# A Reconstruction Game

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Abstract

We propose a game in which the number of players is 2 - the robber and the detective. The detective is not after revealing the identity of the robber but in disclosing what the robber stole, given some hints by the robber himself. The winner is the robber if the detective fails to reveal the stolen property; otherwise the detective wins. We apply this to Ulam's Reconstruction Conjecture, a problem which is still open and which states that for a graph of order three or more, it is possible to reconstruct the original graph $G$ from the deck of one vertex-deleted subgraphs of $G$.

## What is a Graph?

Let us begin by considering the figure below:

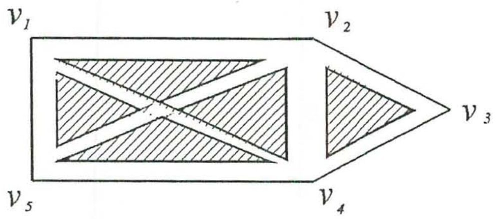


FIGURE 0.4. A road map

It is clear that it can be represented diagrammatically by means of points and lines as in Figure 0.5 below.

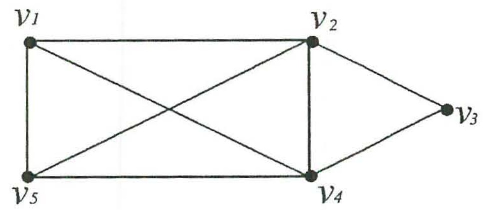


FIGURE 0.5. The corresponding graph

The points $\{v_1, v_2, \ldots, v_n\}$ are called **vertices** and the lines are called **edges**; the whole diagram is called a **graph**. Usually, we stick to the following notation:
In general a graph $G$ has a set $V(G) = \{v_1, v_2, \ldots, v_n\}$ of $n$ vertices and an edge set $E(G)$ of $m$ edges such that every edge joins a pair of distinct vertices. The **degree** or **valency** of a vertex is the number of edges which have that vertex as an endpoint and corresponds in figure 1.1 to the number of roads at an intersection. thus the degree of the vertex $v_2$ is 4. If all the verices have tha same valency $r$, then $G$ is said to be **regular**.

**What is the Reconstruction Game?**

Consider the following game with the following rules:

(1) Number of players is 2 - the robber and the detective. Strangely enough, in our case the detective is not after revealing the identity of the robber but in disclosing what the robber stole, given some hints by the robber himself!
(2) The winner is the robber if the detective fails to reveal the stolen property; otherwise the detective wins.

Let's say that the stolen property is the graph $G$ given below:

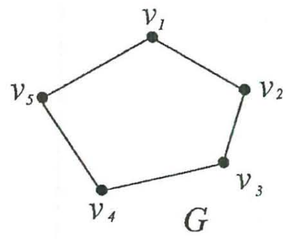


FIGURE 0.6. The stolen property

and the hints given to the detective are the five subgraphs (**cards**) below , commonly known as the **deck** $D(G)$ of $G$. Each card is obtained by stepwise removing $v_1$, $v_2$, ..., $v_5$ and any adjacent edges from $G$.

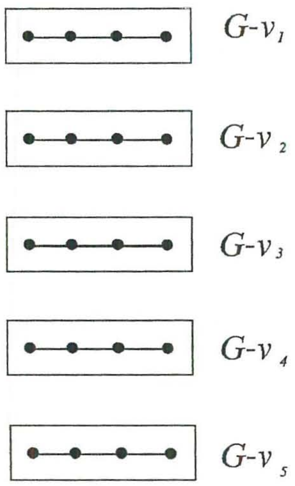


FIGURE 0.7. Deck of cards

The basic question, which is Ulam's Reconstruction Game, is very simple indeed: *is it possible to reconstruct the original graph G ?* In this particular case, the answer is yes. The method how to go about it is as follows:

From the given information deck, we deduce that $n = 5$. Also, the number of edges in each card is 4 so that the same number of edges are deleted with each vertex $v_i$. Thus the parent graph $G$ is regular. Hence, it is clear that regularity of $G$ is recognisable from $D(G)$. To recover $G$, it suffices to add a vertex to any one of the subgraphs in the deck and join it to those vertices having the minimum degree.

Hence, in this case the detective is the winner. The case for regular graphs is very simple but the problem has proved to be very difficult for the arbitrary graph and is still open for about half century of history. In mathematical terms, the game is called "Ulam's Reconstruction Conjecture" and it reads as follows:

*Every graph with at least 3 vertices is reconstructible.*

It is clear that we consider $n \geq 3$ because the problem fails for $n = 2$.

### What is the Polynomial Reconstruction Game?

Now there exists a parallel reasoning using polynomials, rather than graphs. The conjecture is called "The Polynomial Reconstruction Conjecture" and it is a variant of Ulam's reconstruction conjecture originated by D. Cvetković in 1973. It states that:

*Every graph with at least 3 vertices is polynomial reconstructible.*

Equivalently, for $n \geq 3$, given a **p-deck** $PD(G)$ of $n$ cards, each showing a characteristic polynomial $\phi(G - v; \lambda)$ as $v$ runs through the $n$ vertices of $G$, the characteristic polynomial $\phi(G; \lambda)$ can be recovered.

Even in this case, the problem is still open in general, although it has been solved for some classes of graghs such as regular graphs. Hence we will reconsider the previous game, this time using polynomials rather than graphs.

But let us first define what we understand by polynomial:

The **adjacency matrix** of a graph $G$ with vertex set $v_1$, $v_2$, ..., $v_n$ is the (0,1)symmetric $n \times n$ matrix $A(G) = (a_{ij})$ whose $(ij)$-entry $a_{ij}$ is equal to the number of edges the vertex $v_i$ to the vertex $v_j$. As an example, the adjacency matrices of the graph in Figure 1.2 and its vertex deleted subgraph are given below:

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A(G - v_i) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The **characteristic polynomial** of a graph $G$ is defined to be the characteristic polynomial of its adjacency matrix $A = A(G)$ and if $I$ denotes the identity matrix, then

$$\phi(G) = \phi(G; \lambda) = |(\lambda I - A)|$$

is a polynomial $\sum_{i=0}^{n}(a_i \lambda^{n-1})$ with integer coefficients $a_i$.

The characteristic polynomials of $G$ and $G - v_i$ are $x^5 - 5x^3 + 5x - 2$ and $x^4 - 3x^2 + 1$ respectively.

A useful result which enables the recovery of most of the terms of the characteristic polynomial of the parent graph $G$ from the $PD(G)$ is the following:

THEOREM 0.0.4.

$$\phi'(G;\lambda) = \sum_{PG} \phi(G - v_i; \lambda)$$

Thus by integrating the previous result, we obtain $\phi(G;\lambda)$, save for the constant term. This can be checked out by adding up $x^4 - 3x^2 + 1$ for all 5 subgraphs and then integrating with respect to $x$ to obtain the characteristic polynomial of $G$ save for the constant $-2$. Thus a boundary condition is required to determine $\phi(G;\lambda)$ completely.

It is interesting to point out that a positive answer to the Polynomial Reconstruction Game would imply the validity of Ulam's conjecture. But this approach depends on the resolution of a major problem: which graphs are determined by their spectrum? Unfortunately, all non trivial graphs known at present to be characterised by their spectra are regular, while Ulam's conjecture is trivially true for regular graphs. Thus it would be interesting to find some non trivial classes of non regular graphs which are characterised by their spectra.

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30th October

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