The Eigenvalues of Self Complementary Graphs

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Abstract

Self complementary graphs have many interesting properties with reference to their main and non-main eigenvalues. Eigenvalues are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, proper values, or latent roots. We consider the spectra of self complementary graphs.

A *graph* has a set $\mathcal{V}$ of vertices $\{1,2,\ldots,n\}$ and a set $\mathcal{E}$ of edges joining distinct pairs of vertices.

Graph Complement The complement of a graph G is the graph $\overline{G}$with the same vertex set but whose edge set consists of the edges not present in G (i.e., the complement of the edge set of G with respect to all possible edges on the vertex set of G).

Example:

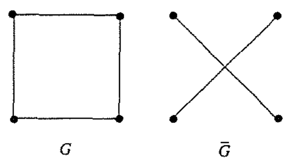

Figure 3: **Graph G and its Complement Graph**

Self Complementary Graphs : A *self-complementary* graph is a graph which is isomorphic to its graph complement.

Next are three examples of self-complementary graphs.

Example 1:

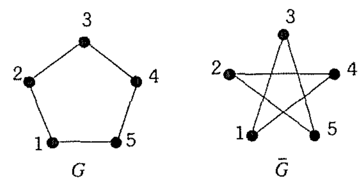

Figure 4: $G = C_5$ **and its compliment** $\overline{G}$

P: $1 \rightarrow 3$
$2 \rightarrow 5$
$3 \rightarrow 2$
$4 \rightarrow 4$
$5 \rightarrow 1$

Example 2:

Figure 5: $G = P_4$ **and its complement** $\overline{G}$

P: $1 \rightarrow 2$
$2 \rightarrow 4$
$3 \rightarrow 1$
$4 \rightarrow 3$

Example 3:

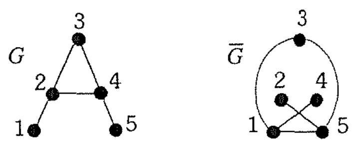

Figure 6: $G = A_G$ **and its compliment** $\overline{G}$

P: 1 → 4
2 → 1
3 → 3
4 → 5
5 → 2

An interesting property follows from the definitions given below of the adjacency matrix and its complement.

A is the *adjacency matrix* of a graph G, if it is the $n \times n$ symmetric matrix such that

$$a_{ij} = \begin{cases} 1 & \{i,j\} \text{ is an edge of G;} \\ 0 & \text{otherwise.} \end{cases}$$

$\overline{\mathbf{A}}$ is the adjacency matrix of the *complement* $\overline{G}$ of G if it is an $n \times n$ symmetric matrix such that

$$a_{ij} = \begin{cases} 0 & \{i,j\} \text{ is an edge of G;} \\ 1 & \text{otherwise.} \end{cases}$$

If **J** is the all 1 matrix and **I** is the identity matrix then

$$\overline{\mathbf{A}} + \mathbf{A} = \mathbf{J} - \mathbf{I} \tag{1}$$

Finding an Antimorphism and an Automorphism

Example1: The adjacency matrix of $\mathbf{C}_5$ is denoted by $\mathbf{A}(\mathbf{C}_5)$.

As we have shown before the mapping from $\mathbf{C}_5$ to its complement may be represented as the permutation $\mathbf{P}$ = (1 3 2 5) (4). By entering the matrices below into Mathematica and using the command Transpose[P].A.P we obtain the following matrices.

$$P = \begin{pmatrix} 0&0&0&0&1 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \end{pmatrix}$$

$$P^T = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \end{pmatrix}$$

$$\mathbf{A}(\mathbf{C})_5 = \begin{pmatrix} 0&1&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{pmatrix}$$

So $\mathbf{P}^T.\mathbf{A}.\mathbf{P}=\overline{\mathbf{A}}$

Therefore **P** is an *antimorphism* since it represents a mapping from **A** to its complement $\overline{\mathbf{A}}$

Let $\mathbf{Q} = \mathbf{P}^2$= (4) (1 3 2 5) . (4) (1 3 2 5) = (4) (1 2) (3 5)

then

$$Q = \begin{pmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \end{pmatrix}$$

So $\mathbf{Q}^{-1}.\mathbf{A}.\mathbf{Q} = \mathbf{A}$.

Therefore $\mathbf{Q}$ represents an automorphism since it is a mapping from A onto itself.

Example 2 : The adjacency matrix of $\mathbf{P}_4$ is denoted by $\mathbf{A}(\mathbf{P}_4)$.

The mapping from $\mathbf{P}_4$ to its complement maybe represented as the permutation

$\mathbf{P} = (\ 1\ 2\ 4\ 3\ )$.

