

# An Infinite Servers Nodes Network in the Study of a Pensions Fund

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**Abstract:** It is suggested a representation of a pensions fund through a stochastic network with two infinite servers nodes. This representation allows to deduce an equilibrium condition of the system with basis on the identity of the random rates, at which contributions arrive to the fund and pensions are paid by the fund, expected values.

**Keywords** - Pensions fund, stochastic network, tandem queues, Poisson process, Wald's equation

## 1. Introduction

Consider two nodes, service centres, *A* and *B* both with infinite servers. The traffic through arches *a* to *e* is as it is schematized in Figure 1.

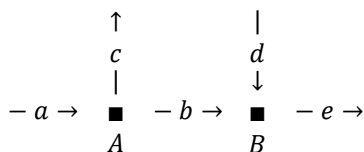


Figure 1. Traffic in the stochastic network

The users arrive to node *A* by arch *a* at rate  $\lambda_A$ . And the service time at this node is a positive random variable with distribution function (d.f.)  $G_A$  and finite mean  $\alpha_A$ . After node *A* the users go to node *B* through *b* with probability *p*. Or just abandon the system through arch *c* with probability  $1 - p$ .

The users coming directly from outside through *d* at rate  $\lambda_B$  have also access to the service supplied at *B*, according to a positive random variable with d.f.  $G_B$  and finite mean  $\alpha_B$ . The system is abandoned by these users through arch *e*.

In [1] this system is suggested as a representation of a pensions fund. So at node *A* arrive individuals that pay, during the service time, their contributions to the fund. The pensioners are at node *B*, which service represents their pensions payment by the fund. This representation reflects also the functions of the common social security funds and that is why it accepts the access of pensioners that have not

formerly participated, at node *A*, in the building of the fund.

The target of this study is, having this representation in mind, to obtain results about the transient behaviour of the system from the point of view of its equilibrium and autonomy.

## 2. The Fund Equilibrium

Let  $N_A(t)$  and  $N_B(t)$  be the random variables (r.v.) that represent the number of individuals by time *t* at nodes *A* and *B*, respectively. Consider also the sets of r.v., i.i.d.:

$$X_{A_1}(t), X_{A_2}(t), X_{A_3}(t), \dots, (X_{B_1}(t), X_{B_2}(t), X_{B_3}(t), \dots)$$

which designate the unitary contributions, pensions by time *t*, with mean  $m_A(t)$  and  $m_B(t)$ <sup>1</sup>.

The system is in equilibrium when the expected values of the rates at which the contributions are being received and the pensions are being payed by the fund are identical:

$$E \left[ \sum_{i=1}^{N_A(t)} X_{A_i}(t) \right] = E \left[ \sum_{j=1}^{N_B(t)} X_{B_j}(t) \right].$$

That is, by Wald's equation:

$$m_A(t)E[N_A(t)] = m_B(t)E[N_B(t)] \tag{1}$$

Eq. (1) just stays that at each instant the mean value of the unitary pension should be proportional to the mean value of the unitary contribution, with the ratio between the averages of the numbers of contributors and pensioners as proportionality factor. Being  $t = 0$  the origin time, its solution corresponds, for  $t > 0$ , to the following pairs:

$$(m_A(t); m_B(t)) = \left( m_A(t); \frac{m_A(t)E[N_A(t)]}{E[N_B(t)]} \right),$$

where  $m_A(t)$  is independent of the equilibrium.

<sup>1</sup> Under the term unitary contribution, pension, it is meant the amount of money that one individual pays, receives, by unit of time.

If the mean value of the unitary pension is initially 1, and grows continuously with an interest rate  $r$ ,

$$m_B(t) = e^{rt}$$

$$m_A(t) = e^{rt}(E[N_B(t)]/E[N_A(t)]).$$

It is elementary, after Eq. (1),

$$E[N_A(t)] < E[N_B(t)] \Rightarrow m_A(t) > m_B(t).$$

So, in equilibrium, the mean value of the unitary pension is smaller than the mean value of unitary contribution whenever the number of pensioners at  $B$  is bigger than the number of contributors at  $A$ .

### 3. The Case of Poisson Arrivals

If the arrivals from outside at nodes  $A$  and  $B$  are according to a Poisson process, with rates  $\lambda_A$  and  $\lambda_B$ , respectively, the system may be seen as a two nodes network where the first node is a  $M/G/\infty$  queue and second a  $M_t/G/\infty$  queue, see for instance [2]. So,  $N_A(t)$  is Poisson distributed with parameter, see [3]

$$\lambda_A \int_0^t (1 - G_A(v)) dv.$$

The output of the first node is a nonhomogeneous Poisson process with intensity function  $\lambda_A G_A(t)$  and, consequently, the global arrivals rate at node  $B$  is  $p\lambda_A G_A(t) + \lambda_B$ . Under this conditions  $N_B(t)$  is Poisson distributed with parameter, see [4]:

$$\int_0^t (p\lambda_A G_A(v) + \lambda_B)(1 - G_B(t - v)) dv.$$

And Eq. (1) is written like this

$$m_A(t)\lambda_A \int_0^t (1 - G_A(v)) dv = m_B(t) \int_0^t (p\lambda_A G_A(v) + \lambda_B)(1 - G_B(t - v)) dv \quad (2).$$

