Optimal estimation of optomechanical couplings

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We apply the formalism of quantum estimation theory to obtain information about the value of the optomechanical coupling in the simplest model of two harmonic oscillators. In particular, we discuss the minimum mean-square error estimator and a quantum Cramér-Rao inequality for the estimation and accuracy of the coupling's value. Our estimation strategy reveals some cases, where quantum statistical inference is inconclusive and only prior expectations on the coupling strength are reassured. We show that this situations involve also the highest expected information losses. It is demonstrated that interaction times in the order of one time period of mechanical oscillations are the most suitable for this type of estimation scenario. We also compare situations involving different initial photon and phonon excitations.

I. INTRODUCTION

Quantum estimation theory seeks the best strategy of learning the value of one or more parameters of the density matrix of a quantum mechanical system [1]. The observation strategy for estimating these parameters is expressed as a positive-operator valued measure (POVM). Elements of the POVM are applied in repeated measurements on the system and the unknown parameters are estimated from the data set. The optimum strategy consists of those POVMs, which minimize the average cost functional, and is typically considered for maximum likelihood or mean square error estimators. The mathematical framework for studying the conditions under which solutions of the optimization problem exists was established by Holevo [2, 3]. The considerable theoretical and experimental developments on quantum statistical inference lead to various applications in quantum tomography and metrology [4].

Radiation pressure is defined as the pressure exerted upon a surface exposed to an electromagnetic radiation field. Therefore, the momentum transfer, inflicted by the electromagnetic radiation, can also have an effect on macroscale mechanical masses (such as cavity mirrors). One of the most prominent presence of this effect is in the laser-based gravitational wave interferometers [5], where it imposes limits on continuous position detection [6, 7]. The subject of this paper, the quantum cavity-optomechanical systems, has drawn also a lot of attention from both sides of theoretical and experimental physics [8]. The motivations are mainly centered around the perspectives of sensitive optical detection of small forces, mechanical motion in the quantum regime, and coherent light-matter interfaces for future hybrid quantum information devices. The most crucial ingredient here is the value of the coupling strength between the optical field and mechanical degrees of freedom. We simply consider that the value of this parameter is best determined from quantum inference techniques, which have been successfully applied lately to phase estimations of quantum states [9, 10]. In our case the parameter to be estimated is not a simple phase parameter, but rather a parameter which appears in the spectrum of the Hamilton operator [11].

In this paper we analyze the basic model of cavity-optomechanical systems, a single-mode of the electromagnetic radiation field coupled via radiation pressure to a vibrational mode of a mechanical oscillator (cavity mirror), subject to adiabatically slow motion of the mirror [12]. The analytical solutions to this simple model results in a density matrix, which is going to depend on the unknown value of the optomechanical coupling. This density matrix describes the joint state of single-mode field and mechanical oscillator. Measurements are usually performed on the field state, emerging from the interaction, and thus we trace out the mechanical degrees of freedom. The obtained state is going to be subject of a quantum estimation procedure. We focus here on a mean-square error estimator and assume that the prior probability density function of the optomechanical coupling is a normal distribution. We set the mean and the standard deviation of this distribution function to values obtained by a canonical quantization procedure with a high frequency cut-off of the radiation field and adiabatically slow motion of the mechanical oscillator. In order to illustrate basic features of our proposal, we consider the mechanical oscillator to be initially in a coherent state and the initial field state to have only a few excitations. We determine the mean-square error estimator which minimizes the cost functional and we study a quantum Cramér-Rao inequality of the estimator's variance. However, the mean-square error estimator which minimizes the cost functional is biased. Furthermore, the emerged field state does not have either a right logarithmic or a symmetrized logarithmic derivative with respect to the optomechanical coupling. Thus, we give a new lower bound for the variance of the estimator by using standard techniques [13].

This paper is organized as follows. In Sec. II we discuss the model of a single-mode radiation field coupled via radiation pressure to a vibrational mode of a mechanical oscillator and its solutions. In Sec. III we introduce the

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quantum estimation theory for minimum mean square error estimators and study in this context the properties of the optomechanical model. We then address the variance of the biased estimators in Sec. IV, derive a lower bound, and employ this result to the optomechanical model. A discussion about our analytical and numerical findings is summarized up in Sec. V.

II. MODEL

We consider a systems composed of two harmonic oscillators, a single-mode of the radiation field and a vibrational mode of a mechanical oscillator. Provided that the field is a high finesse cavity field and one of the mirrors is movable, one is able to derive a radiation pressure interaction operator [12, 14] by using time-varying boundary conditions in the quantization procedure. The single-mode assumption for both quantized field and mirror motion results in the following Hamiltonian ($\hbar = 1$)

$$\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b}), \tag{1}$$

where $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operator of the single-mode radiation field with frequency ω_c and $\hat{b}$ ($\hat{b}^\dagger$) is the annihilation (creation) operator of the moving mirror's mode with frequency ω_m. g is the optomechanical coupling strength.

The time evolution of the system is given by the Schrödinger equation

$$|\Psi(t)\rangle = e^{-i\hat{H}t} |\Psi(0)\rangle.$$

We are interested in a case where no initial correlations are present between the field and the mechanical oscillator. Therefore, we choose an initial state of the form

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty} a_n |n\rangle_c |\alpha\rangle_m \tag{2}$$

with the mechanical oscillator considered initially in a coherent state [15]

$$|\alpha\rangle_m = \sum_{n=0}^{\infty} e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_m, \quad \alpha = \sqrt{\bar{n}}\, e^{i\phi} \tag{3}$$

written in terms of the number states $|n\rangle_m$ ($n \in \mathbb{N}_0$) and where ϕ is the phase of the coherent state. In this section, we treat the coefficients a_n of the photon-number states $|n\rangle_c$ very general and only impose the normalization condition $\sum_n |a_n|^2 = 1$.

The interaction Hamiltonian $g\hat{a}^\dagger \hat{a}(\hat{b}^\dagger + \hat{b})$ commutes with the free Hamiltonian of the radiation field $\omega \hat{a}^\dagger \hat{a}$, which yields

$${}_c\langle n|\hat{H}|m\rangle_c = \left(n\omega_c \hat{I} + \omega_m \hat{b}^\dagger \hat{b} + ng(\hat{b}^\dagger + \hat{b}) \right) \delta_{nm} \tag{4}$$

with δ_{nm} being the Kronecker delta and $\hat{I}$ is the identity operator on the Hilbert space of the mechanical oscillator. Thus, the Hamiltonian in (1) is block-diagonal with respect to the photon-number states $|n\rangle_c$.

