

## COALESCING FIEDLER AND CORE VERTICES

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*Dedicated to the memory of Miroslav Fiedler*

*Abstract.* The nullity of a graph  $G$  is the multiplicity of zero as an eigenvalue in the spectrum of its adjacency matrix. From the interlacing theorem, derived from Cauchy's inequalities for matrices, a vertex of a graph can be a core vertex if, on deleting the vertex, the nullity decreases, or a Fiedler vertex, otherwise. We adopt a graph theoretical approach to determine conditions required for the identification of a pair of prescribed types of root vertices of two graphs to form a cut-vertex of unique type in the coalescence. Moreover, the nullity of subgraphs obtained by perturbations of the coalescence  $G$  is determined relative to the nullity of  $G$ . This has direct applications in spectral graph theory as well as in the construction of certain ipso-connected nano-molecular insulators.

*Keywords:* nullity; core vertex; Fiedler vertex; cut-vertices; coalescence

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## 1. INTRODUCTION

The *adjacency matrix*  $\mathbf{A}(G) = (a_{ij})$  of a labelled simple graph  $G$  on  $n$  vertices is the  $n \times n$  matrix whose entry  $a_{ij}$  is 1 if the vertices  $i$  and  $j$  are adjacent and 0 otherwise. The subgraph of a graph  $G$  formed from a subset  $S$  of the vertices of  $G$  and all the edges connecting pairs of vertices of  $S$  in  $G$  is a subgraph *induced* by  $S$  in  $G$ . The graph  $G - v$  denotes the subgraph of  $G$  obtained by deleting the vertex  $v$  (and all the edges incident to it). If  $G_1$  is an induced subgraph of  $G$ ,  $G - G_1$  is obtained from  $G$  by deleting all the vertices of  $G_1$ . The graph  $G + v$  is obtained from  $G$  by adding a vertex  $v$  adjacent to a nonempty subset of vertices, which is a prescribed set of neighbours of  $v$  in  $G$ .

Two nonadjacent vertices  $i$  and  $j$  of a graph  $G$  are *identified* (or *coalesced*) when these two vertices are replaced by a single vertex incident to all the edges which are

incident in  $G$  to either  $i$  or  $j$ . A graph has a root vertex  $v$  if one of its vertices,  $v$ , is distinguished. Let  $H_1$  and  $H_2$  be two graphs with root vertices  $v_1$  and  $v_2$ , respectively. The *coalescence*  $H_1 \circ H_2$  of the *component graphs*  $H_1$  and  $H_2$  is a connected graph  $G$  obtained by identifying  $v_1$  with  $v_2$  to form the *coalescence vertex*  $v$  in  $G$ . We note that the coalescence vertex  $v$  is a *cut-vertex* of  $G$ . The *coalescence*  $G$  of  $s$  ( $\geq 2$ ) *component graphs* is obtained by identifying a root vertex of each of the  $s$  component graphs to obtain a connected graph.

If  $\mathbf{I}$  is the  $n \times n$  identity matrix, the *characteristic polynomial* of a graph  $G$ , denoted by  $\varphi(G)$ , is the characteristic polynomial  $\det(\lambda \mathbf{I} - \mathbf{A})$  of the  $n \times n$  adjacency matrix  $\mathbf{A}$  ( $= \mathbf{A}(G)$ ). It is independent of the labelling of the vertices of  $G$ . The *eigenvalues of the graph* are the eigenvalues of  $\mathbf{A}$  and they form the *spectrum* of  $G$ . Since  $\mathbf{A}$  is a real symmetric matrix, its eigenvalues are real numbers and the dimension of each eigenspace is equal to the algebraic multiplicity of the respective eigenvalue, that is, the number of times the eigenvalue is repeated as a root of the characteristic equation  $\varphi(G) = 0$ .

A graph is *singular* if its adjacency matrix  $\mathbf{A}$  is not invertible and *nonsingular* otherwise. The multiplicity of zero as an eigenvalue in the spectrum of the graph  $G$  is called the *nullity*, denoted by  $\eta(G)$ . The *nullspace* is the eigenspace associated with the eigenvalue 0. An eigenvector  $\mathbf{x}$  ( $\neq \mathbf{0}$ ) in the nullspace of  $\mathbf{A}$  is called a *kernel eigenvector* of  $G$  and satisfies  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .

We shall have occasion to use a consequence of Cauchy's Interlacing theorem for real symmetric matrices and Schwenk's Coalescence theorem, stated hereunder as Theorem 1.1 and Theorem 1.2, respectively.

**Theorem 1.1** ([13], page 119). *Let  $u$  be any vertex of a graph  $G$  on  $n \geq 2$  vertices. Then*

$$\eta(G) - 1 \leq \eta(G - u) \leq \eta(G) + 1.$$

**Theorem 1.2** ([15]). *Let  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$  be two graphs with root vertices  $v_1$  and  $v_2$ , respectively. The characteristic polynomial of the coalescence  $H_1 \circ H_2$  is given by*

$$\varphi(H_1 \circ H_2) = \varphi(H_1)\varphi(G_2) + \varphi(G_1)\varphi(H_2) - \lambda\varphi(G_1)\varphi(G_2).$$

According to Theorem 1.1, when a vertex is deleted or added to a graph  $G$ , the nullity changes by at most one. We distinguish between a vertex corresponding to a zero entry in each of the kernel eigenvectors and a vertex corresponding to a nonzero entry in some kernel eigenvector.

**Definition 1.3** ([18], [19]). Let  $\mathbf{x}$  be a kernel eigenvector of a singular graph  $G$ , of order  $n \geq 3$ . A subgraph of  $G$  induced by the vertices corresponding to the nonzero entries of  $\mathbf{x}$  is a *core* with respect to  $\mathbf{x}$ .

**Definition 1.4** ([12], [17], [18]). The set of *core vertices* of  $G$  consists of those vertices that lie in some core of  $G$ . A vertex not lying in any core is said to be a *Fiedler vertex*.

Note that the set of core vertices is an invariant of  $G$  in [18]. Fiedler vertices are also referred to as *core-forbidden vertices*. The following result characterises a core vertex in a singular graph.

**Proposition 1.5** ([16]). A vertex  $u$  is a core vertex in a graph  $G + u$  if and only if  $\eta(G + u) = \eta(G) + 1$ .

