Gödel's Theorem

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At the beginning of the 20th century the mathematician David Hilbert posed a set of problems to the mathematical community that should have been the so-called road map of tasks to accomplish during the following hundred years. Among them was a problem which he posed in collaboration with Ackermann dealing with the question of whether a formal system of mathematical logic can be considered complete – where completeness implies that every true statement can be expressed within the system, possibly without a paradox.

This was probably inspired by the recent discovery of a series of paradoxes in Russell and Whitehead's *Principia Mathematica* which is now a *de facto* standard for defining and proving mathematical statements. The well-known Russell's paradox – formulated in a hundred different ways – has been catered for by denying the possibility of having a set being a member of itself. However, other forms of paradoxes are not that easy to eliminate. Epimenides' paradox falls into this category: "I am a liar" or in logic-speak: "This statement is false".

Gödel's seminal work in 1931 not only managed to show that the PM system was inconsistent, but that any sufficiently powerful formal system is bound to be littered with paradoxes. It is worth stating how series this matter is: practically speaking he stated that there might exist theorems that cannot be proved or disproved – theorems about number theory itself, for instance.

The approach to Gödel's proof I am going to use is a simplified version based on the work of Douglas R. Hofstadter, "Gödel, Escher, Bach: an Eternal Golden Braid". A book which I thoroughly recommend to anyone interested in the question of how animate matter can result out of combinations of inanimate matter.

INTRODUCING TNT

TNT stands for typographical number theory and it is basically an arbitrary formal system that is sufficiently complete for our purposes. The reason for which we shall be working in TNT rather than PM or any other established formal system is to emphasise the point that Gödel's theorem can be applied to ANY system whatsoever and still result in a paradox – or contradiction in our case.

The tools:

Logical operators:

A	-	for all
E	-	there exists
V	-	or
Λ	-	and
~	-	not
:	-	such that

Mathematical operators:
+, *, =

Variables:
a, a', a'', a''', ...

Number system:
0, S0, SS0, SSS0, ...

The axioms:

Axiom 1: Aa:~Sa=0
Axiom 2: Aa:(a+0)=a
Axiom 3: Aa:Aa':(a+Sa')=S(a+a')
Axiom 4: Aa:(a*0)=0
Axiom 5: Aa:Aa':(a*Sa')=((a*a')+a)

Without going into exceptional detail it is just sufficient to know that with these axioms it is possible to express any statement in number theory. In fact we are going to **assume** that fact in order to prove the theorem.

The task at hand is to import the statement "This is not a theorem of TNT" into TNT, possibly in a universal manner that is applicable to all formal systems.

GÖDEL NUMBERING

A first step in this direction is to introduce the revolutionary idea of replacing each symbol of the formal system by a number. You might ask yourself what is so revolutionary about such a change in notation. On one hand it represents an interesting way of making statements about numbers by using numbers that is going to be vital in order to prove this theorem, on the other it paved the way for the of Alan Turing and subsequently the invention of computers.

To give you an example, if we choose '123','666','434' and '000' to represent a, =, S and 0 respectively then the statement a=S0 becomes 123666434000.

Now we can use mathematical operations rather than string manipulations in order to express a new theorem.

THE CONCEPT OF THEOREMHOOD

I am now going to give two definitions of theoremhood that we shall use to come up with our final G sentence:

Definition: A number has *theoremhood* if it corresponds to a valid theorem of TNT—or, in other words, to a true statement about numbers.

Alternative definition: A number has *theoremhood* if it is possible to create that number from our small set of axiom-numbers, by the application of our small set of function-rules.

ARITHMOQUINING

This will be our next and final tool for the job. This is basically a way of expressing a theorem as part of itself – again a form of recursion. The method is quite simple: replace every occurrence of the variable 'a' by the Gödel number of the entire sentence.

Example: a statement like a=S0 has Gödel number 123666434000 (using our previous notation) hence the arithmoquined version would be 123666434000=S0 and the Gödelised form would be 123666434000666434000.

PROVING GÖDEL

All we need now is a statement about the impossibility of expressing the statement in TNT whose Gödel number happens to be the number of the sentence. Without further ado I'm going to give you this statement and allow you to ponder upon it on your own.

The arithmoquine of "The arithmoquine of a is not a valid TNT theorem-number" is not a valid TNT theorem-number.

SO WHAT?

I have already explained the implications of this theorem when it comes to logic and provability of theorems. However it is worth noticing that this theorem has some very interesting philosophical implications dealing with the way we think and the possibility / impossibility of artificial intelligence.

Bibliography:

Gödel, Escher, Bach - Douglas R. Hofstadter

Kurt Gödel – Collected Works Volume I (1929-1936)

Kenny's Overview of Hofstadter's Explanation of Gödel's Theorem – Kenny Felder

The Emperor's New Mind - Roger Penrose