The Boolean Power Sequence of Graphs*

Enzo Maria Li Marzi
Department of Mathematics, University of Messina,
Contrada Papardo, Salita Sperone 31,
98166 - Sant'Agata - Messina
emlimarzi@dipmat.unime.it

Irene Sciriha†
Department of Mathematics
Faculty of Science
University of Malta
irene@maths.um.edu.mt

20th February, 2002

Abstract

The adjacency matrix $\mathbf{A}$ of a graph G is a 0-1 matrix. The Boolean power sequence of $\mathbf{A}$ is convergent or periodic of period $p = 2$. The *index* γ of $\mathbf{A}$ is the least integer m such that $\mathbf{A}^m = \mathbf{A}^{m+1}$ if $\mathbf{A}$ converges and the least integer m such that $\mathbf{A}^m = \mathbf{A}^{m+2}$ if $\mathbf{A}$ is periodic. In this paper we determine the index γ of $\mathbf{A}$ if the graph G is bipartite. In the case of non-bipartite connected graphs, we give new lower and upper bounds for γ , which are sharp.

Keywords: *Boolean matrices, index, power sequence of matrices, bipartite graphs, adjacency matrix, periodic matrices, primitive matrices, convergent matrices, k–block partitionable.*

*A research supported by MURST: National Project "Geometric Structures, Combinatorics and their Applications" (**Italy**) and EU funds.

†Corresponding author.

1 Introduction

A *graph* $G(\mathcal{V},\mathcal{E})$ has a set $\mathcal{V}$ of $n\,(= n(G))$ vertices and an edge set $\mathcal{E}$ of $m\,(= m(G))$ edges. Each edge joins a pair of distinct vertices. All the graphs we consider are *simple*, i.e. without multiple edges or loops, and the vertex set is labelled.

In a *bipartite graph* $G(\mathcal{V}_1,\mathcal{V}_2,\mathcal{E})$, the vertex set is partitioned into two sets $\mathcal{V}_1$ and $\mathcal{V}_2$, such that an edge in $\mathcal{E}$ joins a vertex of $\mathcal{V}_1$ to a vertex of $\mathcal{V}_2$. It is clear that in a two–colouring of G, $\mathcal{V}_1$ and $\mathcal{V}_2$ are the two colouring classes.

1.1 Distance in graphs

The *distance* $d_{i,j}$ is the minimum length of a path joining the ith and the jth vertex. The maximum distance for all pairs of vertices of G is the *diameter* of G.

Among all graphs on n vertices, the *complete graph* K_n, in which an edge joins every pair of distinct vertices, the diameter is the least, namely one. The maximum diameter is reached by the *path* P_n on n vertices, which has $n-1$ edges and diameter $n-1$. The *cycle* C_n with n vertices and n edges has diameter $\lfloor \frac{n}{2} \rfloor$.

1.2 Boolean Matrices

The *Boolean algebra* $(\beta_0, +, .)$ is defined on the non-empty set $\beta_0 = \{0,1\}$ such that the two operations $+$ and $.$ defined on β_0 satisfy the following axioms

- each of the operations $+$ and $.$ is commutative;
- each operation is distributive over the other;
- there exist distinct identity elements 0 and 1 relative to the operations $+$ and $.$ respectively.

For $x \in \beta_0$,

$$\begin{array}{ll} x+0=x & \quad x0 = x.0 = 0 \\ x+1=1 & \quad x1 = x.1 = x \end{array}$$

1.3 Boolean Matrices

By a *Boolean matrix* $\mathbf{A} = (a_{ij})$, we mean a matrix over β_0 with the (i, j)-entry of $\mathbf{A}$ denoted by a_{ij}. Let $\mathcal{B}_{mn}$ denote the set of all Boolean matrices of size $m \times n$. In $\mathcal{B}_{mn}$, matrix addition and multiplication are the same as in the case of real or complex matrices but the sums and products of elements are Boolean. We observe that the *Boolean power* $\mathbf{A}^r$ of a Boolean matrix $\mathbf{A} \in \mathcal{B}_{mn}$ can be obtained from the real power of $\mathbf{A}$, by replacing the non-zero entries by 1.