There are three types of vertices, depending on the change in the multiplicity of an eigenvalue allowed by the Interlacing theorem. In [11], [23], a vertex is referred to as *downer*, *neutral* or *Parter* when its deletion decreases, does not change or increases the multiplicity of an eigenvalue, respectively. For the eigenvalue zero, Fiedler vertices and Parter vertices are sometimes referred to as *F-vertices* and *P-vertices*, respectively, see [1]. Following [6], [12], we call a vertex  $u$  a *core vertex* (CV), a *middle core-forbidden vertex* ( $\text{CFV}_{\text{mid}}$ ) or an *upper core-forbidden vertex* ( $\text{CFV}_{\text{upp}}$ ) if the nullity of  $G - u$  is  $\eta(G) - 1$ ,  $\eta(G)$ , or  $\eta(G) + 1$ , respectively.

In this paper we investigate the nullity of a coalescence and derive some concise formulae. There are areas of graph theory that could benefit directly from the study of the change in nullity on deleting a cut-vertex. The graphs with a cut-vertex form a subclass of the class of forbidden subgraphs of Hamiltonian graphs. The line graphs of a trees can be viewed as the iterative coalescence of complete graphs. The results we obtain here enable the nullity of such graphs to be deduced by determining the types of vertices being identified. Coronas and windmills are also subclasses of graphs with a cut-vertex [8], [20].

A further motivation is the interpretation of the electron energy given by Schrödinger's equation in the quantum theory of molecules [10] and its relation to the nullity of a molecular graph. Collatz and Sinogowitz posed the problem of characterizing all graphs with nonzero nullity (that is, the class of singular graphs), see [3]. Most of the work in this regard was done in [18], [19], where certain subgraphs that force a graph to be singular were identified. However, the problem is hard and research is still ongoing.

In chemical graph theory, a molecular graph, which is a  $\pi$ -system of carbon atoms, is a labelled graph whose vertices correspond to the atoms of the compound and whose edges correspond to chemical sigma bonds. The recent flurry of research on the

conductivity of carbon nano molecules used as components in circuits has spurred the development of the theory to understand the factors that cause a molecule to allow or bar the flow of electricity. For a molecule in a circuit with two leads connected to the same atom (referred to as an ipso connection), the concept of the change in nullity when the connecting vertex is deleted from the underlying graph proved to be crucial to predicting the conductivity or insulation of the molecule [5], [14], [21]. From Theorem 4.5 of [5], our results in this paper predict that a molecule with ipso connection at a cut-vertex is an insulator if it is constructed from component graphs of which at least one has a root vertex which is  $\text{CFV}_{\text{upp}}$ .

Brown et al. [2] proved that a graph  $G$  is singular if, and only if,  $G$  possesses a nontrivial zero-sum weighting (which is equivalent to the existence of a kernel eigenvector) and asked what causes a graph to be singular. Gutman and Sciriha [9] introduced nut graphs as the graphs of nullity one having a kernel eigenvector with all its entries being nonzero. In [18], [19], Sciriha determined properties of substructures responsible for a graph to be singular, establishing that in a graph of nullity  $\eta$ , there are  $\eta$  induced subgraphs (termed singular configurations) of nullity one from a prescribed list. In [22], Sharaf and Ali proceeded to determine a sharp lower bound and a sharp upper bound for the nullity of the coalescence of two graphs. They showed that the nullity of the coalescence of  $s$  component graphs varies by at most  $s - 1$  from the sum of the nullities of the component graphs. Other results on the nullity of a graph with a cut-vertex are given in [7].

The rest of this paper is structured as follows. In Sections 2 and 3, we start from the component graphs with a root vertex of prescribed type and determine the type of the coalescence vertex as well as the nullity of the coalescence. The inverse problem is then studied in Section 4. Starting from a given type of a cut-vertex in the coalescence, we determine the possible types of vertices in the component graphs. The subgraphs of  $G$  that we consider in the sequel are  $R = G - v$ ,  $U_i = (G_i + v_i) \cup (R - G_i)$ ,  $G - G_i$ ,  $W_i = (G_i + v_i) \cup (G - G_i)$ ,  $S_i = G_i \cup (G - G_i)$  and  $Y = \bigcup_{j=1}^s (G_j + v_j)$  shown in Figure 1 for the case when  $i = 1$ .

## 2. DETERMINING THE TYPE OF THE COALESCENCE VERTEX

In this section we determine the type of the coalescence vertex starting from the type of the root vertex in one or more of the component graphs. We show that the type of a vertex  $v_i$  (that is, CV,  $\text{CFV}_{\text{mid}}$  or  $\text{CFV}_{\text{upp}}$ ) in  $H_i = G_i + v_i$ , for  $i \in \{1, 2, \dots, s\}$ , determines the possible type of the cut-vertex  $v$  in  $G$  or of  $u$  in  $W_1$  or in  $S_1$ . Furthermore, we study the change in nullity when the coalescence  $G$  of the component graphs  $H_1, H_2, \dots, H_s$  is perturbed.

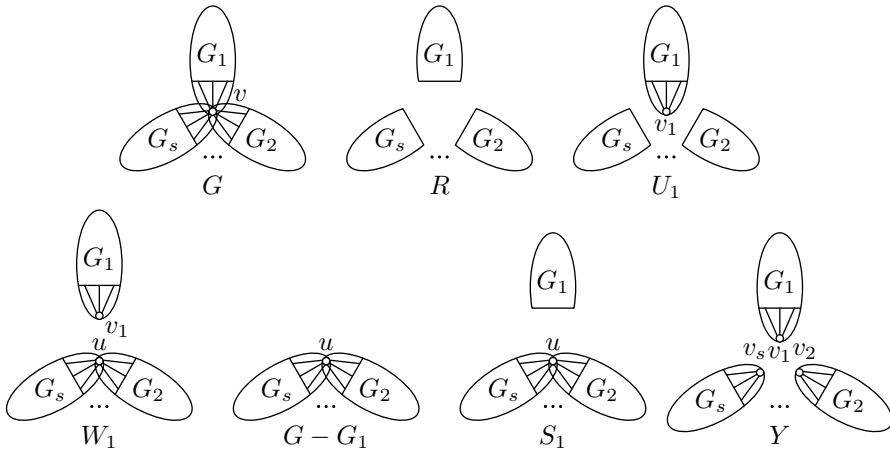


Figure 1. The graphs  $G$ ,  $R$ ,  $U_1$ ,  $W_1$ ,  $G - G_1$ ,  $S_1$  and  $Y$ .

The statements of all the results that follow refer to Figure 1. The following theorem gives the nullity of  $U_1$ .

