The Duplex of a Graph

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1 What is the Duplex?

Definition 1.1 *Consider a graph G with the vertex set $\mathcal{V}(G) = \{v_1, \ldots, v_n\}$ and edge set $\mathcal{E}(G) = \{e_1, \ldots, e_m\}$ as usual. The* **Duplex** *of G (denoted here by $\mathcal{D}G$) is the graph with the following vertex and edge set:*

- $\mathcal{V}(\mathcal{D}G) = \{v_1, \ldots, v_n, v_{1'}, \ldots, v_{n'}\}$
- $\mathcal{E}(\mathcal{D}G) = \{e_{1^*}, \ldots, e_{m^*}, e_{1'}, \ldots, e_{m'}\}$ *such that, for all $e_i \in \mathcal{E}(G)$, if $e_i = \langle v_i, v_j \rangle$, then $e_{i^*} = \langle v_i, v_{j'} \rangle$ and $e_{i'} = \langle v_{i'}, v_j \rangle$*

Clearly, $m(\mathcal{D}G) = 2m(G)$ and $n(\mathcal{D}G) = 2n(G)$ (hence the name 'duplex'). From the definition of the Duplex of G, it is definitely bipartite with vertex partitions $\{v_1, \ldots, v_n\}$ and $\{v_{1'}, \ldots, v_{n'}\}$. This means that $\mathcal{D}G$ always contains no odd circuits. Indeed, any odd circuit in G with v vertices will result in a circuit with $2v$ vertices in $\mathcal{D}G$. Even circuits in G with w vertices result in two circuits, each with w vertices, in $\mathcal{D}G$. Figure 1 clarifies this point.

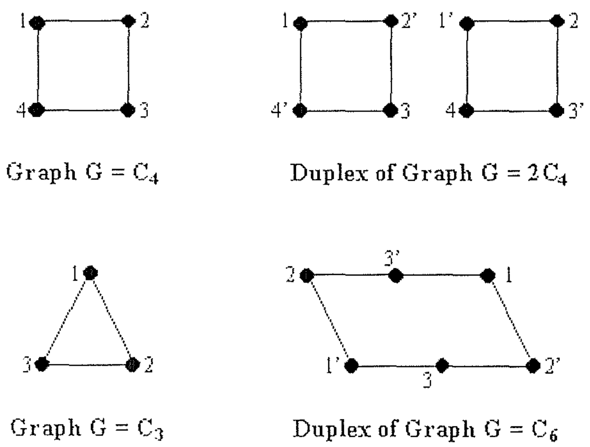

Figure 1: The Duplex of even and odd circuits

Thus we have the following lemma:

Lemma 1.2 *The Duplex is always bipartite.*

2 The Spectrum of the Duplex

A very interesting property that the Duplex has is that its spectrum contains the spectrum of the original graph G. To prove this, however, we first need a lemma from matrix theory, which will be quoted without proof.

Lemma 2.1 *If M is a non-singular square matrix, then*

$$\det \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det(M) \cdot \det(Q - PM^{-1}N)$$

Theorem 2.2 $\phi(G) \mid \phi(\mathcal{D}G)$

Proof: Consider the adjacency matrix of G. We can write it as:

$$\mathrm{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & 0 \end{pmatrix}$$

(remembering that **A** is symmetric.) >From the definition of the duplex of graph G, the adjacency matrix of $\mathcal{D}G$ is:

$$\mathcal{D}\mathrm{A} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & \dots & \dots & 0 & a_{12} & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & 0 & a_{1n} & a_{2n} & a_{3n} & \dots & 0 \\ 0 & a_{12} & a_{13} & \dots & a_{1n} & 0 & 0 & \dots & \dots & 0 \\ a_{12} & 0 & a_{23} & \dots & a_{2n} & 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & 0 & 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

This can be written briefly as:

$$\mathcal{D}\mathrm{A} = \begin{pmatrix} 0_{\mathrm{n}\times\mathrm{n}} & \mathrm{A} \\ \mathrm{A} & 0_{\mathrm{n}\times\mathrm{n}} \end{pmatrix}$$

(Here, the fact that the duplex of any graph is bipartite is very apparent.)