Let $\mathbf{A}, \mathbf{B} \in \mathcal{B}_{mn}$. By $\mathbf{B} \leq \mathbf{A}$, we mean that if $b_{ij} = 1$ then $a_{ij} = 1$ for every i and j. We denote by $\mathbf{A}^t$ the *transpose* of the matrix $\mathbf{A}$.

In $\mathcal{B}_{1n}$, the *Boolean vector space of dimension* n,

- $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$
- $a(b_1, \ldots, b_n) = (ab_1, \ldots, ab_n)$
- $(a_1, \ldots, a_n)^c = (b_1, \ldots, b_n)$, where $b_i = 0$ if $a_i = 1$ and $b_i = 1$ if $a_i = 0$.

for all $a, a_i, b_i \in \beta_0$.

The *row space* $R(A)$ of a matrix $\mathbf{A}$ is the span of the set of all row vectors of $\mathbf{A}$. The *column space* $C(\mathbf{A})$ is defined likewise.
Each entry of the $m \times n$ *universal matrix* $\mathbf{J}$ is 1 and we define the complement $\mathbf{A}^c$ of $\mathbf{A}^c$ as the matrix satisfying $\mathbf{A} + \mathbf{A}^c = \mathbf{J}$. The $m \times n$ *zero matrix* $\mathbf{0}$ is the matrix each of whose entries is zero. We also denote by $\mathbf{j}$ the vector $(1, \ldots, 1) \in \mathcal{B}_{1n}$ and with $\mathbf{0}$ the vector $(0, \ldots, 0) \in \mathcal{B}_{1n}$. The *row (column) rank* of a matrix $\mathbf{A}$ is the cardinality of the **unique** basis of $R(A)(C(A))$. We denote the row (column) rank of $\mathbf{A}$ by $\varrho_r(\mathbf{A})(\varrho_c(\mathbf{A}))$. In general $\varrho_r(\mathbf{A}) \neq \varrho_c(\mathbf{A})$ but if $\mathbf{A} \in \mathcal{B}_{nn}$ is symmetric, then $\varrho(\mathbf{A}) = \varrho_r(\mathbf{A}) = \varrho_c(\mathbf{A}) = \varrho$.

1.4 Powers of a Matrix

Definition 1.1. *A matrix* $\mathbf{A} \in \mathcal{B}_{nn}$ *is said to be* convergent *if in the sequence* $\{\mathbf{A}^h, h = 1, \ldots\}$, $\mathbf{A}^m = \mathbf{A}^{m+1}$ *for some* $m \in \mathbb{Z}^+$. *Then* $\mathbf{A}$ *is said to converge to the matrix* $\mathbf{A}^m$ *and the* index γ *of* $\mathbf{A}$ *is the least integer* m *such that* $\mathbf{A}^m = \mathbf{A}^{m+1}$.

Clearly $\mathbf{A}^{m'} = \mathbf{A}^\gamma$, $\forall m' \geq \gamma$.

Definition 1.2. *A matrix* $\mathbf{A} \in \mathcal{B}_{nn}$ *is called a* primitive matrix *if* $\mathbf{A}$ *converges to the matrix* $\mathbf{J} \in \mathcal{B}_{nn}$*. If* $\mathbf{A}$ *is primitive , the* index γ *of* $\mathbf{A}$ *is the least integer* m *such that* $\mathbf{A}^m = \mathbf{J}$ *.*

Definition 1.3. *A matrix* $\mathbf{A} \in \mathcal{B}_{mn}$ *is said to be* oscillatory (periodic) *of period* p *if in the power matrix sequence of* $\mathbf{A}$*,* $\exists m \in \mathbb{Z}^+$ *such that* $\mathbf{A}^m = \mathbf{A}^{m+p}$*, where* $p > 1$ *is the smallest positive integer for which this holds. The positive integer* p *is called the* period of oscillation *of* $\mathbf{A}$ *. The least integer* m *such that* $\mathbf{A}^{m+p} = \mathbf{A}^m$ *for this positive integer* p *is called the* index γ *of* $\mathbf{A}$*.*

Clearly $\mathbf{A}^{m+p} = \mathbf{A}^m$, $\forall m \geq \gamma$.