In order to evaluate the expression $\exp\{-i\omega_m \hat{b}^\dagger \hat{b} t - ing(\hat{b}^\dagger + \hat{b})t\}$ we employ the Baker-Campbell-Hausdorff formula and obtain (see for example [16])

$$e^{-i\omega_m \hat{b}^\dagger \hat{b} t - ing(\hat{b}^\dagger + \hat{b})t} e^{i\omega_m \hat{b}^\dagger \hat{b} t} = e^{i\Phi_n(t)} e^{\alpha_n(t)\hat{b}^\dagger - \alpha_n^*(t)\hat{b}} \tag{5}$$

where we have introduced the parameters

$$\begin{aligned} \alpha_n(t) &= \frac{ng}{\omega_m} \left(e^{-i\omega_m t} - 1 \right), \\ \Phi_n(t) &= \frac{n^2 g^2}{\omega_m^2} \left[\omega_m t - \sin(\omega_m t) \right]. \end{aligned}$$

This means that the whole time evolution can be viewed as photon-number dependent displacements of the mechanical oscillator and with the help of Eqs. (4) and (5) we get

$$\begin{aligned} |\Psi(t)\rangle &= e^{-i\hat{H}t}|\Psi(0)\rangle = \sum_{n=0}^{\infty} a_n e^{i\varphi_n(t)} |n\rangle_c |\beta_n(t)\rangle_m, \\ \varphi_n(t) &= -n\omega_c t + \frac{n^2 g^2}{\omega_m^2} \left[\omega_m t - \sin(\omega_m t) \right] \\ &+ \frac{ng}{\omega_m} \frac{\alpha^*(1 - e^{i\omega_m t}) - \alpha(1 - e^{-i\omega_m t})}{2i}, \\ \beta_n(t) &= \frac{ng}{\omega_m} \left(e^{-i\omega_m t} - 1 \right) + \alpha e^{-i\omega_m t}, \end{aligned} \tag{6}$$

where we used a corollary of the Baker-Campbell-Hausdorff formula: the product of two displacement operators is also a displacement operator with a phase factor.

The quantum state of (6) yields a complete description of the interaction between the single mode of the radiation field and the single-mode vibration of the mechanical oscillator, i.e., neglecting all decoherence sources. In the subsequent sections we are interested in possible measurement scenarios, performed on the emerged field state, which are capable to estimate the optomechanical coupling g. Therefore, the field to be measured in an estimation scenario reads

$$\hat{\rho}_F = \mathrm{Tr}_m\{|\Psi(t)\rangle\langle\Psi(t)|\} = \sum_{n,m=0}^{\infty} A_{n,m} |n\rangle_c \langle m| \tag{7}$$

with

$$\begin{aligned} A_{n,m} &= a_n a_m^* e^{i\varphi_n(t) - i\varphi_m(t) - \left(|\beta_n(t)|^2 + |\beta_m(t)|^2\right)/2 + \beta_n(t)\beta_m^*(t)} \\ &= a_n a_m^* e^{-g^2 f^{(2)}_{n,m}(t) + g f^{(1)}_{n,m}(t) - f^{(0)}_{n,m}(t)}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} f^{(0)}_{n,m}(t) &= i\omega_c t(n-m), \\ f^{(1)}_{n,m}(t) &= \frac{\alpha^*(1-e^{i\omega_m t})-\alpha(1-e^{-i\omega_m t})}{\omega_m}(n-m), \\ f^{(2)}_{n,m}(t) &= \frac{1-\cos(\omega_m t)}{\omega_m^2}(n-m)^2 \\ &- i\frac{\omega_m t-\sin(\omega_m t)}{\omega_m^2}(n^2-m^2). \end{aligned} \tag{9}$$

The above equations show that the coefficient of the linear term in g contributes to $A_{n,m}$ only when the initial state of the mechanical oscillator is not in the ground state, i.e., $\alpha = 0$.

III. QUANTUM MINIMUM MEAN-SQUARE ERROR ESTIMATION

Quantum estimation theory attempts to find the best strategy for estimating one or more parameters of the density matrix [17]. In our case, the density matrix of the field in (7) depends on the parameter g to be estimated. Any outcome of a measurement on the field is a variable with probability both depending on the estimanda g. As our knowledge is limited, we assume that the estimanda is a random variable with prior probability density function

$$p(g) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(g-g_0)^2}{2\sigma^2}} \tag{10}$$

with mean g_0 and variance σ^2. We shall return to these parameters and their physical meanings later on.

Our estimation problem is to find the best measurements on $\hat{\rho}_F(g)$ to estimate g. In practice, we are looking for a POVM whose elements are defined on the compact intervals of the real line, i.e, the set of all possible values for g, satisfying

$$0 \leqslant \hat{\Pi}(\Delta) \leqslant \hat{I}, \quad \Delta \subset \mathbb{R}, \tag{11}$$

where $\hat{I}$ is the identity operator. We also suppose that the infinitesimal operators $d\hat{\Pi}(g)$ can be formed and by thus yielding

$$\hat{\Pi}(\Delta) = \int_\Delta d\hat{\Pi}(g)$$

and

$$\hat{I} = \int_{-\infty}^{\infty} d\hat{\Pi}(g). \tag{12}$$

In order to solve the estimation problem we have to provide a cost function, a measure of the cost suffered upon making errors in the estimate of g. Here, g is to be estimated with minimum mean square error and thus the cost function is:

$$C(\tilde{g}, g) = (\tilde{g} - g)^2, \tag{13}$$

where $\tilde{g}$ is the estimate of g, a function of the measurement data.

