On Hand Shakes: A Combinatorial Problem

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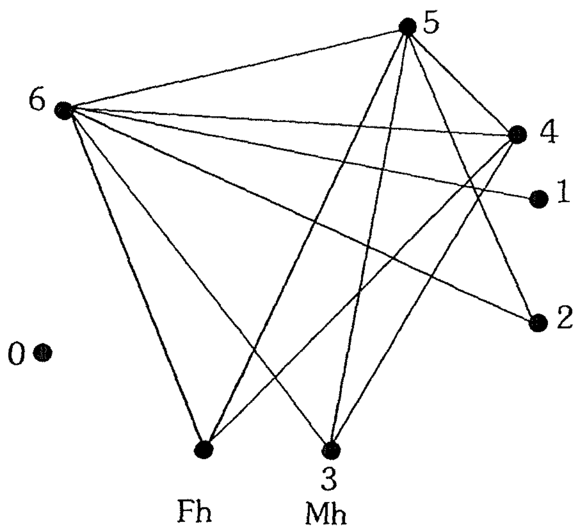
The problem is as follows:

> A number n of couples meet at one of the couples' homes. The female host (Fh) notices that no two of the people present (excluding herself) shake hands with the same number of people. No person shakes hand with the partner. With how many people does her husband (Hh) shake hands?

Solution One

The first solution will solve the problem for $n = 4$, where there are in all 8 persons. Assuming that handshaking never occurs between partners, a person can shake hands with at most 6 people. Construct a graph G having $2n = 8$ vertices each representing a single person. Two vertices are adjacent if a handshake occurs between these two people. Excluding the Fh, no two of the seven people shake the same number of hands. Therefore the possible number of handshakes are 6,5,4,3,2,1 and 0. Thus there must be some person, labelled v_6 who shakes hands with 6 people. The partner of v_6 is v_0 who does not shake hands with anyone. To see this recall that partners do not exchange handshakes; thus v_0 is not adjacent to v_6. Since all the other people shook hands with v_6 and since the number of handshakes is unique (excluding Fh), then v_0 must have shaken hands zero times (i.e v_0 is disconnected from the rest of the graph). Let v_5 be the person who shook hands five times. Then the person who shook hands once (v_1) is his/her partner, since apart from

v_1, v_0 all other persons must have shaken hands at least twice, with v_6 and v_5. Similarly, if v_4 is the person who shook hands four times, then by the same argument v_2 (the person who shook hands twice) is v_4's partner. The remaining vertex is v_3 for the person who shook hands 3 times. Who is v_3's partner? Choosing any partner from $v_0, v_1, v_2, v_3, v_4, v_5, v_6$ would contradict the premise that no two people (Fh excluded) shake hands the same number of times. Thus v_3's partner must be Fh which implies that v_3 is Mh, thus solving the problem. The following graph illustrates the solution. Note that at least two vertices in a graph must have the same degree (number of edges incident to a vertex).



Solution Two

The use of adjacency matrices provide a better model for solving the problem for a general positive integer n. As before the $2n-1$ people apart from the Fh can be labelled according to the number of handshakes $(0 \ldots 2n-2)$ which are all distinct by the above premise. Reserve the label $2n-1$ for the Fh. Construct a $2n-1 \times 2n-1$ matrix where element (i, j) is labelled if the i'th person shakes hands with the j'th person. Using a similar argument to that of solution one, $2n-2, 0$ form a couple; similarly $2n-3, 1$ form a couple. In general, $2n-k, k-2$ will form a couple for $n \geq k-1$. Thus, $n-1$ couples with $(n+1)-2=n-1$, which contradicts the uniqueness of the number of handshakes, unless this couple represents Fh, and Mh. Thus Mh shakes $n-1$ hands.

$$\begin{matrix} 2n-2 \\ 2n-3 \\ \vdots \\ \vdots \\ 0 \end{matrix} \begin{pmatrix} 0&1&1&1&1&0 \\ 1&0&1&1&0&0 \\ 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$$