**Theorem 2.1.** *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If  $v_i$  is a CFV<sub>upp</sub> in  $H_i = G_i + v_i$ , then  $\eta(U_i) = \eta(G)$ .*

*Proof.* Without loss of generality, let  $i = 1$  so that  $v_1$  is a CFV<sub>upp</sub> in  $H_1$  (refer to Figure 1). Since  $v_1$  is a CFV<sub>upp</sub> in  $H_1 = G_1 + v_1$ , hence, by definition,  $\eta(G_1) = \eta(H_1) + 1$ .

The nullity of a disconnected graph is equal to the sum of the nullities of its components. Thus,

$$(2.1) \quad \eta(R) - \eta(U_1) = \eta(G_1) - \eta(H_1) = 1.$$

Let  $\eta(G) = \eta$ . By Theorem 1.1,  $\eta - 1 \leq \eta(R) \leq \eta + 1$ , and hence  $\eta - 2 \leq \eta(U_1) \leq \eta$ .

We use a result given in [4] for the  $n$ -vertex graph  $G = R + v$ . For an orthonormal set of eigenvectors  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n-1)}$  corresponding to the  $n - 1$  eigenvalues of  $R$ , if  $x_k^{(i)}$  is the  $k^{\text{th}}$  entry of  $\mathbf{x}^{(i)}$ , then

$$\varphi(G) = \varphi(R) \left( \lambda - \sum_{i=1}^{n-1} \frac{\sum_{k \in \mathcal{N}} (x_k^{(i)})^2}{\lambda - \lambda_i} \right),$$

where  $\mathcal{N} = N_G(v) = \bigcup_{i=1}^s N_{G_i}(v_i)$ ,  $N_{G_i}(v_i)$  is the neighbourhood of  $v_i$  in  $G_i$ , and  $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$  are the eigenvalues of  $R$ . Now from (2.1),  $\eta(R) > \eta(U_1) \geq 0$ , and thus  $R$  is singular. Since the adjacency matrix  $\mathbf{A}(R)$  of  $R$  is diagonalizable,  $m_R(\lambda)$

is of the form  $\lambda(\lambda - \mu_2) \dots (\lambda - \mu_t)$ , where  $\mu_1 (= 0)$ ,  $\mu_2, \dots, \mu_t$  are the  $t$  ( $t \leq n - 1$ ) distinct eigenvalues of  $R$ .

It follows that  $\varphi(G)$  is of the form

$$\varphi(G) = \varphi(R) \left( \lambda - \sum_{i=1}^t \frac{\delta_i^2}{\lambda - \mu_i} \right),$$

where  $\delta_i \in \mathbb{R}$ . Thus,

$$(2.2) \quad \varphi(G) = \frac{\varphi(R)}{m_R(\lambda)} \left( \lambda m_R(\lambda) - \left( \frac{\delta_1^2}{\lambda} + \frac{\delta_2^2}{\lambda - \mu_2} + \dots + \frac{\delta_t^2}{\lambda - \mu_t} \right) m_R(\lambda) \right).$$

Note that  $\lambda^{\eta(R)}$  divides  $\varphi(R)$ , and thus the rational quotient  $\varphi(R)/m_R(\lambda)$  has  $\lambda^{\eta(R)-1}$  as a factor (because both the numerator and the denominator in the quotient have a common factor of  $\lambda$ ). Also,  $\delta_1^2 m_R(\lambda)/\lambda$  has a nonzero constant term, and hence the right hand side of equation (2.2) has  $\lambda^{\eta(R)-1}$  as a factor but it is not divided by  $\lambda^{\eta(R)}$ . Thus,  $\eta(G) = \eta(R) - 1$ . From (2.1),  $\eta(G) = \eta(U_1)$ .  $\square$

We shall also use the following result which gives a sufficient condition for the coalescence vertex to be  $\text{CFV}_{\text{upp}}$  in  $G$ . We explore the cases for the coalescence vertex to be CV or  $\text{CFV}_{\text{mid}}$  in Theorems 2.5 to 2.8.

**Theorem 2.2** ([7]). *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If there exists a component  $G_i$  among  $G_1, G_2, \dots, G_s$  such that  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in  $H_i = G_i + v_i$ , then  $v$  is a  $\text{CFV}_{\text{upp}}$  in  $G$ .*

We now consider the type of a vertex  $v_i$  other than  $\text{CFV}_{\text{upp}}$  in  $H_i = G_i + v_i$ .

**Lemma 2.3** ([7]). *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If  $v_1$  is a CV in  $H_1 = G_1 + v_1$ , then  $\eta(S_1) = \eta(G_1) + \eta(G - G_1) = \eta(G)$ .*

Since the choice of the component graph  $H_1$  is arbitrary, we have:

**Corollary 2.4.** *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If there exists  $i \in \{1, 2, \dots, s\}$  such that  $v_i$  is a CV in  $G_i + v_i$ , then  $\eta(S_i) = \eta(G_i) + \eta(G - G_i) = \eta(G)$  and  $\eta(W_i) = \eta(S_i) + 1$ .*

We now present a sufficient condition for the cut-vertex  $v$  in  $G$  to be a CV.

**Theorem 2.5.** *Let  $G$  be a graph with a cut-vertex  $v$ , and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If  $v_i$  is a CV in each  $H_i = G_i + v_i$  for  $i \in \{1, 2, \dots, s\}$ , then  $v$  is a CV in  $G$ .*

**Proof.** Let  $v_i$  be a CV in each  $H_i$  for  $i \in \{1, 2, \dots, s\}$ . To show that  $v$  is a CV in  $G$ , by Proposition 1.5 it is sufficient to show that  $\eta(G) = \sum_{i=1}^s \eta(G_i) + 1 = \eta(G - v) + 1$ . We use induction on the number  $s$  of components of  $G - v$ .

For  $s = 2$ , let  $G_1$  and  $G_2$  be the only two components of  $G - v$  such that  $v_1$  is a CV in  $H_1$  and  $v_2$  is a CV in  $H_2$ . Then by Lemma 2.3, we have  $\eta(G) = \eta(G_1) + \eta(G - G_1)$ . Now  $G - G_1 = H_2$ , and thus  $\eta(G) = \eta(G_1) + \eta(H_2)$ , implying that  $\eta(G) = \eta(G_1) + \eta(G_2) + 1$ . Since  $G_1$  and  $G_2$  are the components of  $G - v$ , we have  $\eta(G) = \eta(G - v) + 1$ . Hence,  $v$  is a CV in  $G$ .