Now, we are able to formulate the quantum estimation problem. We are looking for $d\hat{\Pi}(\tilde{g})$, which minimizes the average cost of this estimation strategy

$$\bar{C} = \mathrm{Tr}\Big\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(g) C(\tilde{g}, g)\hat{\rho}_F(g) d\hat{\Pi}(\tilde{g}) dg \Big\} \tag{14}$$

under the constraints (11) and (12). This is a variational problem for the functional $\bar{C}$ and furthermore we consider the estimate $\tilde{g}$ to be an eigenvalue of the Hermitian operator

$$\hat{M} = \int_{-\infty}^{\infty} \tilde{g} d\hat{\Pi}(\tilde{g}) = \int_{-\infty}^{\infty} \tilde{g}|\tilde{g}\rangle\langle\tilde{g}| d\tilde{g} \tag{15}$$

with eigenstates $|\tilde{g}\rangle$. This also means that we are considering only projective POVMs. Thus, the average cost functional in (14) together with the cost function in (13) yields

$$\bar{C}[\hat{M}] = \mathrm{Tr}\Big\{ \int_{-\infty}^{\infty} p(g)\big(\hat{M} - g\hat{I}\big)^2 \hat{\rho}_F(g) dg \Big\}. \tag{16}$$

We also define the following operators

$$\hat{\Gamma}_k = \int_{-\infty}^{\infty} g^k p(g)\hat{\rho}_F(g) dg, \quad k \in \{0, 1, 2\}. \tag{17}$$

Let ϵ be a real number and $\hat{N}$ any Hermitian operator. Let $\hat{M}_{\min}$ be the Hermitian operator which minimizes $\bar{C}[\hat{M}]$. Then, we have

$$\bar{C}[\hat{M}_{\min}] \leqslant \bar{C}[\hat{M}_{\min} + \epsilon\hat{N}], \tag{18}$$

because the sum of Hermitian operators is a Hermitian operator. Evaluating the right hand side of the inequality and using the operators defined in (17) we get

$$\begin{aligned} &\bar{C}[\hat{M}_{\min} + \epsilon\hat{N}] = \bar{C}[\hat{M}_{\min}] \\ &+\epsilon \mathrm{Tr}\left\{ \hat{N}\left(\hat{\Gamma}_0\hat{M}_{\min} + \hat{M}_{\min}\hat{\Gamma}_0 - 2\hat{\Gamma}_1\right)\right\} + \epsilon^2 \mathrm{Tr}\{\hat{\Gamma}_0\hat{N}^2\}. \end{aligned} \tag{19}$$

Differentiating with respect to ϵ one is able to show that the unique Hermitian operator $\hat{M}_{\min}$ minimizing $\bar{C}$ must satisfy [18]

$$\hat{\Gamma}_0\hat{M}_{\min} + \hat{M}_{\min}\hat{\Gamma}_0 = 2\hat{\Gamma}_1. \tag{20}$$

The average minimum cost of error is

$$\begin{aligned} \bar{C}_{\min} &= \mathrm{Tr}\{\hat{\Gamma}_0\hat{M}^2_{\min} - 2\hat{\Gamma}_1\hat{M}_{\min} + \hat{\Gamma}_2\} \\ &= \mathrm{Tr}\{\hat{\Gamma}_2 - \hat{M}_{\min}\hat{\Gamma}_0\hat{M}_{\min}\}, \end{aligned} \tag{21}$$

where we have used the relation in (20). In order to determine $\hat{M}_{\min}$ we have to solve the operator equation (20)and it has been shown by Ref. [18] that its solution the can be written as

$$\hat{M}_{\min} = 2\int_0^{\infty} \exp(-\hat{\Gamma}_0 x)\hat{\Gamma}_1 \exp(-\hat{\Gamma}_0 x) dx. \tag{22}$$

We would like to comment here on this solution, because the operator that we have found, it does not represents the best estimator of g rather than the measurement operator which protects the best against information loss, no matter what the true value of g is [19].

In the next step we evaluate all $\hat{\Gamma}_k$ by using the form of $\hat{\rho}_F(g)$ in (7) and obtain

$$\hat{\Gamma}_k = \sum_{n,m=0}^{\infty} a_n a_m^* A_{n,m}^{(k)} \exp\left(-\gamma_{n,m}\right) |n\rangle_c\langle m|, \quad k \in \{0,1,2\} \tag{23}$$

with

$$A_{n,m}^{(0)} = \frac{1}{\sigma'}$$
$$A_{n,m}^{(1)} = \frac{g_0 + f_{n,m}^{(1)}(t)\sigma^2}{\sigma'^3}$$
$$A_{n,m}^{(2)} = \frac{\left(g_0 + f_{n,m}^{(1)}(t)\sigma^2\right)^2 + \sigma^2\sigma'^2}{\sigma'^5}$$

where we have also introduced

$$\gamma_{n,m} = \frac{2g_0^2 f_{n,m}^{(2)}(t) - 2g_0 f_{n,m}^{(1)}(t) + 2f_{n,m}^{(0)}(t)\sigma'^2 - \left(f_{n,m}^{(1)}(t)\right)^2\sigma^2}{2\sigma'^2}$$
$$\sigma'^2 = 2f_{n,m}^{(2)}(t)\sigma^2 + 1.$$

These results are very general and in the subsequent subsections we are going to investigate some simple cases of the optomechanical model.

A. A case study

The situation can be easily understood for the case, when $a_n = 0$ for $n > 1$ in Eq. (2). In order to maximize the absolute values of the off-diagonal elements of the density matrix we choose $a_0 = a_1 = 1/\sqrt{2}$. This specific choice is due to the fact that the unknown parameter g is only present in the off-diagonal elements (see Eq. (9)). Here, the estimate $\tilde{g}$ is simply one of the two eigenvalues of $\hat{M}_{\min}$, which turn up as a result of applying the two projective measurements defined by their accompanied eigenvectors.

Furthermore, we ought to define g_0 and σ in Eq. (10), the priori probability density function of the estimanda g. We set

$$g_0 = \frac{\omega_c}{L}\sqrt{\langle \hat{x}^2\rangle_0} = \frac{\omega_c}{L}\sqrt{\frac{1}{2m\omega_m}}, \tag{24}$$
$$\sigma^2 = \left(\frac{\omega_c}{L}\right)^2 \sqrt{\langle \hat{x}^4\rangle_0 - \langle \hat{x}^2\rangle_0^2} = \left(\frac{\omega_c}{L}\right)^2 \frac{1}{\sqrt{2}m\omega_m},$$

where L is the length of the cavity, m is the mass of the mechanical oscillator and $\langle \hat{A}\rangle_0$ is the average of operator $\hat{A} \in \{\hat{x}^2, \hat{x}^4\}$, acting only on the Hilbert space of mechanical oscillator, in the ground state [8, 12]. For the sake of simplicity we perform our calculations in the rotating frame of the single-mode field, i.e., $\hat{\rho}_F \to \hat{U}\hat{\rho}_F\hat{U}^\dagger$ with $\hat{U} = \exp\{-i\omega_c t\, \hat{a}^\dagger \hat{a}\}$.

