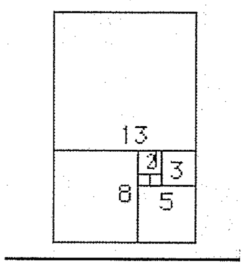


The rectangle shown here is a Golden Rectangle with proportions x/y. The section labeled "a" is a square drawn in the rectangle with proportions x/x. The section labeled "b" is another Golden Rectangle, this one with proportions (y-x)/x. In other words, the ratio of the lengths of the sides of section "b" is the same as the ratio of the length of the sides of the entire large rectangle. This is the characteristic of a Golden Rectangle. When you square it (inscribe a square with lengths the same as the length of the short side of the rectangle), you are left with another rectangle with the same proportions as the original.

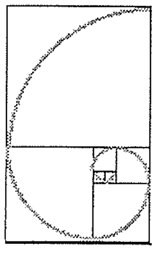
The Fibonacci Rectangles and Shell Spirals

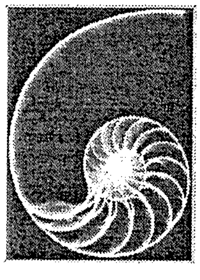
Another picture showing the Fibonacci numbers 1,1,2,3,5,8,13,21,.. can be achieved starting with two small squares of size 1 next to each other. On top of both of these a square of size 2 (=1+1) is drawn.

Now a new square - touching both a unit square and the latest square of side 2 - so having sides 3 units long is drawn; and then another touching both the 2-square and the 3-square (which has sides of 5 units). We can continue adding squares around the picture, *each new square having a side which is as long as the sum of the latest two square's sides*. This set of rectangles whose sides are two successive Fibonacci numbers in length and which are composed of squares with sides which are Fibonacci numbers, we will call this set the **Fibonacci Rectangles**.



The next diagram shows that we can draw a spiral by putting together quarter circles, one in each new square. This is a spiral (the **Fibonacci Spiral**). A similar curve to this occurs in nature as the shape of a snail shell or some sea shells.