Assume that the result is true for  $k < s$ , that is, if  $G_1, G_2, \dots, G_k$  are the components of  $G$  such that  $v_i$  is a CV in each  $H_i$  for  $i \in \{1, 2, \dots, k\}$ , then  $\eta(G) = \sum_{i=1}^k \eta(G_i) + 1$ . We show that the result follows for the case when  $G - v$  has  $(k + 1)$  components  $G_1, G_2, \dots, G_k, G_{k+1}$ . Let  $v_i$  be a CV in each  $H_i$  for  $i \in \{1, 2, \dots, k+1\}$ , and let  $X = G - G_{k+1}$ . Then  $S_{k+1} = X \cup G_{k+1}$  and by Corollary 2.4, we have  $\eta(G) = \eta(S_{k+1}) = \eta(G_{k+1}) + \eta(G - G_{k+1}) = \eta(G_{k+1}) + \eta(X)$ . Now, by the inductive hypothesis,  $\eta(X) = \sum_{i=1}^k \eta(G_i) + 1$ . Thus  $\eta(G) = \eta(G_{k+1}) + \sum_{i=1}^k \eta(G_i) + 1 = \sum_{i=1}^{k+1} \eta(G_i) + 1$ . Finally, since  $G - v = \bigcup_{i=1}^{k+1} G_i$ , we conclude that  $\eta(G - v) = \sum_{i=1}^{k+1} \eta(G_i)$ . Thus,  $\eta(G) = \eta(G - v) + 1$ . The result follows by induction on  $s$ .  $\square$

As we shall see below, the occurrence of a  $\text{CFV}_{\text{mid}}$  among the root vertices of the component graphs is nondeterministic. We discuss the case when the number of component graphs is two in Theorems 2.6 and 2.7. For any finite number of component graphs, the instance when the occurrence of a  $\text{CFV}_{\text{mid}}$  among the root vertices determines the type of coalescence vertex uniquely is given in Theorem 2.8.

**Theorem 2.6.** *Let  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$  be two graphs with root vertices  $v_1$  and  $v_2$ , respectively. If  $v_1$  is a CV in  $H_1$  and  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $H_2$ , then  $\eta(G) = \eta(H_1) + \eta(H_2) - 1$  and the coalescence vertex  $v$  obtained by identifying  $v_1$  and  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $G = H_1 \circ H_2$ .*

**Proof.** By Lemma 2.3,  $\eta(G) = \eta(S_1) = \eta(G_1) + \eta(H_2)$ . Since  $v_1$  is a CV in  $H_1$ , so  $\eta(G_1) = \eta(H_1) - 1$  and thus  $\eta(G) = \eta(H_1) + \eta(H_2) - 1$ . But  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $H_2$ , implying that  $\eta(R) = \eta(S_1)$ . Hence  $\eta(G) = \eta(R)$  and, by definition, the coalescence vertex  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G = H_1 \circ H_2$ .  $\square$

**Theorem 2.7.** *Let  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$  be two graphs with root vertices  $v_1$  and  $v_2$ , respectively, and let  $v$  be the coalescence vertex in  $G = H_1 \circ H_2$  obtained by identifying  $v_1$  and  $v_2$ . If both  $v_1$  and  $v_2$  are  $\text{CFV}_{\text{mid}}$  in  $H_1$  and  $H_2$ ,*

respectively, then either  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G$  and  $\eta(G) = \eta(H_1) + \eta(H_2)$ , or  $v$  is a CV in  $G$  and  $\eta(G) = \eta(H_1) + \eta(H_2) + 1$ .

**Proof.** Since  $v_1$  is a  $\text{CFV}_{\text{mid}}$  in  $H_1$  and  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $H_2$ , we have  $\eta(G_1) = \eta(H_1)$  and  $\eta(G_2) = \eta(H_2)$ . By Theorem 1.2, the characteristic polynomial of the coalescence  $G = H_1 \circ H_2$  is given by

$$\varphi(G) = \varphi(H_1)\varphi(G_2) + \varphi(G_1)\varphi(H_2) - \lambda\varphi(G_1)\varphi(G_2).$$

So we obtain

$$\eta(G) \geq \eta(H_1) + \eta(H_2) = \eta(G_1) + \eta(G_2) = \eta(R).$$

Thus, on the deletion of the cut-vertex  $v$  in  $G$ , the nullity does not increase. Thus  $v$  cannot be a  $\text{CFV}_{\text{upp}}$ . Hence, either  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G$  and  $\eta(G) = \eta(H_1) + \eta(H_2)$ , or  $v$  is a CV in  $G$  and  $\eta(G) = \eta(H_1) + \eta(H_2) + 1$ .  $\square$

Figure 2 shows that each of the situations described in Theorem 2.7 can occur for the coalescence of component graphs with root vertices of the type  $\text{CFV}_{\text{mid}}$ .

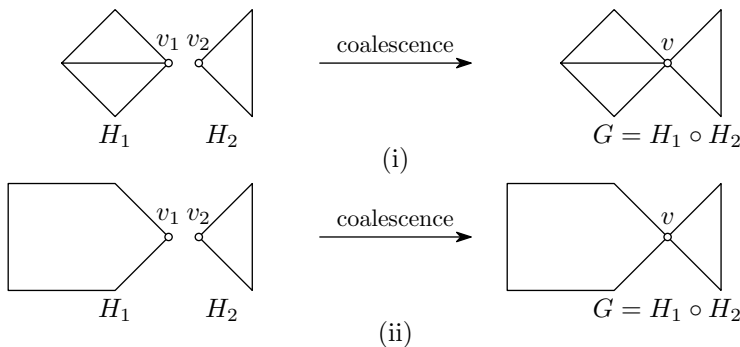


Figure 2. Two graphs illustrating Theorem 2.7 when (i)  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G$ ; and (ii)  $v$  is a CV in  $G$ .

**Theorem 2.8.** For each  $i \in \{1, 2, \dots, s\}$ , let  $H_i = G_i + v_i$  be a graph with root vertex  $v_i$ . If no  $H_i$  has a  $\text{CFV}_{\text{upp}}$  and exactly one  $H_i$  has a  $\text{CFV}_{\text{mid}}$ , then the coalescence vertex  $v$  is a  $\text{CFV}_{\text{mid}}$  in the coalescence  $G$  obtained by identifying all the vertices  $v_i$  in  $H_i$ .