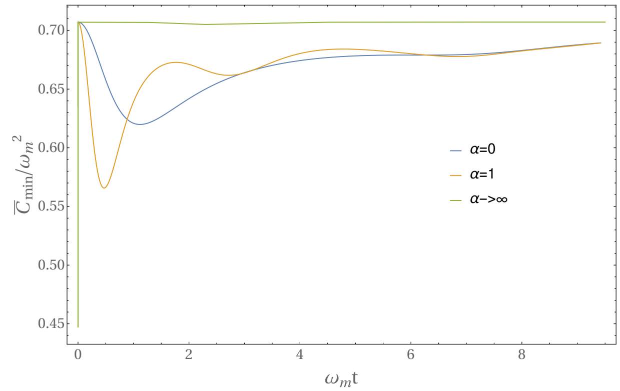

FIG. 1: The average minimum cost of error $\bar{C}_{\min}/\omega_m^2$ as a function of $\omega_m t$. We consider the parameter α of initial coherent state of the mechanical oscillator to be real (see Eq. (9)). We set $g_0/\omega_m = 1$ and $\sigma/\omega_m = 2^{-1/4}$. All curves are characterized by one global minimum which decreases by the increase of α.

In the next step we determine $\hat{M}_{\min}$ from the $\hat{\Gamma}_k$-s in Eq. (23) by using (22). One can obtain analytical results, however due to their complex structure we omit to present them here. Instead, we focus on numerical solutions. First, we investigate the average minimum cost of error $\bar{C}_{\min}$. Fig. 1 shows that the average minimum cost of error $\bar{C}_{\min}$ as a function of time, which decreases until it reaches its minimum and then returns asymptotically to its initial value. This value is equal to σ^2. At $t = 0$, where no interaction occurred, the eigenvalues of $\hat{M}_{min}$ are g_0 and 0. The probability of measuring the eigenvalue 0 is zero and therefore the estimate is g_0. It is immediate from the form of the priori probability distribution $p(g)$ in (10) that the average minimum cost of error is σ^2, or simply the variance of $p(g)$.

In the other limit $t \to \infty$ the average minimum cost of error $\bar{C}_{\min}$ is also σ^2, however the estimates or the eigenvalues of $\hat{M}_{min}$ are g_0, see Fig. 2. This means that for long interaction times the inference of the parameter g from the measurement data can only result in the mean g_0 of the probability distribution $p(g)$. As for both $t = 0$ and $t \to \infty$ the average minimum cost of error attains its maximum, we are going to neglect these situations. Instead we determine the time t^*, when the minimum of $\bar{C}_{\min}$ occurs. This results in a specific M^*_{min} to be measured and this is the case when one is the best protected against information loss.

We turn now to the analysis of t^* and its dependence from α, the parameter of the mechanical oscillator's initial coherent state. In Fig. 1, the limit $\alpha \to \infty$ with $\alpha \in \mathbb{R}$ results $t^* = 0$ and $\bar{C}_{\min} \approx 0.45$. However, the eigenvalues of $\hat{M}^*_{min}$ are still 0 and g_0. Whence it fol-

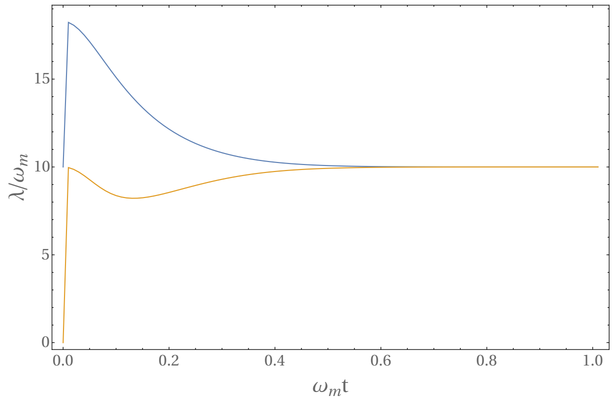

FIG. 2: The two eigenvalues of the operator $\hat{M}_{min}$ to be measured as a function of $\omega_m t$. We set $g_0/\omega_m = 1$, $\sigma/\omega_m = 2^{-1/4}$ and $\alpha = 0$. The initial value of the two eigenvalue are g_0 and 0. There is a jump for in these values when $\omega_m t$ becomes larger than 0, or in other words when the interaction is turned on. For large interaction times, i.e., $t \to \infty$, the eigenvalues tend to the same value g_0.

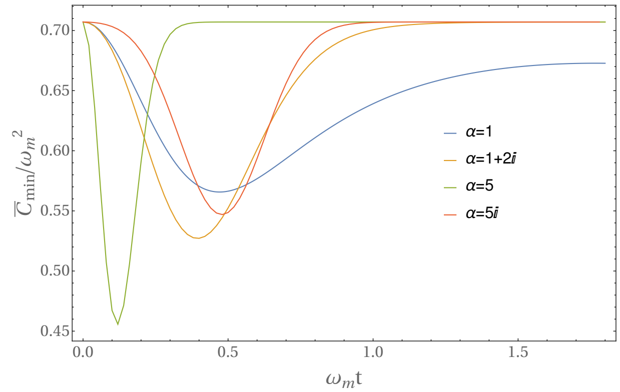

FIG. 3: The average minimum cost of error $\bar{C}_{\rm min}/\omega_m^2$ as a function of $\omega_m t$. We set $g_0/\omega_m = 1$ and $\sigma/\omega_m = 2^{-1/4}$. The imaginary part of α shifts the value of the minimum to the right: compare $\alpha = 1 + 2i, 5$ and $5i$, which have the same absolute value. Higher values of the minimum's position are accompanied with an increase in the value of the minimum.