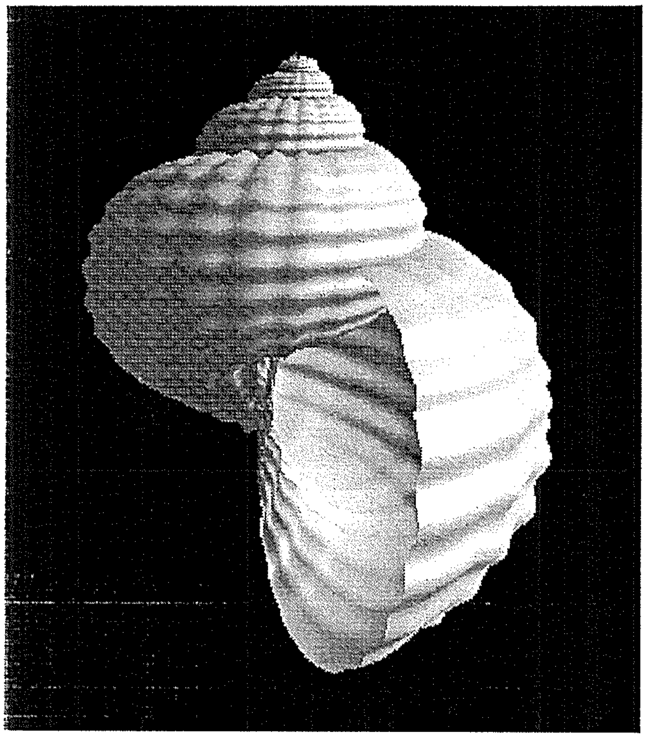


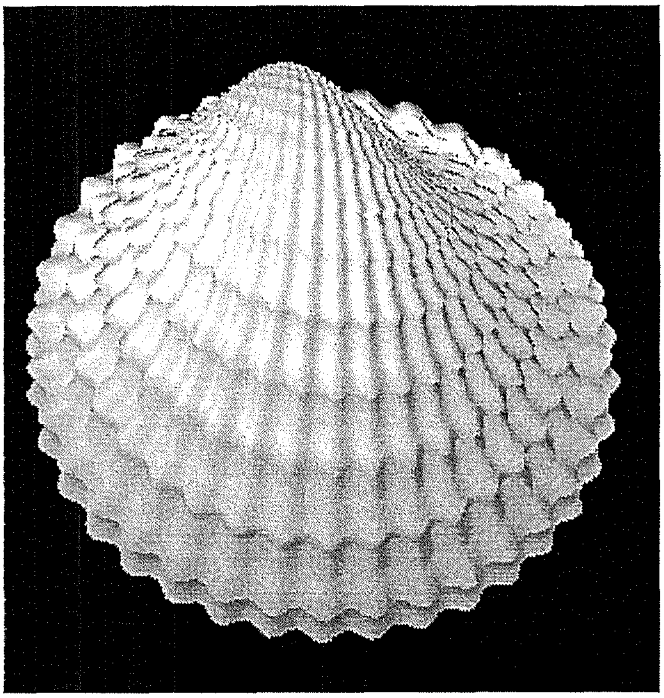


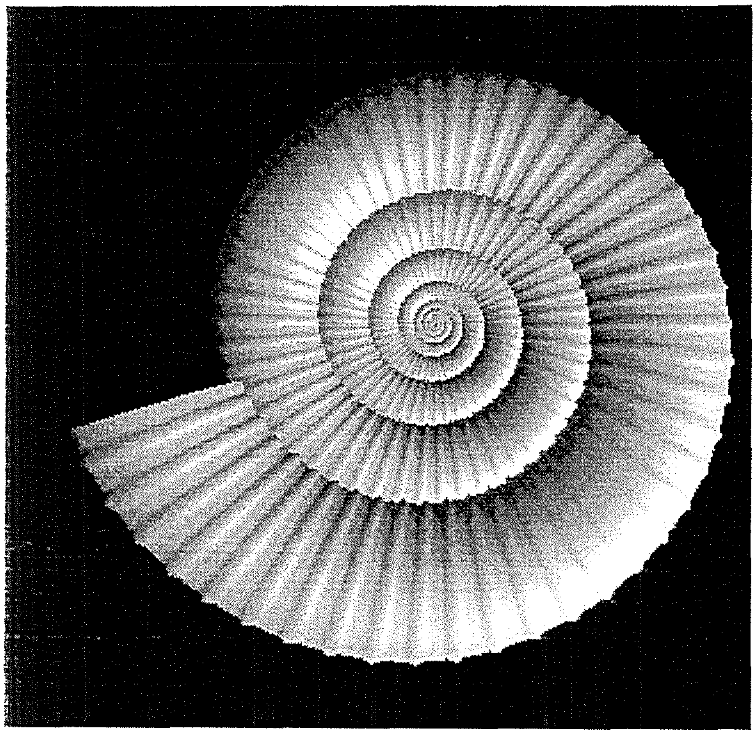
Whereas the Fibonacci Rectangles spiral increases in size by a factor of ϕ (1.618..) in a *quarter of a turn* (i.e. a point a further quarter of a turn round the curve is 1.618... times as far from the centre, and this applies to *all* points on the curve), the Nautilus spiral curve takes a *whole turn* before points move a factor of 1.618... from the centre.