**Proof.** Without loss of generality, let  $H_1$  be the only component graph with a  $\text{CFV}_{\text{mid}}$  root vertex. If no component graph  $H_i$  has a  $\text{CFV}_{\text{upp}}$  root vertex, then the root vertex in each  $H_i$ , for  $i \in \{2, \dots, s\}$ , is a CV. If we let  $u$  be the coalescence vertex in the graph  $G - G_1$  obtained by identifying each root vertex  $v_i$ ,  $i \in \{2, \dots, s\}$ , then by Theorem 2.5,  $u$  is a CV in  $G - G_1$ . Since  $v_1$  is a  $\text{CFV}_{\text{mid}}$  in  $H_1$  and  $u$  is a CV in  $G - G_1$ , by Theorem 2.6,  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G$ , where  $v$  is the coalescence vertex in  $G = H_1 \circ (G - G_1)$ .  $\square$



### 3. DETERMINING THE NULLITY OF A COALESCENCE

At this stage, using results in Section 2, we can determine the nullity of the coalescence from the nullities of the component graphs. For simplicity we start with the coalescence of exactly two component graphs. This enables us to get the generalization to the coalescence of a finite number  $s > 2$  of component graphs, by induction.

In Proposition 3.1 and Theorem 3.2 we consider the case when one or more of the root vertices in the component graphs are CV.

**Proposition 3.1.** *Let  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$  be two graphs with root vertices  $v_1$  and  $v_2$ , respectively. If at least one of the root vertices is a CV in the respective component graph, then  $\eta(G) = \eta(H_1) + \eta(H_2) - 1$  where  $G = H_1 \circ H_2$ .*

*Proof.* Without loss of generality, we let  $v_1$  be a CV in  $H_1$ . Thus,  $\eta(G_1) = \eta(H_1) - 1$ , and by Lemma 2.3,  $\eta(G) = \eta(G_1) + \eta(G - G_1) = \eta(H_1) - 1 + \eta(G - G_1)$ . But  $G - G_1 = H_2$  and thus  $\eta(G) = \eta(H_1) - 1 + \eta(H_2)$ .  $\square$

Now we determine the nullity of  $G - G_i$  (for  $i \in \{1, 2, \dots, s\}$ ) and of  $G$  in terms of the nullities of their subgraphs.

**Theorem 3.2.** *Let  $G$  be a graph with a cut-vertex  $v$ , and let  $G_1, G_2, \dots, G_s$  be all the components of  $G - v$ . If  $v_i$  is a CV in each  $H_i = G_i + v_i$  for  $i \in \{1, 2, \dots, s\}$ , then*

$$(i) \quad \eta(G - G_i) = \sum_{\substack{j=1 \\ j \neq i}}^s \eta(G_j) + 1; \text{ and}$$

$$(ii) \quad \eta(G) = \sum_{i=1}^s \eta(H_i) + (1 - s).$$

*Proof.* Since  $v_i$  is a CV in each  $H_i$ , for  $i \in \{1, 2, \dots, s\}$ , by Theorem 2.5,  $v$  is a CV in  $G$ . Thus,

$$(3.1) \quad \eta(G) = \eta(G - v) + 1 = \sum_{i=1}^s \eta(G_i) + 1.$$

(i) By Corollary 2.4,  $\eta(G) = \eta(G_i) + \eta(G - G_i)$  for  $i \in \{1, 2, \dots, s\}$ . Thus,  $\eta(G - G_i) = \sum_{\substack{j=1 \\ j \neq i}}^s \eta(G_j) + 1$  since  $\eta(G) = \eta(G_i) + \sum_{\substack{j=1 \\ j \neq i}}^s \eta(G_j) + 1$ .

(ii) Since  $v_i$  is a CV in each  $H_i$ , for  $i \in \{1, 2, \dots, s\}$ , by definition,  $\eta(G_i) = \eta(H_i) - 1$ . Thus, from (3.1),

$$\begin{aligned} \eta(G) &= \eta(H_1) - 1 + \eta(H_2) - 1 + \dots + \eta(H_s) - 1 + 1 \\ &= \eta(H_1) + \eta(H_2) + \dots + \eta(H_s) + (1 - s) \\ &= \sum_{i=1}^s \eta(H_i) + (1 - s). \end{aligned}$$

$\square$

The case when one or more of the root vertices are  $\text{CFV}_{\text{upp}}$  is treated in Proposition 3.3 and Theorem 3.4.

**Proposition 3.3.** *Let  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$  be two graphs with root vertices  $v_1$  and  $v_2$ , respectively.*

- (i) *If both the root vertices are  $\text{CFV}_{\text{upp}}$ , then  $\eta(H_1 \circ H_2) = \eta(H_1) + \eta(H_2) + 1$ .*
- (ii) *If  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$  and  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $H_2$ , then  $\eta(H_1 \circ H_2) = \eta(H_1) + \eta(H_2)$ .*
- (iii) *If  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$  and  $v_2$  is a CV in  $H_2$ , then  $\eta(H_1 \circ H_2) = \eta(H_1) + \eta(H_2) - 1$ .*

*Proof.* In each of the three cases (i), (ii) and (iii), at least one root vertex is a  $\text{CFV}_{\text{upp}}$  in the respective component graph. Hence by Theorem 2.2, if  $G = H_1 \circ H_2$ , we have

$$(3.2) \quad \eta(R) = \eta(G - v) = \eta(G) + 1.$$

- (i) Assume that both the root vertices are  $\text{CFV}_{\text{upp}}$ . As shown in Figure 1,  $W_1 = H_1 \cup H_2$ . Hence  $\eta(S_1) = \eta(W_1) + 1 = \eta(H_1) + \eta(H_2) + 1$  and  $\eta(R) = \eta(S_1) + 1 = \eta(H_1) + \eta(H_2) + 2$ . Thus, from (3.2) we get  $\eta(G) + 1 = \eta(H_1) + \eta(H_2) + 2$ . Hence,  $\eta(G) = \eta(H_1) + \eta(H_2) + 1$ .
- (ii) Assume that  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$  and  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $H_2$ . As shown in Figure 1,  $\eta(S_1) = \eta(W_1) + 1 = \eta(H_1) + \eta(H_2) + 1$  because  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$ . Since  $v_2$  is a  $\text{CFV}_{\text{mid}}$  in  $H_2$ , we have  $\eta(R) = \eta(S_1) = \eta(H_1) + \eta(H_2) + 1$ . Thus, from (3.2) we get  $\eta(G) = \eta(H_1) + \eta(H_2)$ .
- (iii) If  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$  and  $v_2$  is a CV in  $H_2$ , then the result follows immediately from Proposition 3.1. □

Generalizing the above result to the case when the number of component graphs is  $s \geq 2$ , we obtain the following.

