A Homomorphism on Musical Notes

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Abstract

The intervals between successive notes in the major and minor scales are not equal so that difficulties arose when modulating to new keys. Adjustment to the tempered scale, in which all intervals are equal, ensured portability in all keys. The tempered intervals form a group under multiplication. Moreover, the musical notes can be partitioned into equivalence classes by the octave. A homomorphism can be defined on the set of tempered intervals. The kernel is the set of exact number of octave and the range isomorphic to C_{12}.

Introduction

This article deals with applying group theory to musical notes. We start by taking note of their mathematical and physical properties.

1. The pitch of a musical note is defined by its frequency, that is the number of vibrations per second.
2. The frequencies of pure musical tones form an infinite set of real numbers. The range between 20 and 20,000Hz is within the lower and upper limits of audibility.
3. Instruments with discrete musical tones are finite subsets of this infinite set. The pianoforte, for example, has a subset with 88 elements.

4. Instruments with continuous musical tones are infinite subsets of this infinite set. Examples of these are the string instruments and some wind instruments like the trombone.

5. There is an order relation, $>$, defined on the musical notes which is antisymmetric and transitive. Given three musical notes a, b and c:

i. $a < b$ does not imply $b < a$ (antisymmetric)

ii. however, if $a < b$ and $b < c$, then $a < c$ (transitive)

6. Musical notes can also be divided into equivalence classes.

Recall that if A is a set and $\sim$ is an equivalence relation, then the equivalence class of $a \in A$ is the set: $\{x \in A : a \sim x\}$.

Also recall that three properties define an equivalence relation $\sim$.

$\forall a, b, c \in A$,

1.$a \sim a$ (reflexive)

2.$a \sim b \Longrightarrow b \sim a$ (symmetric)

3.$a \sim b$ & $b \sim c \Longrightarrow a \sim c$ (transitive).

Classes of Notes

In music, notes with the same name are part of the same equivalence class. The reason behind this is that notes with the same name are related by the following equivalence relation: $a \sim b$ if $\frac{b}{a} = 2^n, n \in \mathbb{Z}$.

1. $\sim$ is reflexive, since $a = 2^0 a$
2. $\sim$ is symmetric, since$a = 2^x b \Longrightarrow b = 2^{-x} a$
3. $\sim$ is transitive, since $a = 2^x b$ & $b = 2^y c \Longrightarrow c = 2^{x+y} a$ given that $x, y \in \mathbb{Z}$.

Thus the frequencies of successive notes in one equivalence class are in the ratio of 2:1 and are said to be an octave apart. They sound similar since they have common vibrations, the higher note doing an extra vibration in between each of two consecutive vibrations of the lower note. Notes in the same class are given the same name. Middle C on the piano is 256 Hz and the frequencies 128Hz and 512Hz are also C notes. On the tempered scale Middle C is adjusted to 261.6Hz.

Problems on Modulating

DEFINITION 0.0.3. The interval beween two notes aHz and bHz with $b > a$ is the ratio $\frac{b}{a}$.

Of the seven Greek modes, the Ionian and Aeolian modes developed into the natural major and minor scales respectively, because the tonal symmetry among the intervals contributes to the ear's ready acceptance of the scale. The intervals given by $\frac{1}{1}$, $\frac{9}{8}$, $\frac{5}{4}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{15}{8}$ and $\frac{2}{1}$ have become accepted as the ones most pleasing to the ear and correspond to unison, major 2nd, major 3rd, perfect 4th, perfect 5th, major 6th, major 7th and the octave resp.

The function $LI : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $LI : (a, b) \mapsto \log_2 \frac{b}{a}$ gives the interval in octaves. Thus if $a \sim b$ then $\frac{b}{a} = 2^n, n \in \mathbb{Z}$, the notes a, b are n octaves apart and belong to the same equivalence class. Now if we have an interval of 12 perfect fifths starting from C, we expect to obtain C again 7 octaves up, that is the interval $2^7 = 128$. But one perfect fifth is $\frac{3}{2}$ which when compounded gives $\left(\frac{3}{2}\right)^{12} = 129.7$, so that successive multiplication by $\frac{3}{2}$ is not closed under successive multiplication by 2. The discrepancy represents the interval between $B^\sharp$ and C which does not figure on the piano. Even with just intonation where each of the major triads $F-A-C$, $C-E-G$ and $G-B-D$ are in the ratio $4:5:6$, the interval between two notes, one tone, is sometimes $\frac{9}{8}$ (e.g.$C-D$) and sometimes $\frac{10}{9}$, (e.g. $D-E$), whereas one semitone (e.g. $B-C$) is $\frac{16}{15}$.

The interval between two notes, say C and D, is divided into two semitones which, in the natural scales, are not exactly equal in size since $\left(\frac{16}{15}\right)^2 \neq \frac{9}{8}$ or $\frac{10}{9}$.

The function $LI_{12} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $LI_{12} : (a, b) \mapsto \log_{2^{\frac{1}{12}}} \frac{b}{a}$ gives the interval in semitones.

