An Upper Bound for the Nullity of Trees and Edge-Colourings

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Abstract

A necessary and sufficient condition for the non-singularity of the adjacency matrix of a tree is given in terms of the existence of a 1-factor in the tree. The result is used to give an upper bound for the nullity of the tree via edge-colourings of bipartite graphs.

Illustrating the basic concepts

$$A(G) = \begin{pmatrix} 0&1&1&0&1&0 \\ 1&0&1&1&0&0 \\ 1&1&0&0&0&1 \\ 0&1&0&0&1&1 \\ 1&0&0&1&0&1 \\ 0&0&1&1&1&0 \end{pmatrix}$$

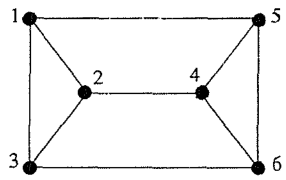

Figure 1: A *graph* G and its *adjacency matrix* $A(G)$

Edges $\{12, 46\}$ in G are *independent* because they share no vertex; they are also called a *matching*.

Independent edges $\{15, 24, 36\}$ are a *1-factor* of G because they cover all vertices; they form a *maximal* matching.

An *edge-colouring* of G is a partitioning of the edge set $E(G)$ of G into matchings, called *colour-classes*. The least number of colour-classes is the *chromatic* *index* $\chi'(G)$. In the example given $\chi' = 3$ and *the* partitioning (the only one possible) is {13, 24, 56} (coloured α), {12, 36, 45} (coloured β) and {15, 23, 46} (coloured γ).

If $\Delta(G)$ is the maximum valency of G, then, clearly $\Delta(G) \leqslant \chi'(G)$; it has been shown by Vizing [2, pp. 30-32] that $\chi'(G) \leqslant \Delta(G) + 1$.

The graph G (above) has odd circuits $\langle 1231 \rangle$, $\langle 124651 \rangle$. If all circuits are even, then G is said to be *bipartite* and the vertex set $V(G)$ of G can be partitioned into $V(G) = A \cup B$, $A \cap B = \phi$ such that $E(G) \subseteq A \times B$.

A *tree* T is a connected graph with no circuits and hence bipartite. If $|V(T)| = n$, then $|E(T)| = n - 1$ and it must have a vertex of valency 1.

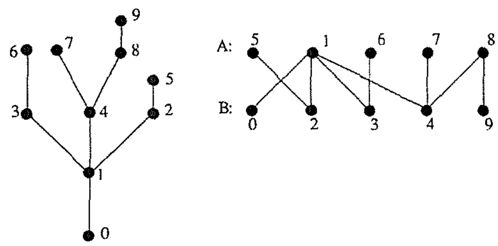

Figure 2: A *tree* T and its bipartition:

König (1916) proved that for a bipartite graph of maximum valency Δ, $\chi' = \Delta$. [2, p.25]

The *spectrum* spec(G) of a graph G is the set of eigenvalues of $A(G)$; since $A(G)$ is real and symmetric, spec(G) is real. Coulson and Rushbrook (1940) proved that the spectrum of a bipartite graph is symmetric about 0. [1, p. 87]

Main Theorem: A tree T has a 1-factor if and only if $A(T)$ is non-singular.

Theorem 1: If a tree T has a 1-factor, then $A(T)$ is non-singular.

Proof: T bipartite $\Rightarrow$ spec(T) symmetric about 0

$\Rightarrow \quad A(T)$ singular if $n(T)$ is odd.

But T has a 1-factor $\Rightarrow n(T)$ even, $n(T) = 2k$.

Proceed by induction on k.

For $k = 1$, there is only one tree on 2 vertices and $\det(A(T)) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$; hence non-singular.

Assuming the assertion is true for k and considering a vertex v of valency 1 with neighbour w in a tree with $2k+2$ vertices, we label its vertices $v = v_1, w = v_2$ so that $A(T) = \begin{bmatrix} 0\ 1 & \mathbf{o} \\ 1\ 0 & \mathbf{u} \\ \mathbf{o}\ \mathbf{u}^T & A(T-v-w) \end{bmatrix}$.