**Theorem 3.4.** *Let  $G$  be a graph with a cut-vertex  $v$ , and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If there exists  $i \in \{1, 2, \dots, s\}$  such that  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in  $H_i = G_i + v_i$ , then  $\eta(G) = \sum_{i=1}^s \eta(G_i + v_i) + (a - b - 1)$ , where  $a$  is the number of component graphs which have the root vertex  $v_i$  as a  $\text{CFV}_{\text{upp}}$  in  $H_i$ , and  $b$  is the number of component graphs which have the root vertex  $v_i$  as a CV in  $H_i$ .*

*Proof.* Without loss of generality, let  $G_i$  be relabelled so that the first  $a$  component graphs have  $v_i$  as a  $\text{CFV}_{\text{upp}}$ , the next  $b$  component graphs have  $v_i$  as a CV, and the last  $c$  component graphs have  $v_i$  as a  $\text{CFV}_{\text{mid}}$ , where  $s = a + b + c$ . Then we have

$\eta(G_i) = \eta(H_i) + 1$  for  $i \in \{1, 2, \dots, a\}$ ,  $\eta(G_i) = \eta(H_i) - 1$  for  $i \in \{a+1, a+2, \dots, a+b\}$  and  $\eta(G_i) = \eta(H_i)$  for  $i \in \{a+b+1, a+b+2, \dots, a+b+c = s\}$ . Since there exists  $v_i$  which is a  $\text{CFV}_{\text{upp}}$  in at least one  $H_i$ , for  $i \in \{1, 2, \dots, s\}$ , by Theorem 2.2,  $v$  is a  $\text{CFV}_{\text{upp}}$  in  $G$ , and thus  $\eta(G) = \eta(G - v) - 1$ . Hence

$$\begin{aligned} \eta(G) &= \sum_{i=1}^s \eta(G_i) - 1 \\ &= \left( \sum_{i=1}^a (\eta(H_i) + 1) + \sum_{i=a+1}^{a+b} (\eta(H_i) - 1) + \sum_{i=a+b+1}^{a+b+c} \eta(H_i) \right) - 1 \\ &= \sum_{i=1}^a \eta(H_i) + a + \sum_{i=a+1}^{a+b} \eta(H_i) - b + \sum_{i=a+b+1}^{a+b+c} \eta(H_i) - 1 \\ &= \sum_{i=1}^s \eta(H_i) + (a - b - 1). \end{aligned}$$

□

The next result follows immediately.

**Corollary 3.5.** *Let  $G$  be a graph with a cut-vertex  $v$ , and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in each  $H_i = G_i + v_i$ , for  $i \in \{1, 2, \dots, s\}$ , then we have  $\eta(G) = \sum_{i=1}^s \eta(H_i) + (s - 1)$ .*

Theorem 3.2 and Corollary 3.5 give closed formulae for  $\eta(G)$  when, for  $i \in \{1, 2, \dots, s\}$ , all the  $v_i$  are CV and when all the  $v_i$  are  $\text{CFV}_{\text{upp}}$ , respectively. The case when all the  $v_i$  are  $\text{CFV}_{\text{mid}}$  will not be considered here. It is more complex and to obtain a closed formula, specific conditions on the component graphs forming the coalescence are required.

#### 4. DETERMINING THE TYPE OF THE ROOT VERTEX IN COMPONENT GRAPHS

Here we investigate the inverse problem:

*Can the type of the cut-vertex  $v$  in the coalescence  $G$  determine the type of the root vertex in the component graphs?*

For the case when the coalescence vertex is  $\text{CFV}_{\text{upp}}$  in  $G$  we give a necessary and sufficient condition for the type of root vertex in at least one of the component graphs.

**Theorem 4.1.** *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . The coalescence vertex  $v$  is a  $\text{CFV}_{\text{upp}}$  in  $G$  if and only*

if there exists at least one component  $G_i$  in  $G$ , for  $i \in \{1, 2, \dots, s\}$ , such that  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in  $H_i = G_i + v_i$ .

**P r o o f.** Sufficiency follows immediately by Theorem 2.2. To prove necessity, let  $v$  be a  $\text{CFV}_{\text{upp}}$  in  $G$ . As shown in Figure 1,  $W_1$  is a disconnected graph with two components  $G - G_1$  and  $H_1$  with root vertices  $u$  and  $v_1$ , respectively. The graph  $G$  is obtained by coalescing the two root vertices of the components of  $W_1$ . Theorem 2.5, Theorem 2.6 and Theorem 2.7 rule out cases where none of the two root vertices is a  $\text{CFV}_{\text{upp}}$ . Thus, either the root vertex  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$  or the root vertex  $u$  is a  $\text{CFV}_{\text{upp}}$  in  $G - G_1$ .

If  $v_1$  is a  $\text{CFV}_{\text{upp}}$  in  $H_1$ , then the proof is complete. If, on the other hand,  $v_1$  is not a  $\text{CFV}_{\text{upp}}$  in  $H_1$ , then  $u$  is a  $\text{CFV}_{\text{upp}}$  in  $G - G_1$ . We repeat the same argument used above with  $G - G_1$  taking the role of  $G$ . This iterative argument is repeated as many times as is necessary until we obtain that:

- ▷ either there exists a vertex  $v_j$  which is a  $\text{CFV}_{\text{upp}}$  in  $H_j$ , for some  $j \in \{2, \dots, s-2\}$ ;
- ▷ or the root vertex  $u$  of  $G - \bigcup_{i=1}^{s-2} G_i = H_{s-1} \circ H_s$  is a  $\text{CFV}_{\text{upp}}$ . In this case, either  $v_{s-1}$  or  $v_s$  is a  $\text{CFV}_{\text{upp}}$  in the respective component graph.

The result follows. □

The cases when the coalescence vertex is either a  $\text{CFV}_{\text{mid}}$  or a  $\text{CV}$  in  $G$  are considered in Theorems 4.2 and 4.3, respectively.

