Equivalent Intervals

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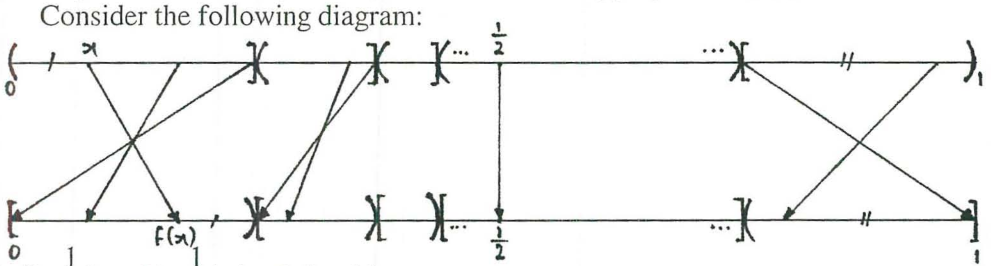
To show that the open interval (0,1) and the closed interval [0,1] are equivalent.

Problem posed by Mr. James Borg.

Proof: Required to find a bijection (1 to 1 and onto mapping) from (0,1) to [0,1].

Consider the following diagram:



Let f: $(0, \frac{1}{2}) \rightarrow [0, \frac{1}{2})$ be defined by

$$f: x \mapsto \left| \begin{array}{ll} \frac{1}{2^2} - x & x \in (0, \frac{1}{2^2}] = I_1 \\ \frac{1}{2^2} + (\frac{1}{2^2} + \frac{1}{2^3} - x) & x \in (\frac{1}{2^2}, \frac{1}{2^2} + \frac{1}{2^3}] = I_2 \\ \frac{1}{2^2} + \frac{1}{2^3} + (\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - x) & x \in (\frac{1}{2^2} + \frac{1}{2^3}, \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}] = I_3 \\ \quad \text{etc.} & \end{array} \right.$$

In general, for $x \in (\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k}, \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{k+1}}]$, $k \geq 2$, we have:

$$f(x) = 2(\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k}) + \frac{1}{2^{k+1}} - x = 1 - \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}} - x$$

and for $x \in (0, \frac{1}{2^2}]$ we have $f(x) = \frac{1}{2^2} - x$.

The first thing to note is that we have partitioned (0,1) into an infinite number of intervals I_k. This is because the sequence $(\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k}) = (\frac{1}{2} - \frac{1}{2^k}) \rightarrow \frac{1}{2}$ as $k \rightarrow \infty$. Also, we are practically mapping an interval of type (a,b] onto an interval of type [a,b) by mapping $x \in (a,b]$ onto $a + (b - x)$; in particular b is mapped onto a. Secondly, one can easily realize that f is well-defined and injective because, on any interval I_k, a unique x value is mapped onto a unique y value. It remains to prove that f is onto. Let $y \in [0, \frac{1}{2})$. If $y \in [0, \frac{1}{2^2})$ then there exists $x \in (0, \frac{1}{2^2}]$ such that $f(x) = y$. Otherwise, there exists a $k \geq 2$ such that $(\frac{1}{2} - \frac{1}{2^k}) \leq y < (\frac{1}{2} - \frac{1}{2^{k+1}})$. So y should lie in an interval $[\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}})$, but f is defined such that there exists an interval $I_k = (\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}}]$ which is mapped onto $[\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}})$, i.e. there exists $x \in I_k$ such that $f(x) = y$.

So we have mapped $(0, \frac{1}{2})$ onto $[0, \frac{1}{2})$. Similarly, we can map $(\frac{1}{2}, 1)$ onto $(\frac{1}{2}, 1]$. And finally, we map $\frac{1}{2} \in (0,1)$ onto $\frac{1}{2} \in [0,1]$. The proof is hence complete.

A second solution by James Borg.

If f maps the irrationals to themselves identically, the rest of the intervals $(0, 1)$ and $[0, 1]$ are equivalent since they are countable.

A third solution by Vincent Mercieca.

This solution is similar to that of Peter Borg. The intervals are divided into sub intervals with end points expressed in Ternary form.

Let f map the interval $[0, 1]$ to $(0, 1)$.