So $\mathbf{P}^T.\mathbf{A}.\mathbf{P} = \overline{\mathbf{A}}$

Let $\mathbf{Q} = \mathbf{P}^2 = (\ 1\ 2\ 4\ 3\ )\ .\ (\ 1\ 2\ 4\ 3\ ) = (\ 1\ 4\ )\ (\ 2\ 3\ )$

So $\mathbf{Q}^{-1}.\mathbf{A}.\mathbf{Q} = \mathbf{A}$

Example 3 : The adjacency matrix of the graph $\mathbf{A}_G$ of Figure 6 is denoted by $\mathbf{A}(\mathbf{A}_G)$.

The mapping from $\mathbf{A}_G$ to its complement may be represented as the permutation

$\mathbf{P} = (\ 1\ 4\ 5\ 2)\ (3)$.

So $\mathbf{P}^T.\mathbf{A}.\mathbf{P} =$

Let $\mathbf{Q} = \mathbf{P}^2 = (3)(1452).(3)(1452) = (3)(15)(42)$

So $\mathbf{Q}^{-1}.\mathbf{A}.\mathbf{Q} = \mathbf{A}$

Special Eigenvalues Properties For Self Complementary Graphs:

An eigenvector is said to be *main* if it is not orthogonal to j.

Example 1: For $\mathbf{A(C)_5}$,
the eigenvalues are: $\{2, -1.61803, -1.61803, 0.618034, 0.618034\}$,
and the eigenvectors are : $\{1, 1, 1, 1, 1\}, \{-1.61803, 1.61803, -1, 0, 1\}$,
$\{-1, 1.61803, -1.61803, 1, 0\}, \{0.618034, -0.618034, -1, 0, 1\}, \{-1, -0.618034, 0.618034, 1, 0\}$

Checking if eigenvectors are main:
Since C_5 is regular the only main eigenvector $\{1, 1, 1, 1, 1\}$

$\mathbf{A(C_5)}$ has non-main eigenvalues λ_2, λ_3, λ_4 and λ_5, which can be paired off as follows:

$$\lambda_2 + \lambda_4 = \lambda_3 + \lambda_5 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = -1$$

This follows from equation 1.

Example 2: For $A(P_4)$
the eigenvalues are equal to: $\{-1.61803, 1.61803, -0.618034, 0.618034\}$,
and the corresponding eigenvectors are: $\{-1, 1.61803, -1.61803, 1\}, \{1, 1.61803, 1.61803, 1\}$,
$\{1, -0.618034, -0.618034, 1\}, \{-1, -0.618034, 0.618034, 1\}$

Checking if eigenvectors are main:
For $\lambda_2 = 1.61803$, the eigenvector $\mathbf{x_2}$ is $\{1, 1.61803, 1.61803, 1\}$ If $\mathbf{j} = \{1, 1, 1, 1, \}$ then $< \mathbf{j}, \mathbf{x_2} > \neq 0$. Hence λ_2 is main.

The non-main eigenvalues are λ_1 and λ_3, which can be paired off as follows:

$$\lambda_1 + \lambda_3 = -1$$

Example 3: For $A(A_G)$
the eigenvalues are: $\{2.30278, -1.61803, -1.30278, 0.618034, 0\}$,
and the corresponding eigenvectors are: $\{1, 2.30278, 2, 2.30278, 1\}, \{-1, 1.61803, 0, -1.61803, 1\}$, $\{1, -1.30278, 2, -1.30278, 1\}, \{-1, -0.618034, 0, 0.618034, 1\}, \{1, 0, -1, 0, 1\}$.

The only non-main eigenvalues are λ_2 and λ_3 which can be paired off as follows:

$\lambda_2 + \lambda_3 = -1$

Justification of the results obtained:

$\overline{\mathbf{A}} + \mathbf{A} = \mathbf{J} - -\mathbf{I}$

$\Rightarrow \mathbf{A} = \mathbf{J} - -\mathbf{I} - \overline{\mathbf{A}}$

$\Rightarrow \mathbf{A}\mathbf{x}_i = \mathbf{J}\mathbf{x}_i - - \mathbf{I}\mathbf{x}_i - - \overline{\mathbf{A}}\mathbf{x}_i$

If λ_i is non-main, then $\mathbf{x}_i.\mathbf{j} = 0$

Thus $\lambda_i\ \mathbf{x}_i = \mathbf{0} - \mathbf{x}_i - - \overline{\mathbf{A}}\mathbf{x}_i$ corresponding to a non-main eigenvalue λ_i

So $\overline{\mathbf{A}}\mathbf{x}_i = (\lambda_i\ -1)\mathbf{x}_i$

Since **G** is *self complementary*, the set of eigenvalues of $\overline{\mathbf{A}}$= set of eigenvalues of **A**

For each λ_i, there exists λ_j = - $(\lambda_i + 1)$

So in *self complementary* graphs non-main eigenvalues are paired s.t. $\lambda_j + \lambda_i = - - 1$. Therefore by just looking at the eigenvalues and by pairing them off, we may find the non-main eigenvalues.