In the scale of C Major the 7 semitones from C to G form the interval $\frac{3}{2}$ whereas the 7 semitones from D to A form the interval $\frac{\frac{5}{3}}{\frac{9}{8}} = \frac{40}{27} < \frac{3}{2}$. Thus modulating to the scale of D reults in a slightly flat fifth which is therefore out of tune.

The Solution: The Tempered Scale

A keyboard instrument in which the scales in all keys have equal intervals would solve this problem. The tempered scale was thus constructed in which there are 12 equal semitones in an octave. These intervals are obtained by placing eleven geometric means between 1 and 2. Since $2^{\frac{1}{12}}$ is irrational, none of the intervals (except the octave) agree with those of the natural scale. However, although the scales in all the keys are slightly out of tune, the average ear is unable to detect a discrepancy of such small dimensions.

The great advantage of the tempered scale is that the set of intervals $\{_aI_b\}$ is now a group under multiplication. Notes of the same name are octaves apart and equivalent intervals in the same class are in a ratio of $2^r : 1,\ r \in \mathbb{N}$. The interval from a to b is equivalent to that between a and $2b$. Thus $\frac{b}{a\times 2^r} \sim \frac{b}{a} \sim \frac{b\times 2^r}{a}$.

The log scale $LI(\{_aI_b\})$ gives fractions of octaves and equivalent intervals differ by an integer. The log scale $LI_{12}\left(\frac{b}{a}\right) = 12 \times LI\left(\frac{b}{a}\right)$ is the simplest and the values form a group under addition homomorphic to $C_{12} = \{0, 1, 2, 3, , 4, 5, , 6, 7, 8, 9, 10, 11\}_{x\ mod\ 12}$.

The CYCLIC GROUP C_{12}

Consider the set of consecutive notes:
$C, C^\sharp, D, D^\sharp, E, F, F^\sharp, G, G^\sharp, A, A^\sharp, B, \ldots..$ Their set of frequencies is not a group; however the set of intervals:

FIGURE 0.3. Musical intervals on a dodecahedron.

${}_C I_C, {}_C I_{C\sharp}, {}_C I_D, {}_C I_{D\sharp}, {}_C I_E, {}_C I_F, {}_C I_{F\sharp}, {}_C I_G, {}_C I_{G\sharp}, {}_C I_A, {}_C I_{A\sharp}, {}_C I_B, \ldots.$ is a group G_C. If the intervals are taken relative to any note N, then G_N will be the same group on the tempered scale. This is the great advantage of tempered intervals: Modulation produces a set of intervals compatible with the default set.

Such a group G will always be of the form:

$\{2^0, 2^{\frac{1}{12}}, 2^{\frac{2}{12}}, 2^{\frac{3}{12}}, 2^{\frac{4}{12}}, 2^{\frac{5}{12}}, 2^{\frac{6}{12}}, 2^{\frac{7}{12}}, 2^{\frac{8}{12}}, 2^{\frac{9}{12}}, 2^{\frac{10}{12}}, 2^{\frac{11}{12}}, \ldots\}$ under multiplication and is a subgroup of $(\mathbb{Z}, \times)$.

A Homomorphism on the Set of Intervals

The homomorphism $\phi : 2^{\frac{n}{12}} \mapsto n \bmod 12$ maps the infinite set G of intervals to their equivalence classes $\{{}_C I_C, {}_C I_{C\sharp}, {}_C I_D, \ldots, {}_C I_B\}$. The kernel $\mathrm{Ker}(\phi)$ is the set consisting of a whole number of octaves. Since G is Abelian, a subgroup is normal. The isomorphism theorems imply that
i) $\phi(G)$ is isomorphic to C_{12}, the set of rotational symmetries of the regular dodecagon;
ii) every normal subgroup of C_{12} corresponds to a normal subgroup of G. Such a subgroup and its cosets represent equivalent chords.

The points on a dodecagon, starting from zero, that are $\frac{2\Pi}{12}$ $\frac{2\Pi}{6}$ and $\frac{2\Pi}{4}$ apart form subgroups of C_{12}. These are C_{12}, $\{0, 2, 4, 6, 8, 10\}$ and $\{0, 3, 6, 9\}$ respectively. The cosets of the latter are the chords of Diminished seventh.

Finally we can define a mapping $f : G \longrightarrow \mathbb{Z}$ by

$f : 2^{\frac{n}{12}} \mapsto \lfloor \frac{n}{12} \rfloor.$

The first twelve elements of the cyclic group G are mapped onto 0, the second twelve elements of the cyclic group mapped onto 1, and so on. The range $\{\lfloor \frac{n}{12} \rfloor\}$ corresponds to the octave above the default (that of middle C) to which the interval $2^{\frac{n}{12}}$ belongs. This mapping, however, is **not** a homomorphism since $f\left(2^{\frac{23}{12}} + 2^{\frac{9}{12}}\right) \neq f\left(2^{\frac{23}{12}}\right) + f\left(2^{\frac{9}{12}}\right)$.