Graph Theory Notes of New York XLIX, 38-40 (2005)
New York Academy of Sciences.

[GTN XLIX:7] MINIMAL CONFIGURATIONS AND INTERLACING

Irene Sciriha[1] and Ivan Gutman[2]

[1]Department of Mathematics
University of Malta
Msida, MALTA
<irene.sciriha-aquilina@um.edu.mt>

[2]Faculty of Science
University of Kragujevac
P.O. Box 60
YU-34000 Kragujevac, YUGOSLAVIA
<gutman@knez.uis.kg.ac.yu>

Abstract

A graph is singular of nullity η if zero is an eigenvalue of its adjacency matrix with multiplicity η. A subgraph that forces a graph to be singular is called a minimal configuration. We show various properties of minimal configurations.

1. Introduction

Let $G = G(V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $\mathbf{A}(G) = \mathbf{A}$ is (a_{ij}), where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. The values of λ for which there exist non-zero vectors $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ are called *eigenvalues* of G. The vectors $\mathbf{x}$ are said to be *eigenvectors* of $\mathbf{A}$. A graph is singular of *nullity* η if zero is an eigenvalue of its adjacency matrix $\mathbf{A}$ with multiplicity η.

There are subgraphs called *minimal configurations* that are found in singular graphs. We study properties of minimal configurations that determine their structure.

2. Singular Graphs

Definition 2.1: A *kernel eigenvector* $\mathbf{x}_0$ of a singular graph with adjacency matrix $\mathbf{A}$ is a non-zero vector in the nullspace of $\mathbf{A}$. ■

Remark 2.2: A kernel eigenvector $\mathbf{x}_0 \in \mathbb{R}^n$ of a singular graph G satisfies $\mathbf{A}\mathbf{x}_0 = \mathbf{0}$.

Definition 2.3: Let $\mathbf{x}_0$ be a kernel eigenvector of a singular graph G of order $n \geq 3$. A subgraph of G induced by the vertices corresponding to the non-zero entries of $\mathbf{x}_0$ is said to be a *core*, χ, (with respect to $\mathbf{x}_0$). The core is sometimes denoted by χ_p, or by $\chi_{\mathbf{x}_0}$, where p, the number of vertices of the core, is called the *core order*. ■

The following definition of minimal configurations—the building blocks of singular graphs—is given in [1].

Definition 2.4: A singular graph Γ of order $n \geq 3$, having a core χ_p and *periphery* $\mathcal{P} := V(\Gamma) - V(\chi_p)$ is said to be a *minimal configuration*, of core order p, if the following conditions are satisfied:
(1) $\eta(\Gamma) = 1$,
(2) $\mathcal{P} = \emptyset$ (that is, $\mathcal{P}$ is empty, no vertex) or $\mathcal{P}$ induces an empty graph (no edge), and
(3) for $\mathcal{P} \neq \emptyset$, $\eta(\mathbf{A}(\Gamma \backslash v)) > 1$, for all $v \in \mathcal{P}(\Gamma)$. ■

We show that condition (3) in the definition can be replaced by the more powerful condition in the following theorem.

Theorem 2.5: A graph Γ is a minimal configuration if and only if

(1) $\eta(\Gamma) = 1$,

(2) $\mathcal{P} = \emptyset$ or $\mathcal{P}$ induces an empty graph, and

(3) $\eta(\chi_p) = 1 + |\mathcal{P}|$.

Proof: By the *interlacing theorem*, addition to or removal of a vertex, from a graph, changes the nullity by at most one.

Let G be a minimal configuration, then $\eta(G) = 1$. By (3) of Definition 2.4, and the interlacing theorem, $\eta(G\backslash v) = 2$ if $v \in \mathcal{P}$.

Case 1: $\mathcal{P} \neq \emptyset$. In this case there are zero entries in x_0, the kernel eigenvector of G, which may be written

$$x_0 = (\alpha_1, \alpha_2, \ldots, \alpha_{|\chi|}, 0_1, 0_2, \ldots, 0_{|\mathcal{P}|}),$$

where α_i, $1 \leq i \leq |\chi|$ are the only non-zero entries of x_0. There is a linear combination, *Rel*, of the rows or columns of A with coefficients equal to the entries of x_0. Deletion of a vertex $v_j \in \mathcal{P}$ leaves *Rel* unaffected since the row or column corresponding to v_j is not involved. This means that although *Rel* remains valid, by (3) of Definition 2.4, *Rel* corresponds to two linearly independent kernel eigenvectors on deletion of any vertex of $\mathcal{P}$. Since no two vertices of $\mathcal{P}$ are adjacent, $v_1, v_2, \ldots, v_{|\mathcal{P}|}$ can be successively removed from G in any order, increasing the nullity by one with each deletion, until the core χ is obtained, so that $\eta(\chi) > 1$.