Definition 1.4. *The* adjacency matrix $\mathbf{A}(G)$ *(or* $\mathbf{A}$ *) of a graph* G *having vertex set* $\mathcal{V}(G) = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ *is an* $n \times n$ *symmetric matrix* (a_{ij}) *such that* $a_{ij} = 1$ *if* $\mathbf{v_i}$ *and* $\mathbf{v_j}$ *are adjacent and* 0 *otherwise.*

Remark 1.5. A graph G determines and is determined by its adjacency matrix. The adjacency matrix $\mathbf{A}$ of a labelled graph G can be considered to be a Boolean matrix. We refer to the Boolean power sequence $\{\mathbf{A}^h : h = 1, 2, ...\}$ as the *Boolean power sequence* of G.

In section **??**, we see that the power sequence of the adjacency matrix distinguishes bipartite from non-bipartite graphs. Since the index of disconnected graphs depends on that of its components, we study first the index of connected graphs, in section **??**. We see that the diameter plays a crucial role. By concentrating on graphs, rather than digraphs, the bounds for the index of a graph given in [1] can be markedly improved, as we see in Theorem **??**.

Main Theorem *Let* G *be a connected graph of diameter* D *and index* γ*.*

(a) G *is bipartite if, and only if,* $\gamma = D - 1$*.*

(b) G *is not bipartite if, and only if,* $D \leq \gamma \leq 2D$*.*

In section **??**, we see that the index of disconnected graphs is not only determined by the nature of the components, depending on whether they are bipartite or otherwise, but also on the maximum value of the index of the components. The rank of A^γ is deduced easily by writng the power of the adjacency matrix, written in block form, with each diagonal block corresponding to a connected component. We conclude, in section **??** by showing that the bounds for the index are sharp.

2 Two Types of Matrices

The entries of the rth power of the adjacency matrix give the number of walks of length r of G. [Theorem 16.8 of [2]]The following theorem follows immediately.

Theorem 2.1. *Let G be a labelled graph with adjacency matrix $\mathbf{A}$. Then the (i,j)-entry of the Boolean power $\mathbf{A}^r$ of $\mathbf{A}$ is 1 if, and only if, there exists a walk of length r from i to j.*

Remark 2.2. We recall that a 0-1 matrix represents a digraph. If the matrix is symmetrical, then it represents a graph. The graph G is bipartite if and only if it has no odd cycles. The proof of the following theorem, then, follows from Proposition 5.4.25 of [3].

Theorem 2.3. *Let G be a connected graph with adjacency matrix $\mathbf{A}$.*

(a) $\mathbf{A}$ is primitive if, and only if , G is not bipartite ;

(b $\mathbf{A}$ is oscillatory if, and only if, G is bipartite, and the period of oscillation of $\mathbf{A}$ is 2.

Remark 2.4. We say that G is *o*scillatory (periodic) or *primitive* if $\mathbf{A}$ is oscillatory or primitive respectively.

3 The index of connected graphs

Remark 3.1. By $\{\mathbf{A}^r : r = 1, 2, \ldots\}$, we shall henceforth mean the Boolean power sequence of a matrix $\mathbf{A}$.

Lemma 3.2. *Let G be a connected graph with diameter D. Then*
(i) $\mathbf{A}^r \leq \mathbf{A}^{r+2}$ for every $r \geq 1$ and
(ii) $\mathbf{A}^{D-1} + \mathbf{A}^D = \mathbf{J}$.