The interval $[0, 1]$ is divided into subintervals $[0, 0.1)$, $[0.1, 0.11)$, $[0.11, 0.111)$,. . . $(0.2, 1)$, $(0.12, 0.2])$, $(0.112, 0.12]$,. . . Now f(0.1)=0.11, f(.11)=0.111, . . . and f(0.2)=0.12, f(.12)=0.112, . . .

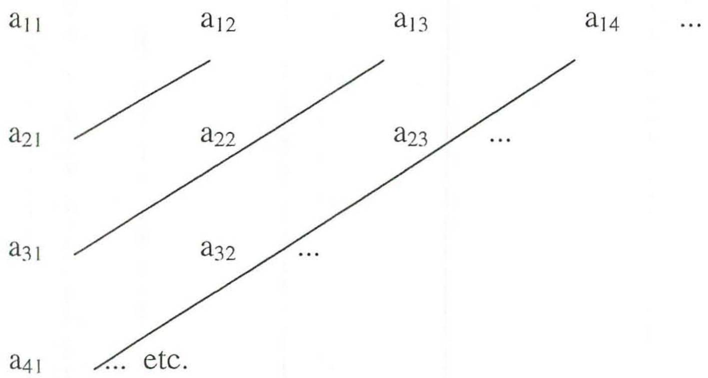
All other points are mapped identically.

Proof (1a):

Let $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$, $A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$, ..., $A_k = \{a_{k1}, a_{k2}, a_{k3}\ldots\}$, ...

For all i, A_i is a countable set. $A = A_1 \cup A_2 \cup A_3 \cup \ldots$ is a countable union of countable sets.

Hence the elements of A can be listed in the following way:



Let f: $A \to N$ be defined by $f : a_{ij} \mapsto \sum_{n=0}^{i+j-2}(n) + i = \frac{(i+j-2)(i+j-1)}{2} + i$, where $i, j > 0$.

It is required to prove that f is well-defined, one-to-one, and onto, i.e. f is bijective.

f is obviously well-defined since $f(a_{ij})$ can take only one value.

To prove f is one-to-one suppose that $f(a_{ij}) = f(a_{pq})$. Hence $\sum_{n=0}^{i+j-2}(n) + i = \sum_{n=0}^{p+q-2}(n) + p$.

Suppose $(i + j) \neq (p + q)$. Therefore either $(i + j) > (p + q)$ or $(i + j) < (p + q)$, but it is enough to just consider $(i + j) > (p + q)$. $f(a_{ij}) = f(a_{mn}) \Rightarrow p - i = \sum_{n=0}^{i+j-2}(n) - \sum_{n=0}^{p+q-2}(n)$. Hence $p - i = \sum_{n=p+q-1}^{i+j-2}(n) \geq p + q - 1$. So $1 - i \geq q$. But $i \geq 1$ $(i > 0)$, and hence $0 \geq q$. This is a contradiction since $q \geq 1$ $(q > 0)$. So $(i + j) = (p + q)$, and from $\sum_{n=0}^{i+j-2}(n) + i = \sum_{n=0}^{p+q-2}(n) + p$ it follows that $i = p$. Hence $j = q$. So $a_{pq} = a_{ij}$.

To prove f is onto let us consider any natural number k. We need to find i and j such that $f(a_{ij}) = k$, where $i, j > 0$. Let m be such that $\sum_{n=0}^{m} n = \frac{m(m+1)}{2} < k \leq \sum_{n=0}^{m+1} n = \frac{(m+1)(m+2)}{2}$.

Hence $0 < i = k - \sum_{n=0}^{m} n \leq m + 1$. Let $j = m - i + 2$ (i.e. $m = i + j - 2$). Since $m - i \geq -1$

then $j = (m - i) + 2 \geq -1 + 2 = 1$. Hence $j > 0$. So we have found i and j such that

$$f(a_{ij}) = \sum_{n=0}^{i+j-2}(n) + i, \text{ where } i, j > 0 \text{ as required.}$$

However, it was not necessary to prove onto in order to prove that A ~ N. First of all, for all i N ~ A_i. Suppose f was not onto N but onto N', an infinite proper subset of N, then we have that A ~ N' $\subset$ N $\subset$ A. In fact, by definition, an infinite set is equivalent to a proper subset of itself. So A must be equivalent to N.