**Theorem 4.2.** *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G$ , then no  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in  $H_i = G_i + v_i$ , for  $i \in \{1, 2, \dots, s\}$ , and  $v_i$  is a  $\text{CFV}_{\text{mid}}$  in at least one component graph  $H_i$ .*

**P r o o f.** Let  $v$  be a  $\text{CFV}_{\text{mid}}$  in  $G$ . By Theorem 4.1, no  $v_i$  can be a  $\text{CFV}_{\text{upp}}$  in  $H_i$ , for  $i \in \{1, 2, \dots, s\}$ . Since  $G$  is obtained from  $W_1$  by identifying the vertex  $u$  in  $G - G_1$  with the vertex  $v_1$  in  $H_1$  (refer to Figure 1), then by Theorems 2.2 and 2.5, either  $u$  is a  $\text{CFV}_{\text{mid}}$  in  $G - G_1$  or  $v_1$  is a  $\text{CFV}_{\text{mid}}$  in  $H_1$ .

If  $v_1$  is a  $\text{CFV}_{\text{mid}}$  in  $H_1$ , then the result follows. If, on the other hand,  $v_1$  is not a  $\text{CFV}_{\text{mid}}$  in  $H_1$ , then  $u$  is a  $\text{CFV}_{\text{mid}}$  in  $G - G_1$  and we repeat the same argument used above with  $G - G_1$  taking the role of  $G$ . This iterative argument is repeated as many times as is necessary to test if there is a component graph  $H_j$  for which  $v_j$  is a  $\text{CFV}_{\text{mid}}$  for  $j \in \{2, \dots, s-2\}$ . If there is none, then the coalescence vertex of  $H_{s-1} \circ H_s$  is  $\text{CFV}_{\text{mid}}$ . By Theorems 2.2, 2.5 and 4.1, at least one of the root vertices of  $H_{s-1}$  or  $H_s$  is a  $\text{CFV}_{\text{mid}}$ . □

**Theorem 4.3.** *Let  $G$  be a graph with a cut-vertex  $v$  and let  $G_1, G_2, \dots, G_s$  be the components of  $G - v$ . If  $v$  is a CV in  $G$ , then no  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in  $H_i = G_i + v_i$ , for  $i \in \{1, 2, \dots, s\}$ , and*

- (i) *either  $v_i$  is a  $\text{CFV}_{\text{mid}}$  in at least two component graphs  $H_i$  for  $i \in \{1, 2, \dots, s\}$ ;*
- (ii) *or  $v_i$  is a CV in each of the  $H_i$  for  $i \in \{1, 2, \dots, s\}$ .*

*Proof.* By Theorem 4.1, Theorem 2.2 and Theorem 2.6,  $u$  and  $v_1$  in  $W_1$  can be of the types given in the following two cases:

- (i) both  $u$  and  $v_1$  are  $\text{CFV}_{\text{mid}}$  in  $G - G_1$  and in  $H_1$ , respectively; or
- (ii) both  $u$  and  $v_1$  are CV in  $G - G_1$  and in  $H_1$ , respectively.

In case (i), by Theorem 4.2,  $G - G_1$  has at least one component  $G_i$ , for  $i \in \{2, \dots, s\}$ , such that  $v_i$  is a  $\text{CFV}_{\text{mid}}$  in  $H_i$ . Thus  $v_i$  is a  $\text{CFV}_{\text{mid}}$  in at least two component graphs.

In case (ii), we consider  $G - G_1$  and use the same argument used above with  $G - G_1$  taking the role of  $G$ . This iterative argument is repeated as follows. If we let  $k \in \{2, \dots, s - 1\}$  be the smallest integer (if it exists) such that  $v_k$  is not a CV in the component graph  $H_k$ , then  $v_k$  is a  $\text{CFV}_{\text{mid}}$ . By (i), the coalescence vertex of the coalescence of the remaining component graphs  $H_{k+1}, \dots, H_s$  must also be a  $\text{CFV}_{\text{mid}}$ , and by Theorem 4.2, a root vertex of at least one of the component graphs  $H_{k+1}, \dots, H_s$  is a  $\text{CFV}_{\text{mid}}$ . Thus there are at least two component graphs  $H_i$  having a root vertex  $v_i$  being a  $\text{CFV}_{\text{mid}}$ . If, on the other hand, there is no  $k$  such that the vertex  $v_k$  is a  $\text{CFV}_{\text{mid}}$  in  $H_k$ , then each vertex  $v_i$ , for  $i \in \{2, \dots, s\}$ , is a CV in the component graph  $H_i$ . Thus  $v_i$  is a CV in all component graphs  $H_i$ , for  $i \in \{1, 2, \dots, s\}$ , completing the proof.  $\square$

Since a vertex can be one of three types, namely CV,  $\text{CFV}_{\text{upp}}$  and  $\text{CFV}_{\text{mid}}$ , there are 3! different ways of choosing the pairs of root vertices when coalescing two component graphs. Table 1 summarizes the results obtained above. The third column

Type of $v_1$ in $H_1$	Type of $v_2$ in $H_2$	Type of $v$ in $H_1 \circ H_2$	Reason	$\eta(H_1 \circ H_2)$	Reason
$\text{CFV}_{\text{upp}}$	$\text{CFV}_{\text{upp}}$	$\text{CFV}_{\text{upp}}$	Theorem 2.2	$\eta_1 + \eta_2 + 1$	Proposition 3.3 (i)
$\text{CFV}_{\text{upp}}$	$\text{CFV}_{\text{mid}}$	$\text{CFV}_{\text{upp}}$	Theorem 2.2	$\eta_1 + \eta_2$	Proposition 3.3 (ii)
$\text{CFV}_{\text{upp}}$	CV	$\text{CFV}_{\text{upp}}$	Theorem 2.2	$\eta_1 + \eta_2 - 1$	Proposition 3.1
CV	CV	CV	Theorem 2.5	$\eta_1 + \eta_2 - 1$	Proposition 3.1
CV	$\text{CFV}_{\text{mid}}$	$\text{CFV}_{\text{mid}}$	Theorem 2.6	$\eta_1 + \eta_2 - 1$	Proposition 3.1
$\text{CFV}_{\text{mid}}$	$\text{CFV}_{\text{mid}}$	CV or $\text{CFV}_{\text{mid}}$	Theorem 2.7	$\eta_1 + \eta_2 + 1$ or $\eta_1 + \eta_2$	Theorem 2.7

Table 1. The type of the coalescence vertex  $v$  and the nullity  $\eta(H_1 \circ H_2)$  of the coalescence of two graphs  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$  with root vertices  $v_1$  and  $v_2$ , respectively.

gives the type of coalescence vertex  $v$  in  $H_1 \circ H_2$  obtained when identifying the root vertices  $v_1$  and  $v_2$  in  $H_1 = G_1 + v_1$  and  $H_2 = G_2 + v_2$ , respectively. The nullity  $\eta(H_1 \circ H_2)$  of the coalescence of  $H_1$  and  $H_2$  in terms of the nullity  $\eta(H_1) = \eta_1$  and  $\eta(H_2) = \eta_2$  is given in the fifth column, for the distinct pairs of types of root vertices. The appropriate link to the respective result in the previous sections is presented for ease of reference.