Proof. i) If the (i,j)-entry of $\mathbf{A}^r$ is 1, then by Theorem **??**, there exists a walk of length r from i to j. Therefore there exists a walk of length $r+2$ between the same vertices, obtained by continuing to traverse an edge incident to j backward and forward. Thus the (i,j)- entry in $\mathbf{A}^{r+2}$ is 1 .
ii) If, a_{ij}^D, the (i,j)- entry in $\mathbf{A}^D$, is 0, then by (i) above, there exists a walk of length d_{ij}, less than D and of different parity to D, from i to j. Hence d_{ij} is of the same parity as $D-1$ and the (i,j)-entry in $\mathbf{A}^{D-1}$ is 1. Thus either $a_{ij}^D = 1$ or $a_{ij}^{D-1} = 1$ or both. $\square$

Lemma 3.3. *Let G be a connected graph. The following assertions are equivalent :*

(a) G is bipartite;

(b) there exists $\mathbf{v} \notin \{\mathbf{0},\ \mathbf{j}\}$, such that $R(\mathbf{A}^{d-1}) = R(\mathbf{A}^{d}) = \{\mathbf{0},\ \mathbf{v},\ \mathbf{v^c},\ \mathbf{j}\}$;

(c) there exist two row vectors $\mathbf{v}_i, \mathbf{v}_j \in R(\mathbf{A}^d)$, such that $\mathbf{v}_i = \mathbf{v}_j^c$, $\mathbf{v}_i \notin \{\mathbf{0},\ \mathbf{j}\}$.

Proof. (a) implies (b). Indeed, if G is bipartite, then it has no odd cycles. Also the adjacency matrix can be written as $\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{L} \\ \mathbf{L}^t & \mathbf{0} \end{pmatrix}$ where the two diagonal blocks of $\mathbf{A}$ conform with the two colouring classes of G. Hence for $i, j \in \mathcal{V}_k$, $k \in \{1, 2\}$, d_{ij} is even and $a^r_{ij} = 1$, $r \geq d_{ij}$ implies that r is even. Also, for i, j in distinct monochromatic classes, d_{ij} is odd and $a^r_{ij} = 1, r \geq d_{ij}$ implies that r is odd.

By Lemma **??** ii, $\mathbf{A}^{D-1} + \mathbf{A}^D = \mathbf{J}$. Therefore the power sequence of $\mathbf{A}$ is of the form

(1) $\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{L} \\ \mathbf{L}^t & \mathbf{0} \end{pmatrix}$, $\mathbf{A}^2 = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}$,..., $\mathbf{A}^{D-1} = \begin{pmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J} & \mathbf{0} \end{pmatrix}$,
$\mathbf{A}^D = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{pmatrix}$, if D is even ;

(2) $\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{L} \\ \mathbf{L}^t & \mathbf{0} \end{pmatrix}$, $\mathbf{A}^2 = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}$,..., $\mathbf{A}^{D-1} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{pmatrix}$,
$\mathbf{A}^D = \begin{pmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J} & \mathbf{0} \end{pmatrix}$, if D is odd.

We note that the row spaces of $\mathbf{A}^{D-1}$ and of $\mathbf{A}^D$ contain a vector $\mathbf{v} \notin \{\mathbf{0},\ \mathbf{j}\}$ and its complement $\mathbf{v}^c$. Also $\mathbf{v} \pm \mathbf{v}^c \in \{\mathbf{0},\ \mathbf{j}\}$. Hence (b) follows.

(b) implies (c) trivially.

(c) implies (a).
Let (c) be true. We can divide the vertices of G into two classes, corresponding to the entries 0, 1, respectively, of the vector $\mathbf{v}_i$. We show that two vertices k, ℓ in the same class are not adjacent. Suppose, for contradiction, that k and ℓ are adjacent. Since a^D_{ki}, $a^D_{\ell i}$ are either both 0 or both 1, then $a^{D-1}_{ki} = a^{D-1}_{\ell i}$. If $a^D_{ki} = a^D_{\ell i} = 0$, then $a^D_{kj} = a^D_{\ell j} = 1$. But then there is a

walk of length $D-1$ from i to k and therefore a walk of length D from i to ℓ passing through k, a contradiction. If, on the other hand, $a_{ki}^D = a_{\ell i}^D = 1$, then $a_{kj}^D = a_{\ell j}^D = 0$. But then there is a walk of length $D-1$ from j to k and hence a walk of length D from j to k and then to ℓ, a contradiction. Hence the vertex set of G is partitioned into two classes, the vertices in each class being mutually non-adjacent. Thus (a) is true. $\square$

Remark 3.4. *From the proof of Lemma* **??**, *it is clear that oscillation of* $\mathbf{A}^r$ *starts from* $r = D-1$.

