Breeding Rabbits

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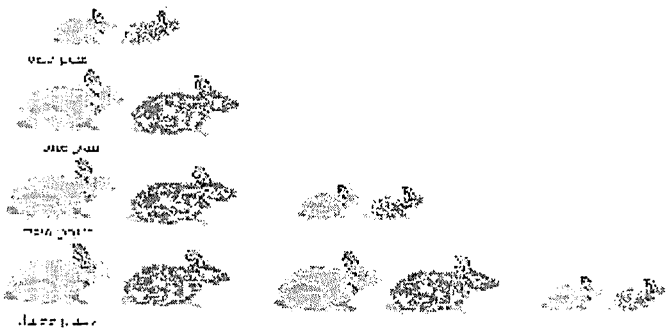
It is known that the Fibonacci sequence, which has been found to apply so often in nature, is defined by $a_{n+1} = a_n + a_{n-1}$ with the following suffices initial conditions $a_0 = 0$ and $a_1 = 1$.

> Problem: How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair, which becomes productive from the second month onwards?



It is easy to see that 1 pair will be produced in the first month, and 1 pair also in the second month (since the new pair produced in the first month is not yet mature). While in the third month 2 pairs will be produced, one by the original pair and one by the pair which was produced in the first month. In the fourth month 3 pairs will be produced, and in the fifth month 5 pairs. After this things expand rapidly, and we get the following sequence of numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

This is an example of a recursive sequence, obeying the simple rule that to calculate the next term one simply sums the preceding two.

Recurrence Relation

The Fibonacci sequence can be described by the recurrence relation which can be represented in matrix form as shown below.

$$\begin{pmatrix} a_{G+1} \\ a_G \end{pmatrix} = \mathbf{A}^m \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

The matrix $\mathbf{A}$, known as the Fibonacci matrix is found to be

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Now our aim is to determine the value of m for a given parameter G.
It is found by induction that $m = G + 1$
Now to verify this result for m, let us take an example for $G = 3$. Hence $m = 4$.

Therefore the recurrence relation becomes

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now the Fibonacci matrix to the power of 4 is

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

which agrees with the required formula.

We can also use the software Mathematica to find the eigenvalues and their corresponding eigenvectors for the Fibonacci matrix. The two commands used in Mathematica are

Eigenvalues[A]

Eigenvectors[A]

Hence the minimum eigenvalue is $\frac{1}{2}(1 - \sqrt{5})$ with the corresponding minimum eigenvector $\left\{ \begin{matrix} \frac{1}{2}(1 - \sqrt{5}) \\ 1 \end{matrix} \right\}$, while the maximum eigenvalue is $\frac{1}{2}(1 + \sqrt{5})$ with $\left\{ \begin{matrix} \frac{1}{2}(1 + \sqrt{5}) \\ 1 \end{matrix} \right\}$ as a maximum eigenvector.

According to theory $P^{-1}AP = D$.
But this is equivalent to $AP = PD$

Where D is the diagonal matrix.

Now the diagonal matrix $D = \begin{pmatrix} \frac{1}{2}(1 + \sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1 - \sqrt{5}) \end{pmatrix}$

And the matrix $P = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{pmatrix}$

Hence $AP = PD$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}$$

$$\begin{pmatrix} 1+\frac{1}{2}(1+\sqrt{5}) & 1+\frac{1}{2}(1-\sqrt{5}) \\ 1+\frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(1+\sqrt{5})^2 & \frac{1}{4}(1-\sqrt{5})^2 \\ \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}$$

$$\begin{pmatrix} 2.61803 & 0.381966 \\ 1.61803 & -0.618034 \end{pmatrix} = \begin{pmatrix} 2.61803 & 0.381966 \\ 1.61803 & -0.618034 \end{pmatrix}$$

The following are the pairs of rabbits after 12, 52 and 356 months

$$a_{12} = 233$$

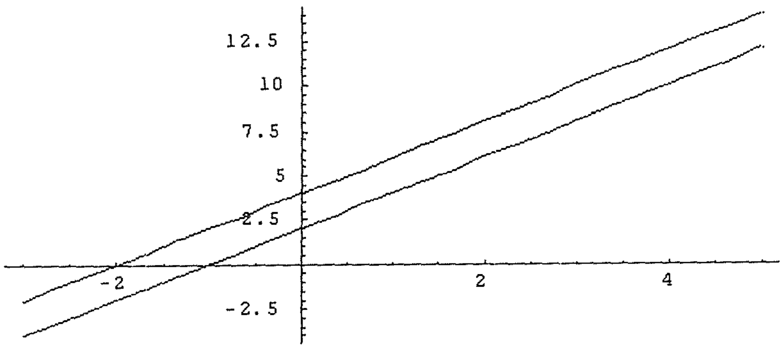
$$a_{52} = 53316291173$$

$$a_{356} = 181594448268301656413948075911105052760867948344134387820089804440720816962$$

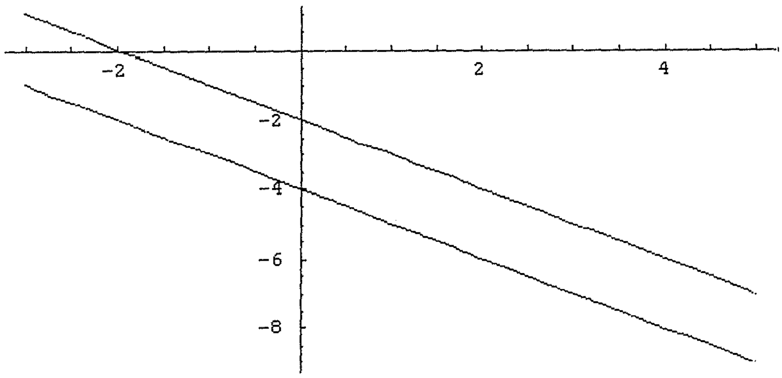
The eigenvector for the Dominant eigenvalue

An invertible matrix **M** maps two parallel lines to another two parallel lines. We illustrate this rule by an example:

let $t(x) = mx + c, h(x) = mx + c'$ be two parallel lines which can represented graphically as shown below and let $M = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$



Hence $t(x)$ and $h(x)$ are mapped to $t'(x) = -(x+4)$ and $h'(x) = -(x+2)$ respectively.



Now both have the same slope, hence these are parallel. The matrix **A** has two eigenvalues, one maximum and the other minimum. By taking powers A^n of **A** as the minimum eigenvalue λ^k_{min} decreases, the maximum eigenvalue λ^k_{max} increases. Starting with any vector $x = \alpha v_{min} + \beta v_{max}$, where v_{min} and v_{max} are eigenvectors corresponding to λ_{min} and λ_{max} respectively, $A^n(x) = (\alpha\lambda^n_{min}v_{min} + \beta\lambda^n_{max}v_{max})$ $\rightarrow \beta\lambda^n_{max}v_{max}$ as $n \rightarrow \infty$ Thus the vector x is transformed by successive application of **A** till it is on the eigenline. Hence we can conclude that $\lim_{n\rightarrow\infty} Av = kv_{max}$.

This application is used for the power method to find the dominant eigenvalue of **A**.