Divisibility Magic

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Abstract

To check whether a number is divisible by 9 (or 3), it is enough to check whether the sum of its digits is a multiple of 9 (or 3). More generally, if numbers are represented with base b, then the divisors of $b-1$ will have this property.

Introduction In a letter to a local newspaper [1], a reader[9] noted that the number 9 has the following "magical" property: if a number is a multiple of nine then the sum of its digits is itself a multiple of nine. He asked what the mathematical explanation of this phenomenon is.

The property mentioned, together with its converse, are well-known facts. To test an integer for divisibility by 9 we sum the digits and check whether the answer obtained is a multiple of nine. If the digit sum is still very large, the process may be repeated until a two-digit integer is obtained.

However, nine is not the only integer with this property. If we take any multiple of three, then the sum of its digits is also a multiple of three. Similarly, if the digit sum of any number is a multiple of three, then the number itself is a multiple of three.

Since the representation of a number as a string of digits depends on the base chosen, the following question immediately arises: will 3 and 9 still have the same property if numbers are represented in a different base?

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[9]See p.12

A quick check reveals that this is not so. For example, the number written as 45 in base 10 will be written as 140 in base 5 and 55 in base 8. In both cases, the digit sum is not a multiple of nine (or of three).

Clearly, the numbers 9 and 3 have this magical property only when integers are represented in base 10. We formulate two questions which shall be answered in the rest of this talk.

(i) Why do the numbers 3 and 9 have the above property in base 10?
(ii) Which numbers (if any) have the same property in other bases?

Main Theorem In the sequel, we shall only consider positive integers.

Let us start with a definition. Recall that the number a is represented by the string of digits $a_n a_{n-1} \ldots a_0$ in base b if $a = \sum_{k=0}^{n} a_k b^k$.

Definition : The integer c has the **sum-of-digits divisibility property in base** b if for any integer a,

$$c|a \Leftrightarrow c|\sum_{k=0}^{n} a_k, \text{ where } a = \sum_{k=0}^{n} a_k b^k.$$

The theorem we shall prove is:
Theorem : The integer c has the sum-of-digits divisibility property in base b if and only if $c|(b-1)$.

Proof of Sufficiency for Base 10 Let us start with the proof for the numbers 3 and 9 i.e. in base 10.

If $a = a_n a_{n-1} \ldots a_0$ then

$$\begin{aligned} a &= \sum_{k=0}^{n} a_k 10^k \\ &= \sum_{k=0}^{n} a_k (9+1)^k \\ &= \sum_{k=0}^{n} a_k (9^k + k9^{k-1} + \cdots + k9 + 1) \\ &= \sum_{k=0}^{n} a_k (9^k + k9^{k-1} + \cdots + k9) + \sum_{k=0}^{n} a_k \\ &= 9\sum_{k=0}^{n} a_k (9^{k-1} + k9^{k-2} + \cdots + k) + \sum_{k=0}^{n} a_k \end{aligned}$$

Hence 9 (or 3) divides a if and only if 9 (or 3) divides the digit sum $\sum_{k=0}^{n} a_k$. In fact, we have proved a slightly stronger result: the remainder left when a number is divided by nine is the same as the remainder left when the digit sum is divided by nine. In the language of modular arithmetic, $a \equiv \sum_{k=0}^{n} a_k$ (mod 9).

Proof of Sufficiency in an Arbitrary Base The above proof can be adapted very easily to an arbitrary base b. Then

$$\begin{aligned} a &= \sum_{k=0}^{n} a_k b^k \\ &= \sum_{k=0}^{n} a_k \left((b-1)+1\right)^k \\ &= \sum_{k=0}^{n} a_k \Big((b-1)^k + k(b-1)^{k-1} + \cdots + k(b-1) + 1\Big) \\ &= \sum_{k=0}^{n} a_k \Big((b-1)^k + k(b-1)^{k-1} + \cdots + k(b-1)\Big) + \sum_{k=0}^{n} a_k \\ &= (b-1)\sum_{k=0}^{n} a_k \Big((b-1)^{k-1} + k(b-1)^{k-2} + \cdots + k\Big) + \sum_{k=0}^{n} a_k \end{aligned}$$

Once again it is clear that $b-1$ divides a if and only if $b-1$ divides the digit sum $\sum_{k=0}^{n} a_k$. Furthermore, any number c will possess the same property if c divides $b-1$. As before, if $c|(b-1)$ then $a \equiv \sum_{k=0}^{n} a_k \pmod{c}$.

For example, in base 8, it is the number 7 which has the "magical" divisibility property, while in base 13, the numbers 2, 3, 4, 6 and 12 will have this property.

Proof of Necessity To prove necessity, it is enough to show that if c does not divide $(b-1)$, then there is *either* at least one multiple of c whose digit sum is not a multiple of c, *or* at least one non-multiple of c whose digit sum is a multiple of c.

Two cases arise:

(a) if $c < b$, then let a be a two digit number such that
  (i) the first digit is 1, and
  (ii) the second digit is $c-1$.
  Then the sum of digits is c, but the number is $b+(c-1) = c+(b-1)$ which is not a multiple of c because $b-1$ is not a multiple of c. [2]

(b) if $c \geq b$, then the digit sum of c is positive but strictly less than c, and hence cannot be a multiple of c.

Hence we prove that only those numbers which divide $b-1$ have the sum-of-digits divisibility property in base b.

Taking it Further The proof for divisibility by 9 may be adapted to obtain a proof for the divisibility test for multiples of 11. This test consists of finding the alternating sums of the digits i.e. sum the 1st, 3rd, 5th etc. digits and then sum the 2nd, 4th, 6th etc. digits. If these two sums are equal or differ by a multiple of 11, then the number is a multiple of 11. The proof of this is left as an exercise.

The more adventurous readers may even develop tests for divisibility by 7 and 13 in the same fashion.

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References 1. M. Pace, "Any Maths Wizards?" in *The Times*, 28th January 1999.

2. J. Muscat, private communication.