Theorem 3.5. *The index of a bipartite graph is D-1.*

Remark 3.6. In [1], an upperbound for the index γ when $\mathbf{A}$ is the adjacency matrix of a convergent digraph is given as $2n-2$. The following theorem gives a marked improvement for convergent graphs, particularly for graphs of small diameter.

Theorem 3.7. *Let G be a connected graph, with diameter D. If G is not bipartite, then the index γ of* $\mathbf{A}$ *satisfies the inequality* $D \leq \gamma \leq 2D$.

Proof. For $r < D$, the ijth entry, $a_{ij}^{(r)}$, of $\mathbf{A}^r$ is zero. Hence by definition of γ (as the least positive integer r for which $\mathbf{A}^r = J$), $\gamma \geq D$. It remains to prove that $\mathbf{A}^{2D} = \mathbf{J}$.

Let $\mathbf{A}^D = (a_{ij}^{(D)})$ and $\mathbf{A}^{2D} = \mathbf{A}^D\mathbf{A}^D = (a_{ij}^{(2D)})$.

Clearly $a_{ij}^{(2D)} = \sum_{h=1}^{n} a_{ih}^{(D)} a_{jh}^{(D)}$. Suppose, for contradiction, that for some pair (i,j) of vertices, $a_{ij}^{(2D)} = 0$. Then $a_{ih}^{(D)} = 1, a_{jh}^{(D)} = 0$ or $a_{ih}^{(D)} = 0$, $a_{jh}^{(D)} = 1$ or $a_{ih}^{(D)} = a_{jh}^{(D)} = 0$ for all $h = 1, \ldots, n$. Thus $a_{ih}^{(D)}$ and $a_{jh}^{(D)}$ are not 1 simultaneously. Because the graph G is not bipartite, from Lemma **??**, no two row vectors of $\mathbf{A}^D$ are complementary. Thus for some vertex h, $a_{ih}^{(D)} = a_{jh}^{(D)} = 0$. Then there are no walks of length D from i to h and from j to h. So there exist walks of length $D-1$ from i to h, and from j to h. But then there exists a walk of length $2D-2$ from i to j. By Lemma **??**, $a_{ij}^{(2D)} = 1$, contradicing the premise. $\square$

Remark 3.8. From Theorems **??** and **??**, we have a characterization of bipartite and non-bipartite connected graphs by considering the index.

Theorem 3.9. *Let G be a connected graph with diameter D and index γ.*

(a) G is bipartite if, and only if, $\gamma = D - 1$.

(b) G is not bipartite if, and only if, $D \leq \gamma \leq 2D$.

Definition 3.10. *A matrix $\mathbf{A} \in \mathcal{B}_{nn}$ will be called* h-block partitionable *if there exists an orthogonal matrix $\mathbf{P}$ such that*

$$\mathbf{B} = \mathbf{P^t}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{A}_1 & . & . & . & . & . \\ . & \mathbf{A}_2 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & \mathbf{A}_h & . \\ . & . & . & . & . & . \end{bmatrix}$$

where each $\mathbf{A}_i$, $i = 1, \ldots h$, is square and the other entries of $\mathbf{B}$, not belonging to $\mathbf{A}_i$, are 0.

Remark 3.11. If a graph without isolated vertices has h connected components and is labelled so that the vertices of a component take consecutive values, then its adjacency matrix $\mathbf{A}$ is h-block partitionable. It is clear that a higher power $\mathbf{A}^r$, $r \geq 2$ of $\mathbf{A}$ is k-block partitionable, where $k \geq h$.

