An Old Japanese Theorem

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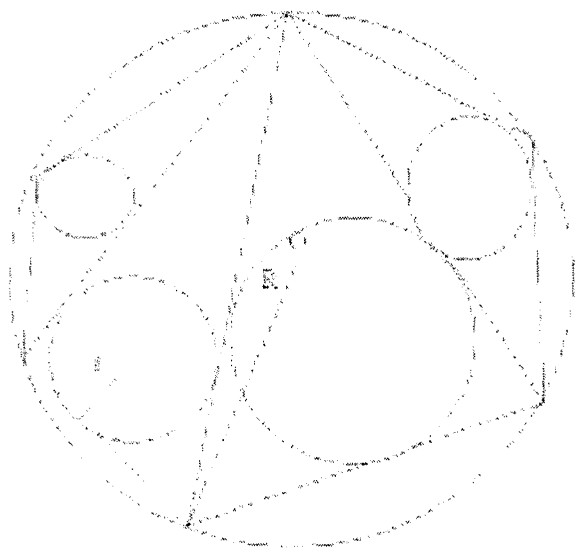
The theorem states the following:

Theorem 1.1. *Let a convex polygon (a shape is convex if with every pair of points that belong to the shape, the shape contains the whole straight line segment connecting the two points) which is inscribed in a circle, be triangulated by drawing all the diagonals from one of the vertices and let the inscribed circle be drawn in each of the triangles. Then the sum of the radii of all these circles is a constant which is independent of which vertex is used to form the triangulation.*

As an illustration of the theorem consider the following figure.

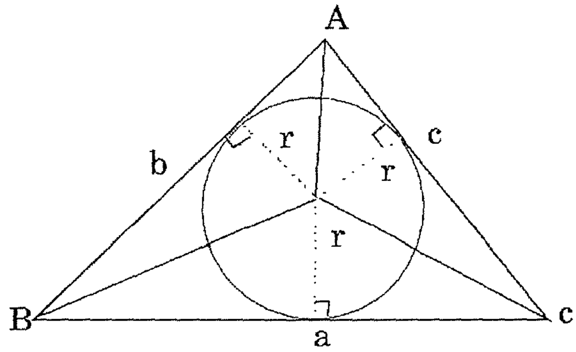


R is the radius of the outer circle while r is the radius of the inner circle inscribed in the triangle.

To prove the theorem, we must first prove the following two lemmas.

Lemma 1.2. *To show that* $r(a+b+c) = 2Area(ABC)$

Proof. Consider this diagram:



Then Area$(ABC) = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)$
$\therefore r(a+b+c) = 2$Area(ABC).

□

Lemma 1.3. (Carnot's Theorem) *In any triangle ABC, the sum of the distances from the circumcenter O to the sides is $R+r$, (i.e The sum of the circumradius with the inradius).*

$$OM_a + OM_b + OM_c = R + r$$

Proof. Only the case of an acute angle will be considered

Given that:

$$a = BC$$
$$b = AC$$
$$c = AB$$

and M_a, M_b and M_c are the intersecting points of the perpendicular lines from the origin O to the sides of the triangle.

For the inradius: $r(a+b+c) = 2$Area(ABC), which was proved in lemma 1.

Also:

$$\begin{aligned}\text{Area}(ABC) &= \text{Area}(0AB) + \text{Area}(OAC) + \text{Area}(OBC)\\ &= \frac{1}{2}cOM_c + \frac{1}{2}OM_b + \frac{1}{2}OM_a\end{aligned}$$

$$\therefore r(a+b+c) = aOM_a + bOM_b + cOM_c$$

O begin the circumcenter of ABC, in the isosceles triangle AOB, $\angle A0B = 2\angle C$. Similarly, $\angle BOC = 2\angle A$ and $\angle AOC = 2\angle B$. Equipped with this knowledge, one may consider several triples of similar (right-angled) triangles:

- ABH_b, ACH_c, and BOM_a (or an equal COM_a)
- BAH_a, BCH_c and COM_b (or an equal AOM_b)
- CBH_b, CAH_a and AOM_c (or an equal BOM_c)