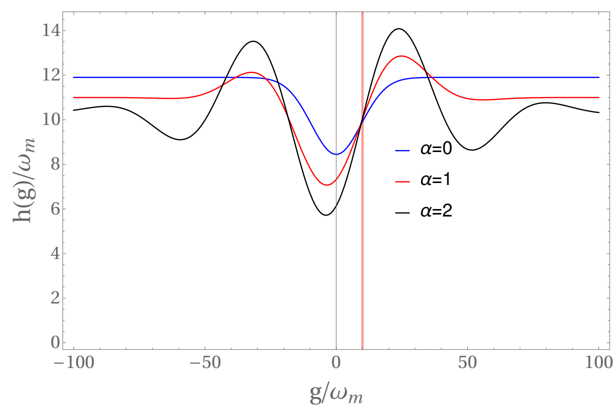

FIG. 4: The average estimator $h(g)/\omega_m$ as a function of g/ω_m. We consider the parameter α of initial coherent state of the mechanical oscillator to be real. We set $g_0/\omega_m = 1$ and $\sigma/\omega_m = 2^{-1/4}$. The time is considered to be such that the average minimum cost of error $\bar{C}_{\rm min}$ attains its minimum as a function of time. The mean value g_0 of the prior probability distribution function $p(g)$ is depicted by a vertical line.

lows that highly excited initial states of the mechanical oscillator with $\alpha \in \mathbb{R}$ result in an estimation scenario, where only the mean g_0 of the priori probability distribution $p(g)$ can be obtained from measuring M^*_{min}. In the next step, we investigate the position of the minimum for $\alpha \in \mathbb{C}$. Fig. 3 shows a shift of t^* towards higher values and an increase of the minimum value of $\bar{C}_{\rm min}$ as the imaginary part of α gets larger. However, for $\alpha = 1$ and $\alpha = 1 + 2i$ the curves almost overlaps, which means that highly excited states with large mean phonon number $|\alpha|^2$ may support the estimation scenario, if $\mathrm{Re}[\alpha] \approx \mathrm{Im}[\alpha]$. This is a critical statement, because there is a limitation on the final phonon number due to the effective thermal occupation of the cooling field [22]. Nonetheless, the extreme cases of $\mathrm{Re}[\alpha] \gg \mathrm{Im}[\alpha]$ or $\mathrm{Im}[\alpha] \gg \mathrm{Re}[\alpha]$ lead to inconclusive measurement scenarios. In our successive plots we employ these findings and in regard to the free parameter of time we use only t^* and $\hat{M}^*_{min}$.

Every outcome of the measurement of $\hat{M}^*_{min}$ is an estimate of g. The most important quantity for a possible experimental implementation is the average estimate

$$h(g) = \mathrm{Tr}\{\hat{M}^*_{min} \hat{\rho}_F(g)\}. \tag{25}$$

The collected measurement data determines the value of $h(g)$ and from which one may deduce the value of g. In Fig. 4, we show the curves of $h(g)$ for different values of the real parameter α. t^* is independently calculated for all initial states. When $\alpha = 0$, the average estimator is symmetric with respect to the y-axis. This is a direct consequence of our particular choice of the cost function (13), which is also an even function.

B. Comparing cases with different initial photonic states

So far we have discussed in detail the estimation problem of the optomechanical coupling g for the simplest initial state of the single-mode field. In the following, we consider more than one available excited photon number states. $\hat{\rho}_F(g)$ depends on g only in the off-diagonal elements and therefore we set the amplitude of all participating photon number states to be equal. This ensures the maximum allowed absolute value for the off-diagonal elements in the density matrix. Due to the added complexity of dealing with Eq. (22) we compare cases with

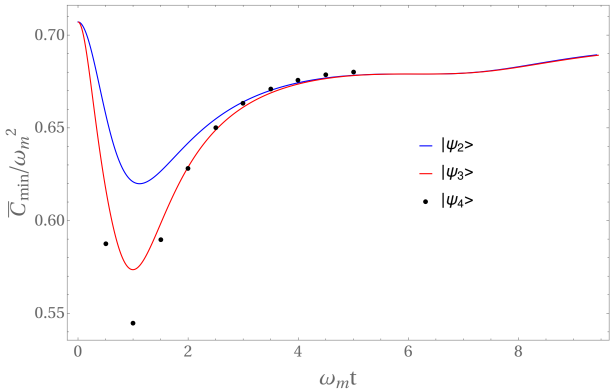

FIG. 5: The average minimum cost of error $\bar{C}_{\min}/\omega_m^2$ as a function of $\omega_m t$ for different initial photonic states. We set $g_0/\omega_m = 1$, $\sigma/\omega_m = 2^{-1/4}$ and $\alpha = 0$. $|\psi_N\rangle_c$ defined in Eq. 26 is the initial photonic state.

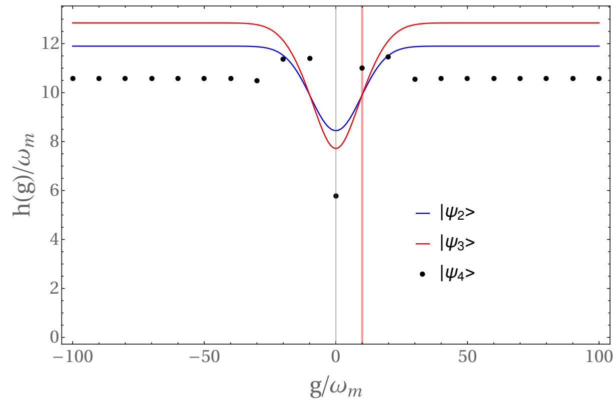

FIG. 6: The average estimator $h(g)/\omega_m$ as a function of g/ω_m. We set $g_0/\omega_m = 1$, $\sigma/\omega_m = 2^{-1/4}$ and $\alpha = 0$. The time is considered to be such that the average minimum cost of error $\bar{C}_{\min}$ attains its minimum as a function of time. The mean value g_0 of the prior probability distribution function $p(g)$ is depicted by a vertical line. $|\psi_N\rangle_c$ defined in Eq. 26 is the initial photonic state.

the following initial states of the single-mode field

$$|\psi_N\rangle_c = \sum_{n=0}^{N-1} a_n |n\rangle_c = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |n\rangle_c, \quad N = 2, 3, 4. \tag{26}$$

Fig. 5 shows that the average minimum cost of error is reduced by the increase of the photon number states in the initial state. This can be understood by examining carefully the Hamilton operator in Eq. (1), which reveals that the interaction between the single mode field and the mechanical oscillator gets stronger with increased number of participating photons. Thus, we have a better chance to estimate the optomechanical coupling g. The time t^* when $\bar{C}_{\min}$ attains its minimum is approximately the same, because we have set α equal to zero. We have also calculated the average estimator $h(g)$ for t^* and Fig. 6 shows the three different curves depending on the initial conditions in (26). Thus, all curves are even function.