Theorem 3.12. *Let G be a connected graph with diameter D. Then G is bipartite if, and only if, for $r \geq D - 1$, the Boolean even powers $\mathbf{A}^r$ are 2-block partionable.*

Proof. If G is bipartite, then the Boolean power sequence of the adjacency matrix is (1) or (2) of Lemma **??**. Therefore, for $r \geq D - 1$, the Boolean even powers of $\mathbf{A}$ are 2-block partionable.
Conversely if the Boolean even powers of $\mathbf{A}$ are 2-block partitionable then the adjacency matrix $\mathbf{A}$ is oscillatory. Consequentely, from Lemma **??**, G is bipartite. □

4 Disconnected Graphs

Remark 4.1. From the definition of index, and the proof of Lemma **??**, it is clear that if G has a bipartite component, then it is oscillatory. If G is labelled so that the isolated vertices are represented by the k bottom rows of $\mathbf{A}$, then $\mathbf{A}^r$ has k zero rows for $r \geq 1$. For the purpose of determining the index, we can therefore concentrate on graphs with non-empty connected components when considering the index. It is clear that if each component of G is non-bipartite, then the index of G is that of the component with the maximum index.

Let G have h_1 bipartite connected components and h_2 non-bipartite connected components.

Lemma 4.2. *(a) If $h_1 \neq 0$, then the Boolean power sequence of $\mathbf{A}$ is oscillatory of period $p = 2$, the odd powers of $\mathbf{A}$ are $(h_1 + h_2)$-block partionable, and the even powers of $\mathbf{A}$ are $(2h_1 + h_2)$-block partionable.*

(b) If $h_1 = 0$, the Boolean power sequence of $\mathbf{A}$ is convergent, and all powers of $\mathbf{A}$ are $(h_1 + h_2)$-block partionable.

Proof. Let $h = h_1 + h_2$.
(a): The graph G can be labelled so that

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & . & . & . & . & . \\ . & \mathbf{A}_2 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & \mathbf{A}_h & . \\ . & . & . & . & . & . \end{pmatrix}$$

where $\{\mathbf{A}_i,\ i = 1, \ldots, h\}$ is the set of the adjacency matrices of the connected components of G. By Theorem **??**, the adjacency matrix, $\mathbf{A}_i$, of each connected component of G, either converges to $\mathbf{J}$ or else is periodic of period 2 depending on whether its corresponding component is a non-bipartite or a bipartite subgraph respectively. Then for $h_1 \neq 0$, $\mathbf{A}$ is periodic of period 2. Because two vertices i, j, belonging to two different connected components of G, are not connected by a walk, then any entries a_{ij} of a power $\mathbf{A}^r$ is zero. Consequentely an odd power of $\mathbf{A}$ is h-block partitionable. By Theorem **??**, the even powers of $\mathbf{A}$ are $(2h_1 + h_2)$-block partitionable.

(b) If G has no biparite components, then each matrix $\mathbf{A}_i$ converges to the matrix $\mathbf{J}$ and the result follows.

□

Remark 4.3. Let us denote the diameters of the h_1 bipartite connected components of G by $\{D'_i, i \in \{1, \ldots, h_1\}\}$, and the diameters of the h_2 non- bipartite connected components of G by $\{D''_i, i \in \{1, \ldots, h_2\}\}$. Let $D' = max\{D'_i\}$ and $D'' = max\{D''_i\}$. It is clear that $\gamma = max\{\gamma_1 \ldots, \gamma_h\}$, where γ_i is the index of the adjacency matrix, $\mathbf{A}_i$, corresponding to the i-th connected component of G. The values of the index for connected graphs, established in Theorem **??**, prompt us to consider the cases for $D' - 1$ lying within and outside the interval $[D'',\ 2D'']$.

Theorem 4.4. *Let D be the diameter of a graph G.*

(a) If $2D'' < D' - 1$, then $D = D'$ and $\gamma = D' - 1$;

(b) if $D'' \leq D' - 1 \leq 2D''$, then $D = D'$ and $D' - 1 \leq \gamma \leq 2D''$;

(c) if $D' - 1 < D''$, then $D = D''$ and $D \leq \gamma \leq 2D$.