Generalizing to cases when the number  $s$  of component graphs  $H_i = G_i + v_i$  is greater than two and  $G$  is obtained by identifying the root vertices  $v_i$  in each  $H_i$ , for  $i \in \{1, 2, \dots, s\}$ , we conclude that

- (i) if  $v_i$  is a CV in each  $H_i$ , then the coalescence vertex  $v$  is a CV in  $G$ ;
- (ii) if  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in each  $H_i$ , then the coalescence vertex  $v$  is a  $\text{CFV}_{\text{upp}}$  in  $G$ ;
- (iii) if  $v_i$  is a  $\text{CFV}_{\text{mid}}$  in each  $H_i$ , then the coalescence vertex  $v$  cannot be a  $\text{CFV}_{\text{upp}}$  in  $G$ .

Also,

- (iv) if  $v_i$  is a  $\text{CFV}_{\text{mid}}$  in exactly one  $H_i$  and each  $H_j$ , for  $j \neq i$ , have a CV, then the coalescence vertex  $v$  is a  $\text{CFV}_{\text{mid}}$  in  $G$ ;
- (v) if  $v_i$  is a  $\text{CFV}_{\text{upp}}$  in at least one  $H_i$ , then the coalescence vertex  $v$  is a  $\text{CFV}_{\text{upp}}$  in  $G$ .

#### References

- [1] *M. Andelić, C. M. Da Fonseca, R. Mamede*: On the number of P-vertices of some graphs. *Linear Algebra Appl.* *434* (2011), 514–525. [zbl](#) [MR](#)
- [2] *M. Broun, J. W. Kennedy, B. Servatius*: Graph singularity. *Graph Theory Notes N. Y.* *25* (1993), 23–32.
- [3] *L. Collatz, U. Sinogowitz*: Spektren endlicher Grafen. *Abh. Math. Semin. Univ. Hamb.* *21* (1957), 63–77. (In German.) [zbl](#) [MR](#)
- [4] *D. Cvetković, M. Doob*: Developments in the theory of graph spectra. *Linear Multilinear Algebra* *18* (1985), 153–181. [zbl](#) [MR](#)
- [5] *P. W. Fowler, B. T. Pickup, T. Z. Todorova, M. Borg, I. Sciriha*: Omni-conducting and omni-insulating molecules. *J. Chem. Phys.* *140* (2014), no. 054115, 12 pages.
- [6] *P. W. Fowler, B. T. Pickup, T. Z. Todorova, R. De Los Reyes, I. Sciriha*: Omni-conducting fullerenes. *Chem. Phys. Lett.* *568/569* (2013), 33–35.
- [7] *S. C. Gong, G. H. Xu*: On the nullity of a graph with cut-points. *Linear Algebra Appl.* *436* (2012), 135–142. [zbl](#) [MR](#)
- [8] *I. Gutman, I. Sciriha*: Spectral properties of windmills. *Graph Theory Notes N. Y.* *38* (2000), 20–24. [MR](#)
- [9] *I. Gutman, I. Sciriha*: Graphs with maximum singularity. *Graph Theory Notes N. Y.* *30* (1996), 17–20. [MR](#)
- [10] *E. Hückel*: Quantentheoretische Beiträge zum Benzolproblem I. Die Elektronenkonfiguration des Benzols und verwandter Verbindungen. *Z. Phys.* *70* (1931), 204–286. (In German.) [zbl](#)
- [11] *C. R. Johnson, B. D. Sutton*: Hermitian matrices, eigenvalue multiplicities, and eigenvector components. *SIAM J. Matrix Anal. Appl.* *26* (2004), 390–399. [zbl](#) [MR](#)

- [12] *I.-J. Kim, B. L. Shader*: On Fiedler- and Parter-vertices of acyclic matrices. *Linear Algebra Appl.* *428* (2008), 2601–2613. [zbl](#) [MR](#)
- [13] *M. Marcus, H. Minc*: A Survey of Matrix Theory and Matrix Inequalities. Allyn and Bacon, Boston, 1964. [zbl](#) [MR](#)
- [14] *B. T. Pickup, P. W. Fowler, M. Borg, I. Sciriha*: A new approach to the method of source-sink potentials for molecular conduction. *J. Chem. Phys.* *143* (2015), #194105, 20 pages.
- [15] *A. J. Schwenk*: Computing the characteristic polynomial of a graph (R. A. Bari, F. Harary, eds.). *Graphs and Combin., Proc. Capital Conf., Washington, Lect. Notes Math.* 406, Springer, Berlin, 1974, pp. 153–172. [zbl](#) [MR](#)
- [16] *I. Sciriha*: Maximal core size in singular graphs. *Ars Math. Contemp.* *2* (2009), 217–229. [zbl](#) [MR](#)
- [17] *I. Sciriha*: Coalesced and embedded nut graphs in singular graphs. *Ars Math. Contemp.* *1* (2008), 20–31. [zbl](#) [MR](#)
- [18] *I. Sciriha*: A characterization of singular graphs. *Electron. J. Linear Algebra* *16* (2007), 451–462. [zbl](#) [MR](#)
- [19] *I. Sciriha*: On the construction of graphs of nullity one. *Discrete Math.* *181* (1998), 193–211. [zbl](#) [MR](#)
- [20] *I. Sciriha*: The characteristic polynomials of windmills with an application to the line graphs of trees. *Graph Theory Notes N. Y.* *35* (1998), 16–21. [MR](#)
- [21] *I. Sciriha, M. Debono, M. Borg, P. W. Fowler, B. T. Pickup*: Interlacing-extremal graphs. *Ars Math. Contemp.* *6* (2013), 261–278. [zbl](#) [MR](#)
- [22] *K. R. Sharaf, D. A. Ali*: Nullity and bounds to the nullity of dendrimer graphs. *Raf. J. of Comp. & Math's* *10* (2013), 71–86.
- [23] *S. K. Simić, M. Andelić, C. M. Da Fonseca, D. Živković*: On the multiplicities of eigenvalues of graphs and their vertex deleted subgraphs: old and new results. *Electron. J. Linear Algebra (electronic only)* *30* (2015), 85–105. [zbl](#) [MR](#)

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