IV. QUANTUM CRAMÉR-RAO INEQUALITY

In the previous sections we have discussed the properties of the optimum Hermitian operator $\hat{M}_{\min}$, which minimizes the average cost in Eq. (14), and where its eigenvalues are the estimates of the unknown optomechanical coupling g. An important task is to find out the accuracy with which g can be estimated. We would like to employ here the quantum Cramér-Rao inequality for an unbiased estimator [13, 20], however in our situation we have a biased estimator

$$\mathrm{Tr}\left\{\hat{\rho}_F(g)(\hat{M}_{\min} - g\hat{I})\right\} = f(g), \tag{27}$$

where $f(g)$ is the bias of the estimation. Therefore, we have to review the derivation of the Cramér-Rao inequality, and furthermore we have to deal in addition with an extra issue in regard to the derivative of the density matrix $\hat{\rho}_F(g)$ with respect to the parameter g. Let us recall the density matrix $\hat{\rho}_F(g)$ from Eq. (7) and observe that

$$\hat{\rho}_F(g) = \tag{28}$$
$$= \sum_{n,m=0}^{\infty} a_n a_m^* e^{-a_1(n-m)^2 + a_2(n^2-m^2) - a_3(n-m)} |n\rangle_c\langle m|,$$

where

$$\begin{aligned} a_1 &= \frac{g^2}{\omega_m^2}\left(1 - \cos(\omega_m t)\right), \\ a_2 &= i\frac{g^2}{\omega_m^2}\left(\omega_m t - \sin(\omega_m t)\right), \\ a_3 &= i\omega_c t - \frac{g}{\omega_m}\left(\alpha^*(1 - e^{i\omega_m t}) - \alpha(1 - e^{-i\omega_m t})\right). \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned} \frac{\partial \hat{\rho}_F(g)}{\partial g} &= -\frac{\partial a_1}{\partial g}\left[\hat{a}^\dagger\hat{a}, [\hat{a}^\dagger\hat{a}, \hat{\rho}_F(g)]\right] + \frac{\partial a_2}{\partial g}\left[(\hat{a}^\dagger\hat{a})^2, \hat{\rho}_F(g)\right] \\ &- \frac{\partial a_3}{\partial g}[\hat{a}^\dagger\hat{a}, \hat{\rho}_F(g)] = \mathcal{L}\left(\hat{\rho}_F(g)\right), \end{aligned} \tag{30}$$

which demonstrates that $\mathcal{L}$ is neither a right logarithmic nor a symmetrized logarithmic derivative of the density matrix $\hat{\rho}_F(g)$, a deviation from the standard analysis [13]. In the standard proof, a Cauchy-Schwartz-Bunyakovsky inequality is employed, which suggests that in our new situation we have to introduce the operator $\hat{\rho}_F^{-1/2}(g)$. This operator does not exist when the spectrum of $\hat{\rho}_F(g)$ contains zero (e.g., a pure state). We avoid this situation by following a different path.

In order to derive a lower bound for the estimator's variance,

$$\mathrm{Var}\left(\hat{M}_{\min} - g\hat{I}\right) = \mathrm{Tr}\left\{\hat{\rho}_F(g)(\hat{M}_{\min} - g\hat{I})^2\right\}, \tag{31}$$

we introduce

$$\mathrm{Tr}\left\{\hat{\rho}_F^2(g)\hat{M}_{\min}\right\}=x_1(g) \tag{32}$$

and then we differentiate both sides with respect to the parameter g,

$$\mathrm{Tr}\left\{\left(\frac{\partial\hat{\rho}_F(g)}{\partial g}\hat{\rho}_F(g)+\hat{\rho}_F(g)\frac{\partial\hat{\rho}_F(g)}{\partial g}\right)\hat{M}_{\min}\right\}=x_1'(g). \tag{33}$$

We also consider $\mathrm{Tr}\left\{\hat{\rho}_F^2(g)\right\}=x_2(g)$, and

$$\mathrm{Tr}\left\{\left(\frac{\partial\hat{\rho}_F(g)}{\partial g}\hat{\rho}_F(g)+\hat{\rho}_F(g)\frac{\partial\hat{\rho}_F(g)}{\partial g}\right)g\hat{I}\right\}=gx_2'(g). \tag{34}$$

Subtracting (34) from (33), we obtain

$$\mathrm{Tr}\left\{\left(\frac{\partial\hat{\rho}_F(g)}{\partial g}\hat{\rho}_F(g)+\hat{\rho}_F(g)\frac{\partial\hat{\rho}_F(g)}{\partial g}\right)(\hat{M}_{\min}-g\hat{I})\right\}= \\ =x_1'(g)-gx_2'(g)=x(g). \tag{35}$$

We make use of (30) and write (35) as

$$\mathrm{Tr}\left\{\left(\mathcal{L}\left(\hat{\rho}_F\right)\hat{\rho}_F+\hat{\rho}_F\mathcal{L}\left(\hat{\rho}_F\right)\right)(\hat{M}_{\min}-g\hat{I})\right\}=x(g), \tag{36}$$

where for the sake of simplicity we have neglected the argument of $\hat{\rho}_F(g)$.