Now it is well known that, for the case when **A** is an adjacency matrix,

$$\lambda^2\mathrm{I} - \mathrm{A}^2 = (\lambda\mathrm{I} - \mathrm{A})(\lambda\mathrm{I} + \mathrm{A})$$

With this in mind, we can show what is required to prove, thus:

$$\begin{aligned}
\phi(\mathcal{D}\mathbf{A}) &= \det\begin{pmatrix} -\lambda\mathbf{I} & \mathbf{A} \\ \mathbf{A} & -\lambda\mathbf{I} \end{pmatrix} \\
&= \det(-\lambda\mathbf{I})\det(-\lambda\mathbf{I} - \mathbf{A}\left(-\frac{1}{\lambda}\mathbf{I}\right)\mathbf{A}) \text{ using Lemma 2.1} \\
&= (-\lambda^n)\det\left(-\frac{1}{\lambda}(\lambda^2\mathbf{I} - \mathbf{A}^2)\right) \\
&= (-\lambda^n)\left(-\frac{1}{\lambda^n}\right)\det(\lambda^2\mathbf{I} - \mathbf{A}^2) \\
&= \det(\lambda^2\mathbf{I} - \mathbf{A}^2) \\
&= \det((\lambda\mathbf{I} - \mathbf{A})(\lambda\mathbf{I} + \mathbf{A})) \\
&= \det(\lambda\mathbf{I} - \mathbf{A})\det(\lambda\mathbf{I} + \mathbf{A}) \\
&= \phi(\mathbf{A})\det(\lambda\mathbf{I} + \mathbf{A})
\end{aligned}$$

Therefore, $\phi(\mathcal{D}\mathbf{A}) \mid \phi(\mathbf{A})$, as required.

>From the above theorem, we can easily find what is the spectrum of the duplex of a graph from its spectrum. We showed above that $\phi(\mathcal{D}\mathbf{A}) = \phi(\mathbf{A})\det(\lambda\mathbf{I} + \mathbf{A})$, where $\mathcal{D}\mathbf{A}$ and $\mathbf{A}$ are the adjacency matrices of $\mathcal{D}G$ and G respectively. Since $\phi(\mathbf{A}) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$ (where $\lambda_i : 1 \le i \le n$ are the elements of the spectrum of G with possible repetitions), it is intuitive to conclude that $\det(\lambda\mathbf{I} + \mathbf{A}) = (\lambda + \lambda_1)(\lambda + \lambda_2)\cdots(\lambda + \lambda_n)$. Indeed, this is the case.

Lemma 2.3 *If $\phi(\mathbf{A}) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$, where $\lambda_i : 1 \le i \le n$ are the elements of the spectrum of G (with possible repetitions), then $\det(\lambda\mathbf{I} + \mathbf{A}) = (\lambda + \lambda_1)(\lambda + \lambda_2)\cdots(\lambda + \lambda_n)$.*

Proof: Consider $\phi(\mathbf{A}) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n$. In the case of $\det(\lambda\mathbf{I} + \mathbf{A})$, it will be exactly the same, except that the coefficients of the odd powers of λ will be $-\alpha_i$ instead of α_i, i.e. it will be equal to $\alpha_0 - \alpha_1\lambda + \alpha_2\lambda^2 - \cdots + (-1^n)\alpha_n\lambda^n$. Also, $(\lambda - \lambda_i)$ for all i was a factor for $\det(\lambda\mathbf{I} - \mathbf{A})$, which means that $\alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \cdots + \alpha_n\lambda_i^n = 0$. This implies that for $\det(\lambda\mathbf{I} + \mathbf{A})$, $(\lambda + \lambda_i)$ for all i is a factor, since $\alpha_0 - \alpha_1(-\lambda_i) + \alpha_2(-\lambda_i)^2 + \cdots + (-1)^n\alpha_n(-\lambda_i)^n = 0$, as required.

The following result follows immediately from the above lemma:

Proposition 2.4 *If the spectrum of G is $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (with possibly repeated elements), then the spectrum of $\mathcal{D}G$ is $(\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n)$ (with possibly repeated elements).*

Remark: A graph is bipartite if its spectrum is symmetric about 0. From the above corollary, we confirm again that the duplex of any graph is always bipartite.