Proof 1(b):

Let $a_{ij} = {}^{j}/_{i}$ in proof 1(a). The denominators in *Q* are elements of *N* and for all denominator $i \in N$ there exists a set A_i which is the set of all positive rationals with denominator i. Hence the union $A = A_1 \cup A_2 \cup A_3 \cup \ldots$ covers all the denominators and hence forms the set of all positive rationals. Hence by result 1(a) Q^+ is countable. The set Q^- of negative rationals is equivalent to the set of positive rationals, hence also countable. Again, by result 1(a), $Q = Q^+ \cup Q^-$ is countable.

2. (a) The set of real numbers, *R*, is uncountable.
   (b) The set of irrational numbers, *I*, is uncountable.

Proof 2(a):

Suppose *R* is countable. Therefore *R* is equivalent to *N*, and hence its elements can be listed. Let us just consider the real interval [0,1]. Hence let the elements in [0,1] be listed as $\{a_1, a_2, a_3, \ldots\}$. Also let each rational element in [0,1] be written in its infinite decimal expansion, e.g. 0.5 = 0.4999... Hence we have the following list:

$$a_1 = 0.a_{11}\, a_{12}\, a_{13}\, a_{14}\ldots$$

$$a_2 = 0.a_{21}\, a_{22}\, a_{23}\, a_{24}\ldots$$

$$a_3 = 0.a_{31}\, a_{32}\, a_{33}\, a_{34}\ldots$$

etc.

Let $b = 0.b_1b_2b_3\ldots$ be a real number in [0,1] such that $b_i = 1$ if $a_{ii} = 0$ and $b_i = 0$ if $a_{ii} = 1$. Hence b is not in the set $\{a_1, a_2, a_3, \ldots\}$. This is a contradiction and [0,1] is therefore uncountable. So, obviously, *R* is uncountable.

Proof 2(b):

$R = Q \cup I$. Suppose I is countable. Again, by result 1(a), this implies that R is countable since Q is also countable. So this is a contradiction and I is therefore uncountable.

3. (a) N has measure 0. Q (or any countable set) has measure 0.

(b) $I \cap [0,1]$ has measure 1.

Some properties of Measure:

- The measure of the empty set $\varnothing$ is 0.
- For any real interval [a,b], b > a, the (Lebesgue) measure is given by (b - a).
- Let M_A denote the measure of set A. If $A = B \cup C$ then $M_A = M_B + M_C - M_{B \cap C}$.

Proof 3(a):

Let $n \in N$ be covered by a real interval of radius $\varepsilon/(2^n)$, i.e. $[n - \varepsilon/(2^n), n + \varepsilon/(2^n)]$. For any $\varepsilon > 0$, all natural numbers are covered. Taking all the covers we get that the measure of N is less than $2\sum_{n=0}^{\infty}\frac{\varepsilon}{2^n} = 2\varepsilon\left(\frac{1}{1-1/2}\right) = 4\varepsilon$. We can let ε tend to 0 without uncovering any natural number, whereby the measure tends to 0. Hence N has 0 measure.

Since (by definition) any countable set is equivalent to N, then any countable set has 0 measure. In particular, Q has 0 measure.

Proof 3(b):

The (Lebesgue) measure of the real interval [0,1] is given by 1 - 0 = 1. By definition, $I = Q^c = R \setminus Q$ (R without Q), i.e. the set of all real numbers which are not rational (irrational). $R = I \cup Q$ and $I \cap Q = \varnothing$ (empty set).

$[0,1] = [0,1] \cap R = [0,1] \cap (I \cup Q) = ([0,1] \cap I) \cup ([0,1] \cap Q)$.

Let $A = [0,1]$, $B = [0,1] \cap I$, $C = [0,1] \cap Q$. $B \cap C = \varnothing$. Hence $M_{B \cap C} = 0$. Also, since C is countable, $M_C = 0$. Therefore $M_A = M_B + M_C - M_{B \cap C} = M_B + 0 - 0 = 1$. So $M_B = 1$, i.e. the measure of the irrationals in [0,1] is 1.

Problem: **<u>If M is an uncountable set in [0,1], does it necessarily have measure 1??!!</u>**

Remark: This will be tackled in a future workshop.