Proof. (a) From Lemma **??**, the Boolean power sequence of $\mathbf{A}$ is oscillatory and from Theorems **??** and **??**, for every i, $\gamma_i \leq D' - 1$.
(b) Although $D' > D''$, the index for a non-bipartite component with diameter D'' exceeds $D' - 2$. Thus if the maximum index of the non-bipartite components of G is γ'', then γ= max$\{\gamma'', D' - 1\}$.
(c) If G has a bipartite component, then oscillation of $\mathbf{A}^r$ sets in when r reaches γ'' (defined in (b)). □

5 The Rank of $\mathbf{A}^r$

Theorem 5.1. *Let G be a connected graph with adjacency matrix $\mathbf{A}$, and index γ. Then $\varrho(A) \geq \varrho(A^2) \geq \ldots \geq \varrho(\mathbf{A}^\gamma)$, and $\varrho(\mathbf{A}^\gamma) = 1$ if $\mathbf{A}$ is primitive, $\varrho(\mathbf{A}^\gamma) = 2$ if $\mathbf{A}$ is oscillatory.*

Proof. If $\mathbf{v_i} + \mathbf{v_j} = \mathbf{v_h}$, $\mathbf{v_i}, \mathbf{v_j}, \mathbf{v_h} \in R(\mathbf{A}^r)$, then $\mathbf{A}(\mathbf{v_i} + \mathbf{v_j}) = \mathbf{A}\mathbf{v_i} + \mathbf{A}\mathbf{v_j} = \mathbf{A}\mathbf{v_h}$, $\mathbf{A}\mathbf{v_i}, \mathbf{A}\mathbf{v_j}, \mathbf{A}\mathbf{v_h} \in R(\mathbf{A}^{r+1})$. Thus the rank of $\mathbf{A}^{r+1}$ never exceeds that of $\mathbf{A}^r$. Theorem **??** determines the lower bound $\varrho(\mathbf{A}^\gamma) = 2$ if G is bipartite, and that for non-bipartite graphs follows from Theorem **??**; that is $\varrho(\mathbf{A}^\gamma) = 1$. □

Remark 5.2. Since the rank of G is the sum of the ranks of its connected components, the following theorem follows immediately.

Theorem 5.3. *Let G be a graph, with h_1 connected bipartite components and h_2 connected non-bipartite components. If $\mathbf{A} = \mathbf{A}(G)$ and γ is the index of $\mathbf{A}$, then $\varrho(\mathbf{A}) \geq \varrho(\mathbf{A}^2) \geq \ldots \geq \varrho(\mathbf{A}^\gamma)$, and $\varrho(\mathbf{A}^\gamma) = 2h_1 + h_2$.*

6 Conclusion

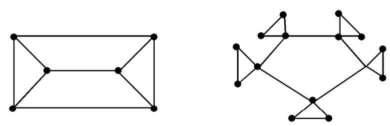

Figure 1: *Graphs with index equal to D.*

The graphs in Figure 1 are examples of non-bipartite graphs with index D and those in Figure 2 have index $2D$, verifying that the bounds on γ are sharp. Other graphs with index D are the maximal planar graphs and the cycles C_n.

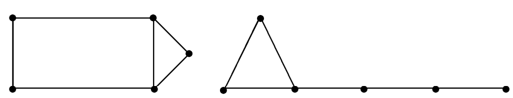

Figure 2: *Graphs with index equal to $2D$.*

The graph in Figure 3 has index strictly between the two extremes. The characterization of the graphs with the various indices is still open.

Theorem 6.1. *Let G be a non-bipartite graph with diameter D and index γ. The the bounds D and $2D$ on γ are sharp.*

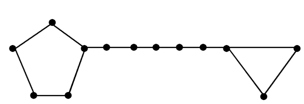

Figure 3: *A graph with $D < \gamma < 2D$.*

For graphs of order n and $D < \frac{n}{2}$, γ is necessarily less than n. In view of the fact that, for the graphs with index greater than $n-1$ which we studied, γ is even, we have the following conjecture.

Conjecture 6.2. *If $\gamma \geq n$, then γ is even.*

References

1 M. Zhengke and Z. Kemin, *The Minimum Norm of Solutions of a Boolean Matrix Equation*, Graph Theory Notes of New York XXXVIII, New York Academy of Sciences (2000). pp.7-11.

2 F. Harary, Graph Theory ,Addison Wesley, New York (1968).

3 Ki Hang Kim, Boolean Matrix Theory and Applications , Marcel Dekker inc., New York and Basel (1982).