Boolean Rings

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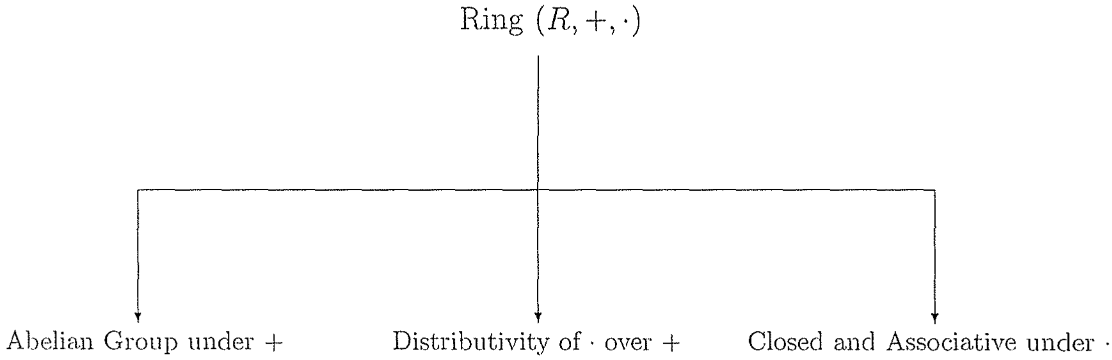

Figure 1: The definition of a Ring.

Definition of a Ring:
A *ring* is a triple comprising a set R and two binary operations $+$ and $\cdot$ satisfying the following properties (refer to Figure 1):

1. R is an Abelian group under $+$
2. R is closed and associative under $\cdot$
3. $\cdot$ is distributive over $+$

Remark: We write ab for $a \cdot b$ and x^2 for $x \cdot x$.

Definition of a Boolean Ring:
R is said to be a *Boolean Ring* if $x^2 = x \ \forall x \in R$

Theorem 1 *Let R be a Boolean Ring. Then* $\forall x \in R, -x = x$

Proof: It can be proved that if R is a ring, then $\forall a, b \in R, (-a)(-b) = ab$ and $(-x)^2 = (-x)(-x) = (x)(x) = x^2$.

$$\begin{aligned} \text{From the definition of a Boolean Ring, } x^2 &= x \\ \text{Thus } (-x)^2 &= -x \\ \text{But } (-x)^2 &= x^2 \\ \Rightarrow x &= -x, \text{ as required.} \end{aligned}$$

Theorem 2 *Let R be a Boolean Ring. Then R is commutative under* $\cdot$

Proof: Let $x, y \in R$. We need to show that $xy = yx$.

$$\begin{aligned}(x+y)(x+y) &= (x+y) \text{ from } x^2 = x \\ x^2 + xy + yx + y^2 &= x + y \\ \text{But } x^2 = x \ &, \ y^2 = y \\ \Rightarrow x + y + xy + yx &= x + y \\ \Rightarrow xy + yx &= 0 \\ \Rightarrow yx &= -xy \\ \text{But } x &= -x \text{ from Theorem 1} \\ \text{Hence } yx &= xy, \text{ as required.}\end{aligned}$$

Theorem 3 *Let R be a Boolean Ring. Then R is a field* $\iff R = \{0, 1\}$

Proof: ($\Longrightarrow$) Let R be a field and let $x \neq 0$ be in R. We need to show that $R = \{0, 1\}$.

Since R is a field, x has an inverse.
Also $x^2 = x$ since R is also a Boolean Ring.
Premultiplying both sides by x^{-1}, we get $x^{-1}x^2 = x^{-1}x \Rightarrow x = 1$

Hence if $x \neq 0$, $x = 1$. Therefore, $R = \{0, 1\}$, as required.

($\Longleftarrow$) Let $R = \{0, 1\}$. We need to show that R is a field.

It can be shown that any field has only two ideals, $\{0\}$ and itself.

Now in R the possible ideals are $\{0\}$, $\{1\}$ and $\{0, 1\}$.

- Is $\{0\}$ an ideal?

  Subgroup under + $0 \pm 0 = 0$ (closure and inverse)
  Absorption under · $0 \cdot 1 = 0$
  Hence $\{0\}$ is an ideal.

- Is $\{1\}$ an ideal?

  Absorption under · $0 \cdot 1 = 0$, hence absorption does not hold.
  Hence $\{0\}$ is NOT an ideal.

- Is $\{0, 1\}$ an ideal?

  Subgroup under + Follows since R is a ring.
  Absorption under · Follows since R is a ring.
  Hence $\{0, 1\}$ is an ideal.

Therefore, the only ideals of R are $\{0\}$ and R. Hence R is a field, as required.