By a sequence of row and column operations of the kind: $\begin{matrix} R_i \mapsto R_i - R_1 \\ C_j \mapsto C_j - C_1 \end{matrix}$

vectors $\mathbf{u}$ and $\mathbf{u}^T$ can be 'killed' without affecting the sub-matrix $A(T-v-w)$ and without changing the value of $|A(T)|$.

A final row-operation $R_1 \leftrightarrow R_2$ changes the sign of the resulting determinant and yields $|A(T)| = -|A(T-v-w)| \neq 0$, by the inductive hypothesis.

Thus $A(T)$ is non-singular. ■

Theorem 2 If a tree T has a matching M of maximum size μ (covering 2 μ vertices $v_1, \ldots, v_{2\mu}$) and if v is any other vertex, then the row R_v in $A(T)$ corresponding to v is linearly dependent on the rows $R_{v_1}, \ldots, R_{v_{2\mu}}$ corresponding to the vertices in the matching.

Proof: Let v have neighbours $v_{i_1}, \ldots, v_{i_s}$.

If some v_{i_t} $(1 \leqslant t \leqslant s)$ is not covered by M, then the edge $v\, v_{i_t}$ could have been added to M, contradicting maximality. Thus all of $v_{i_1}, \ldots, v_{i_t}$ are in M and deleting v from T yields a disconnected graph with s components $C_1, \ldots, C_s$ with
$v_{i_t} \in C_t$ $(1 \leqslant t \leqslant s)$. Thus, $A(T)$ can be represented by:

$$\left(\begin{array}{cccc|c} A(C_1) & 0 & 0 & 0 & \\ 0 & A(C_2) & 0 & 0 & \\ 0 & 0 & \ddots & 0 & * \\ 0 & 0 & 0 & A(C_s) & \\ \hline 1 & 1 & \ldots & 1 & 0 \\ \hline & & * & & \end{array}\right)$$

for an appropriate labelling of its vertices.

One notes that the top right-hand submatrix must be zero; otherwise if there exist v_j (in C_1 say) that is not covered by this matching, then there exists a path in G starting in v ending in v_j with edges alternately "not in" / "in" the matching, contradicting the maximality of M.

But by Theorem 1, the principal sub-matrix of size $2\mu \times 2\mu$ is non-singular so that by suitable elementary row-operations the first 2μ rows of $A(T)$ can be reduced to

$$B := \left(I_{2\mu} \middle| * \right)$$

Thus the $(2\mu + 1)^{th}$ row corresponding to v in seen to be the sum of the rows $Rv_{i_1} + Rv_{i_2} + \ldots + Rv_{i_s}$ of B. ■

Since the vertex v was arbitrarily chosen from $V(T) \setminus V(M)$, we have the following:

Corollary: The rank $rk(T)$ of T equals 2μ. ■

The main theorem follows from this corollary and Theorem 1.

Now let $\Delta = \Delta(T)$ and $n = |V(T)|$. Since T is bipartite, T has an edge-colouring with Δ colours (by König), that is, $E(T)$ can be partitioned into Δ colour-classes $\Gamma_1, \ldots, \Gamma_\Delta$. It is clear that a colourclass consists of independent edges which form a matching. Hence the size of the largest colour class in the graph is less than that of a maximum matching.

Thus $(n-1) = |E(T)| = \sum_{i=1}^{\Delta} |\Gamma_i| \leqslant \Delta \max_{1 \leqslant i \leqslant \Delta} |\Gamma_i| \leqslant \Delta\mu$,

$$\Rightarrow \quad \left\lceil \frac{n-1}{\Delta} \right\rceil \leqslant \mu$$
$$\Rightarrow \quad rk(T) \geqslant \left\lceil \frac{n-1}{2\Delta} \right\rceil$$

Thus, the nullity of T is at most $n - \left\lceil \frac{n-1}{2\Delta} \right\rceil$.

Open Problem: Investigate the nullity of bipartite graphs.

Bibliography

[1] D. Cvetkovich, M. Doob, H. Sachs, *Spectra of Graphs* (New York: Academic, 1979)

[2] S. Fiorini, R. J. Wilson, *Edge-Colourings of Graphs*, (London: Pitman, 1977)