The Banach space of the Hilbert-Schmidt operators is defined as

$$\mathcal{B}_2(\mathcal{H}):=\left\{\hat{X}\in\mathcal{B}(\mathcal{H}):\ \mathrm{Tr}\{\hat{X}^\dagger\hat{X}\}<\infty\right\} \tag{37}$$

where $\mathcal{B}(\mathcal{H})$ is Banach space of all bounded operators defined on the Hilbert space $\mathcal{H}$. $\mathcal{B}_2(\mathcal{H})$ with the inner product

$$\langle A,B\rangle=\mathrm{Tr}\{A^\dagger B\},\quad A,B\in\mathcal{B}_2(\mathcal{H}) \tag{38}$$

is a Hilbert space [21]. The Cauchy-Schwartz-Bunyakovsky inequality for $A,B\in\mathcal{B}_2(\mathcal{H})$ reads

$$|\mathrm{Tr}\{A^\dagger B\}|\leqslant\mathrm{Tr}\{A^\dagger A\}\mathrm{Tr}\{B^\dagger B\}. \tag{39}$$

In our case the Hilbert space is the symmetric Fock space, i.e., $\mathcal{H}=\Gamma_s(\mathbb{C})$ and $\mathcal{L}$ contains powers of $\hat{a}^\dagger\hat{a}$, which is an unbounded operator. This clearly show that we are bounded in our proof to density matrices which fulfill the conditions: $\hat{\rho}_F^{1/2}(\hat{a}^\dagger\hat{a})^2\hat{\rho}_F,\hat{\rho}_F^{1/2}\hat{a}^\dagger\hat{a}\hat{\rho}_F\hat{a}^\dagger\hat{a}\in\mathcal{B}_2\big(\Gamma_s(\mathbb{C})\big)$. These conditions together with the cyclic property of the trace imply that $\hat{\rho}_F^{1/2}\mathcal{L}\left(\hat{\rho}_F\right)$ is a Hilbert-Schmidt operator. Similarly the condition $\hat{\rho}_F^{1/2}\hat{M}_{\min}\in\mathcal{B}_2\big(\Gamma_s(\mathbb{C})\big)$ may restrict further the set of the density matrices, i.e, there are conditions for choosing the values a_n in the initial state (2). In the case of finite dimensional considerations, i.e, there exist an $N>0$ such that $a_n=0$ for $n\geqslant N$, these complications do not arise, because all matrices are Hilbert-Schmidt operators. This is the typical case of numerical simulations.

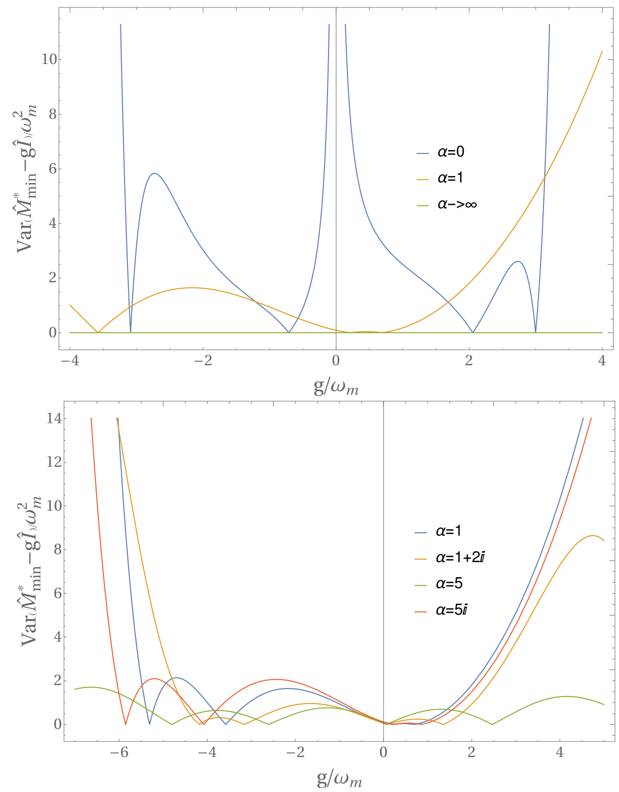

FIG. 7: The lower bound of the estimators's variance $\mathrm{Var}\left(\hat{M}^*_{\min}-g\hat{I}\right)/\omega_m^2$ as a function of g/ω_m. We set $g_0/\omega_m=1$ and $\sigma/\omega_m=2^{-1/4}$. The time is considered to be such that the average minimum cost of error $\bar{C}_{\min}$ attains its minimum as a function of time, which results in the estimator $\hat{M}^*_{\min}$. Compare the top and bottom figures with Fig. 1 and Fig. 3.

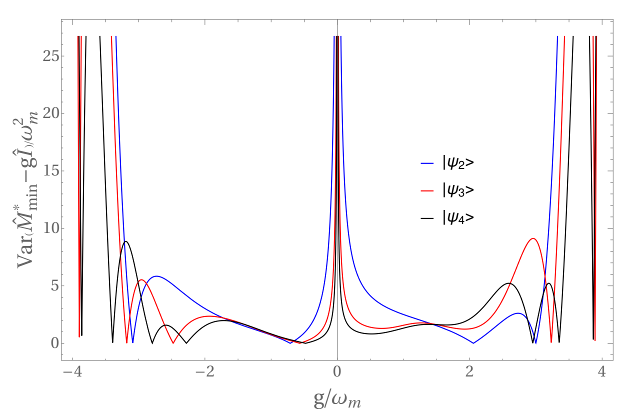

FIG. 8: The lower bound of the estimator's variance $\mathrm{Var}\left(\hat{M}^*_{\min}-g\hat{I}\right)/\omega_m^2$ as a function of g/ω_m. We set $g_0/\omega_m=1$, $\sigma/\omega_m=2^{-1/4}$, and $\alpha=0$. The time is considered to be such that the average minimum cost of error $\bar{C}_{\min}$ attains its minimum as a function of time, which results in the estimator $\hat{M}^*_{\min}$, see Fig. 5.

Now, provided that $\hat{\rho}_F^{1/2}\mathcal{L}\left(\hat{\rho}_F\right)$ and $\hat{\rho}_F^{1/2}\hat{M}_{\min}$ are Hilbert-Schmidt operators, (36) reads

$$|x(g)| = \Big|\mathrm{Tr}\left\{\mathcal{L}(\hat{\rho}_F)\hat{\rho}_F^{1/2}\hat{\rho}_F^{1/2}(\hat{M}_{\min} - g\hat{I})\right\} + \mathrm{Tr}\left\{\hat{\rho}_F^{1/2}\mathcal{L}(\hat{\rho}_F)(\hat{M}_{\min} - g\hat{I})\hat{\rho}_F^{1/2}\right\}\Big| \quad (40)$$

and applying first the subadditivity of the absolute value and then the Cauchy-Schwartz-Bunyakovsky inequality (39) twice we get

$$|x(g)| \leqslant 2\mathrm{Tr}\left\{\hat{\rho}_F\mathcal{L}^2(\hat{\rho}_F)\right\}\mathrm{Var}\left(\hat{M}_{\min} - g\hat{I}\right), \quad (41)$$

where we have used the fact that $\left[\mathcal{L}(\hat{\rho}_F)\right]^\dagger = \mathcal{L}(\hat{\rho}_F)$ (see Eq. (30)).