This is possible due to the similarity properties of the angles.
From the first triple one derives:

$$AH_b/c = AH_c/b = OM_a/R$$

which leads to: $OM_a(b+c) = R(AH_b + AH_c)$
Similarly,

$$\begin{aligned}OM_b(a+c) &= R(BH_a + BH_c)\\ OM_c(a+b) &= R(CH_a + CH_b)\end{aligned}$$

Summing these three up we get:

$$OM_a(b+c) + OM_b(a+c) + OM_c(a+b) = R(AH_b + AH_c + BH_c + CH_a + CH_b)$$

Grouping like subscript terms on the right hand side we get:

$$OM_a(b+c) + OM_b(a+c) + OM_c(a+b) = R(BH_a + CH_a + AH_b + CH_b + AH_c + BH_c)$$

Thus,

$$OM_a(b+c) + OM_b(a+c) + OM_c(a+b) = R(a+b+c)$$

Adding this to (*) and dividing by $(a+b+c)$ implies:

$$OM_a(a+b+c)+OM_b(a+b+c)+OM_c(a+b+c)=r(a+b+c)+R(a+b+c)$$

i.e. $OMa+OMb+OMc=r+R$

$\square$

We can now proceed with the original theorem:

First, observe that any triangulation of an n-gon with its diagonals consists of $(n-2)$ triangles. Next assume that the triangles in the triangulation are numbered and that the inradius of the circle in triangle i is r_i.

From Carnot's theorem, i.e lemma 2, one can prove that $R+r_i=OO_i$, where OO_i is equal to the addition of the perpendicular distance from O to each side of the triangle i.

That is:

$$\begin{aligned}
OO_1 &= OO_{1a}+OO_{1b}+OO_{1c}\\
OO_2 &= OO_{2a}+OO_{2b}+OO_{2c}\\
&\vdots\\
OO_i &= OO_{ia}+OO_{ib}+OO_{ic}\\
&\vdots\\
OO_{(n-2)} &= OO_{(n-2)a}+OO_{(n-2)b}+OO_{(n-2)c}
\end{aligned}$$

where

- $O_{1a}\dots O_{(n-2)a}$ are identical to M_a in Carnot's Theorem.
- $O_{1b}\dots O_{(n-2)b}$ are identical to M_b in Carnot's Theorem.
- $O_{1c}\dots O_{(n-2)c}$ are identical to M_c in Carnot's Theorem.

The only difference is that in Carnot's Theorem one is considering one triangle, while in the original theorem one is considering a number of triangles that are formed by the n-gon.

$$\therefore \sum_{i=1}^{n-2} r_i + R = \sum_{i=1}^{n-2} OO_i$$

Thus $r_1 + \ldots + r_{n-2} = OO_1 + \ldots + OO_{n-2} - (n-2)R$

But $OO_1 + \ldots + OO_n - 2$ is identical to the perpendiculars of the sides of polygon counted once and the perpendiculars to the diagonals counted twice since every diagonal is the side of two adjacent triangles.

Furthermore, considering such internal perpendiculars, each is taken once as a positive value and once as a negative value and thus contributes nothing to the sum. Therefore, this sum is equal to the perpendiculars from O to the sides of the polygon. That is the right-hand side is a constant.

Thus the left-hand side i.e r_i is also a constant.

Figure 1

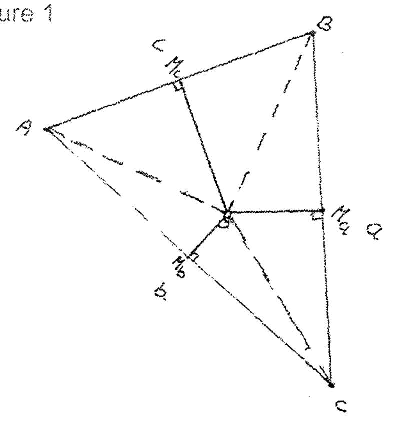


Figure 2