Case 2: $\mathcal{P} = \emptyset$. In this case $\Gamma = \chi$ and $\eta(\chi) = 1$; Γ is said to be a *nut graph* [2].

In both cases, $\eta(\chi) = 1 + |\mathcal{P}|$, as required.

Conversely, let $\eta(\chi) = 1 + |\mathcal{P}|$ and $\eta(\chi + v_1 + v_2 + \ldots + v_{|\mathcal{P}|}) = \eta(G) = 1$. Thus, $\eta(\chi) \geq 1$. If $\eta(\chi) = 1$, then $\mathcal{P} = \emptyset$ and $\chi = G$ is a nut graph. If $\eta(\chi) > 1$, then $|\mathcal{P}| \geq 1$. By the Interlacing theorem,

$$|\mathcal{P}| - (j-1) \leq \eta(\chi + v_1 + v_2 + \ldots + v_j) \leq |\mathcal{P}| + j + 1.$$

For $j = |\mathcal{P}|$, $1 \leq \eta(\chi + v_1 + v_2 + \ldots + v_{|\mathcal{P}|}) \leq 2|\mathcal{P}| + 1$, independently of the sequence in which the vertices of $\mathcal{P}$ are added to χ. However, the nullity of G is one. This forces the nullity to decrease by unity with each addition of a vertex to χ. In particular, $\eta(G\backslash v_j) = 2$ for all $v_j \in \mathcal{P}$ as required. ∎

Theorem 2.6: The deletion of a vertex of the core, χ, from a minimal configuration Γ produces a non-singular graph.

Proof: Let Γ be a minimal configuration with kernel eigenvector x_0 and let $v \in \chi$, the core of Γ. Suppose $\Gamma\backslash v$ is singular, then Γ has another kernel eigenvector with a zero entry corresponding to v, so that $\eta(\Gamma) > 1$, a contradiction. ∎

Remark 2.7: A minimal configuration, with kernel eigenvector x_0, corresponding to a core χ_{x_0}, can be considered to be a graph of nullity one with a minimal number of edges and vertices. The labelled vertices in the Figure refer to the vertices of the core $\overline{K}_4$ in each minimal configuration.

If a minimal configuration Γ is a subgraph of a graph G such that the non-zero entries of the kernel eigenvector are preserved, then G is singular.

For G to be singular with minimal configuration Γ, it is sufficient that the core vertices of Γ have the same degrees as they do in G.

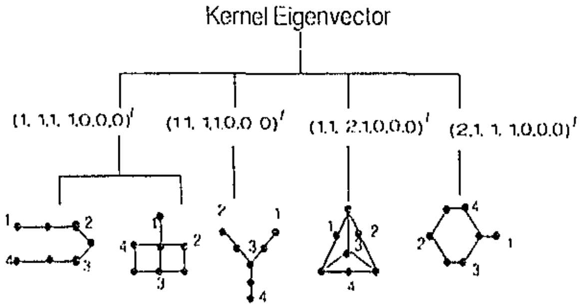

Figure: Minimal configurations with core $\overline{K}_4$.

Corollary 2.8: A minimal configuration is connected.

Proof: Suppose that a minimal configuration Γ has at least two components. Since $\eta(\Gamma) = 1$, only one component, H, is singular and $V(\chi) \subseteq V(H)$. The vertices in the periphery $\mathcal{P}$ induce an empty graph $\Gamma\backslash V(\chi)$. However, Γ then has isolated vertices and $\eta(\Gamma) > 1$, a contradiction. ∎

References

[1] I. Sciriha; On the construction of graphs of nullity one. *Discrete Mathematics*, 181, 193-211 (1998).

[2] I. Sciriha and I. Gutman; Nut graphs—maximally extending cores. *Utilitas Mathematica*, 54, 257–272 (1998).

Received: March 16, 2004