Finally, we get the lower bound for the estimator's variance

$$\mathrm{Var}\left(\hat{M}_{\min} - g\hat{I}\right) \geqslant \frac{|x(g)|}{2\mathrm{Tr}\left\{\hat{\rho}_F\mathcal{L}^2(\hat{\rho}_F)\right\}}. \quad (42)$$

The quantity on the right is very similar to the standard quantum Cramér-Rao bound. Here, in the nominator the function of $x(g)$ represents the fact that the estimator is biased and involves information about the purity of the density matrix $\hat{\rho}_F(g)$. The denominator has a little more complicated structure than the quantum Fischer information [23] due to the non-existence of either a right logarithmic or a symmetrized logarithmic derivative of $\hat{\rho}_F(g)$ with respect to the optomechanical coupling g.

Now, we investigate numerically the lower bound of the estimator's variance. We consider g_0 and σ to be the same as in (24). In Fig. 7, we recall the results of Figs. 1 and 3 and show the behavior of the lower bound as a function of g/ω_m. The most interesting feature occurs when $\mathrm{Re}[\alpha] > \mathrm{Im}[\alpha]$ and for which cases the lower bound of the variance is the smallest. This suggests measurement strategies with better accuracy, however this in contrast with our findings in Sec. III, where we identified the condition $\mathrm{Re}[\alpha] \approx \mathrm{Im}[\alpha]$ for avoiding the eigenvalues of $\hat{M}^*_{\min}$ to be approximately either 0 or g_0. This indicates that measurement scenarios with condition $\mathrm{Re}[\alpha] > \mathrm{Im}[\alpha]$ may simply show that the optomechanical coupling is g_0, our apriori expectation. These setups have to be avoided.

We compare also the lower bound for different initial single-mode field states given in Eq. (26) and with $\alpha = 0$. Here, the time t^* when the average minimum cost of error $\bar{C}_{\min}$ attains its minimum as a function of time is approximately the same. Fig. 8 shows that the accuracy is increased by the increase of the photon number states in the initial state. This is in agreement with our findings in Fig. 5, namely the average minimum cost of error is reduced by the increase of the photon number states. This also suggests that the initial preparation of the single-mode is also very crucial on the outcome of the estimation and equally weighted superposition of many photon number states is preferable. We conclude from this qualitative assessment that preparation of initial states of both the single-mode field and mechanical oscillator is a key element to obtain high precision with reasonable estimations.

V. CONCLUDING REMARKS

In this paper we have investigated the simplest optomechanical model, a single-mode field interacts with a vibrational mode of a mechanical oscillator from the perspective of quantum estimation theory. The optomechanical interaction between these two oscillators has been considered with a unknown optomechanical coupling strength. The time evolution of the joint state has been determined and tracing out the degrees of freedom of the mechanical oscillator one obtains the state of the single-mode field to be measured. We have introduced a quantum estimation scenario, in which one seeks for the best estimator minimizing the mean square error cost functional. This Bayesian inference approach requires a prior probability density function of the optomechanical coupling. We have considered this distribution to be a normal distribution, where the two parameters, mean and standard deviation, have been set to their values emerging from the derivation of the radiation pressure's interaction operator. This derivation consists several approximations [12], which also motivates our analysis, and therefore the estimation procedure results in a updated posterior probability density function of the coupling.

We have mainly focused on the average of the mean-square error estimator, which has been determined for those interaction time where the average minimum cost of error has a minimum as a function of time. The estimates are the eigenvalues of this estimator and the eigenvectors determine a projective POVM for the measurement strategy. Our analysis has shown that highly excited initial coherent states of the mechanical oscillator limits the estimation procedure, unless the imaginary and the real parts of the coherent state's parameter are approximately equal. A large imaginary part shifts the minimum in time towards higher values, whereas a large real part shifts this time towards zero. Both cases reveal inconclusive scenarios because the eigenvalues are either zero or the mean of the prior normal distribution. It is demonstrated that the most promising estimations can be done within a time period of the mechanical oscillation. Furthermore, the increase of the photon number states in the initial single-mode field's state with equally weighted superposition of photon number states, Eq. (26), also reduces the average information loss.

Third, we have investigated the accuracy of the mean-square error estimator. As this estimator has turned out to be biased, we have derived new lower bound, Eq. (42), for its variance. We have also encountered another deviation from the standard derivation of the quantum Cramér-Rao inequality, namely the non-existence of either a right logarithmic or a symmetrized logarithmic derivative of the single -mode's density matrix with respect to the optomechanical coupling. Apart from these issues we have used the standard steps of the quantum Cramér-Rao inequality's derivation. The numerical investigations have shown again that the characteristic interaction times for the best estimations are within a time period of the mechanical oscillation, where we have used initial single-mode field's states with equally weighted superposition of many photon number states. However, the lowest bounds for the estimation accuracy have been found for those limiting cases when the eigenvalues of the estimator are either 0 or the mean of the prior normal distribution. In particular, we have found that the initial state of the mechanical oscillator has to be carefully prepared, otherwise our apriori expectation on the strength of the optomechanical coupling it will be simply confirmed.

Finally let us make some comments on our approach. The analysis clearly indicates a characteristic set of parameters when the estimation of the optomechanical coupling can be done with minimum loss of information. Although, our results pinpoint some important results for an experimental measurement scenario the question of how to implement the detection of the mean-square error estimator has not been answered. Another critical point is the preparation of the initial state of the single-mode field. This has been considered to be an equally weighted superposition of many photon number states. Nonetheless, we have studied a unitary, lossless, model of optomechanical systems. These unanswered questions define the direction of our future investigations. As a final word we think that the presented analysis may offer an interesting perspective and viewpoint on quantum optomechanical systems.

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