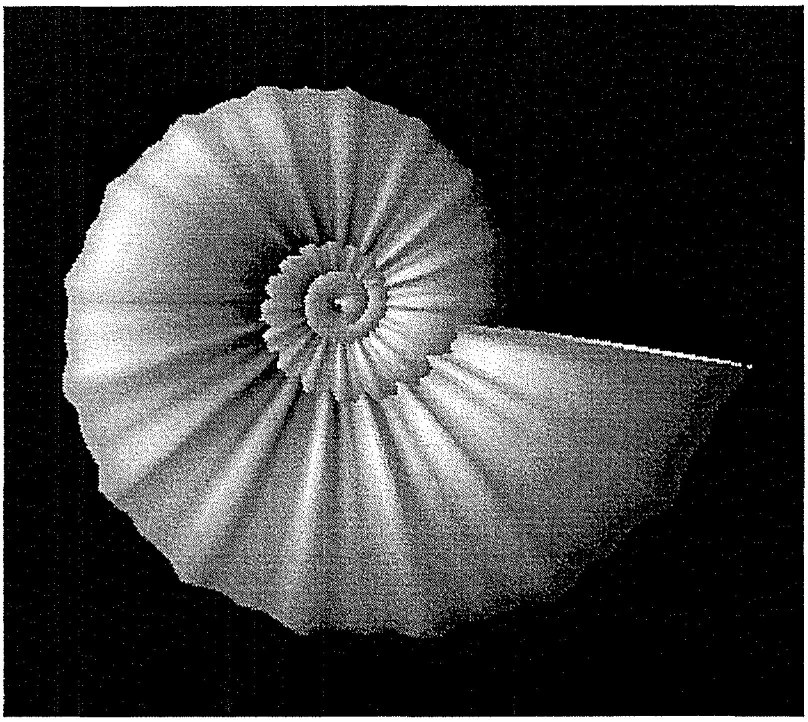
These spiral shapes are called Equiangular or Logarithmic spirals.

<u>The following are some examples of fibonacci spirals in nature</u>







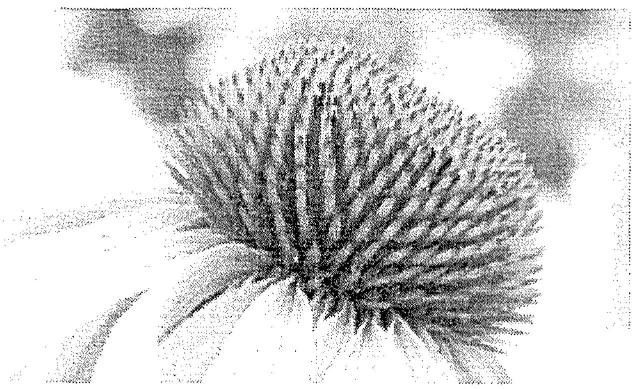


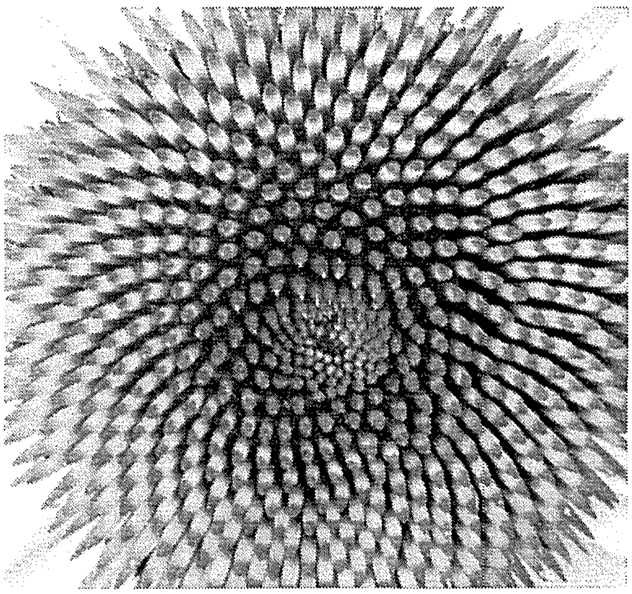
Petals on flowers

On many plants, the number of petals is a Fibonacci number: Buttercups have 5 petals; lilies and iris have 3 petals; some delphiniums have 8; corn marigolds have 13 petals; some asters have 21 whereas daisies can be found with 34, 55 or even 89 petals.

Some species are very precise about the number of petals they have - eg buttercups, but others have petals that are very near those above, with the average being a Fibonacci number.

Seed heads





Fibonacci numbers can also be seen in the arrangement of seeds on flower heads. The picture here is a photograph of a Coneflower.

You can see that the orange "petals" seem to form spirals curving both to the left and to the right. At the edge of the picture, if you count those spiralling to the right as you go outwards, there are 55 spirals. A little further towards the centre and you can count 34 spirals. The pair of numbers are neighbours in the Fibonacci series.

Pine cones

Pine cones show the Fibonacci Spirals clearly. Here is a picture of an ordinary pinecone seen from its base where the stalk connects it to the tree.