When  $t \rightarrow \infty$  the equilibrium conditions assumes the following form where  $m_i = \lim_{t \rightarrow \infty} m_i(t)$ ,  $i = A, B$ :

$$m_A \lambda_A \alpha_A = m_B (\rho \lambda_A + \lambda_B) \alpha_B \quad (3).$$

If the service times at nodes  $A$  and  $B$  have d.f. concentrated in the intervals  $[0, a]$  and  $[0, b]$ ,  $m_A \lambda_A \alpha_A = m_B (\rho \lambda_A + \lambda_B) \alpha_B$  for  $t \geq a + b$ .

## 4. Examples

In this section some concrete examples of service times distributions will be considered.

### 4.1 Uniformly Distributed Service Times

If the service times are uniformly distributed,

supposing that  $\alpha_B < \alpha_A$ , it is obtained for Eq. (2) in  $0 \leq t < 2\alpha_A + 2\alpha_B$ , not to repeat what has just been mentioned:

$$i) \quad m_A(t)\lambda_A \left(t - \frac{t^2}{4\alpha_A}\right) = m_B(t)\lambda_B \left(t - \frac{t^2}{4\alpha_B}\right) +$$

$$m_B(t)p\lambda_A \left(\frac{t^2}{4\alpha_A} - \frac{t^3}{24\alpha_A\alpha_B}\right), \quad \text{if } 0 \leq \frac{t}{2} < \alpha_B$$

$$ii) \quad m_A(t)\lambda_A \left(t - \frac{t^2}{4\alpha_A}\right) = m_B(t)\lambda_B \alpha_B +$$

$$m_B(t)p\lambda_A \left(-\frac{\alpha_B^2}{3\alpha_A} - \frac{t\alpha_B}{2\alpha_A}\right), \quad \alpha_B \leq \frac{t}{2} < \alpha_A$$

$$iii) \quad m_A(t)\lambda_A \alpha_A =$$

$$m_B(t)\lambda_B \alpha_B + m_B(t)p\lambda_A \left(-\alpha_A - \frac{\alpha_A^2}{12\alpha_B} +$$

$$t - \frac{(t-\alpha_A)^2}{4\alpha_B} + \frac{(t-2\alpha_B)^3}{24\alpha_A\alpha_B}\right), \quad \text{if } \alpha_A \leq \frac{t}{2} < \alpha_A + \alpha_B.$$

### 4.2 Exponentially Distributed Service Times

If the service times are exponentially distributed the equilibrium distribution is given by:

$$i) \quad m_A(t)\lambda_A \alpha_A \left(1 - e^{-\frac{t}{\alpha_A}}\right) = m_B(t)(p\lambda_A +$$

$$\lambda_B)\alpha_B \left(1 - e^{-\frac{t}{\alpha_B}}\right) - m_B(t) \frac{\rho\lambda_A\alpha_A\alpha_B}{\alpha_A - \alpha_B} \left(e^{-\frac{t}{\alpha_A}} - e^{-\frac{t}{\alpha_B}}\right), \quad \text{if } \alpha_A \neq \alpha_B$$

$$ii) \quad m_A(t)\lambda_A \alpha_A \left(1 - e^{-\frac{t}{\alpha_A}}\right) = m_B(t) (p\lambda_A +$$

$$\lambda_B)\alpha_B \left(1 - e^{-\frac{t}{\alpha_A}}\right) - m_B(t)p\lambda_A t e^{-\frac{t}{\alpha_A}},$$

if  $\alpha_A = \alpha_B$

### 4.3 Service Times with a Particular Distribution Function

Solving Eq. (2) in the way presented above becomes quite difficult with other standard distributions for the service times. So now it will be considered a collection of d.f.'s, see [5] and [6], for the service times given by

$$G_i(v) = 1 - \frac{(1 - e^{-\rho_i})(\gamma_i + \beta_i)}{\gamma_i e^{-\rho_i}(e^{(\gamma_i + \beta_i)v} - 1) + \gamma_i}, \quad v \geq 0, \gamma_i >$$

$$0, \rho_i > 0, -\gamma_i \leq \beta_i \leq \frac{\gamma_i}{e^{-\rho_i} - 1}, i = A, B.$$

The mean distribution is  $\alpha_i = \rho_i/\gamma_i$ . In this case Eq. (2) becomes

$$\begin{aligned}
m_A(t) &= \frac{\lambda_A}{\gamma_A} \ln \frac{e^{(\gamma_A + \beta_A)t}}{e^{-\rho_A}(e^{(\gamma_A + \beta_A)t} - 1) + 1} \\
&= m_B(t) \frac{p\lambda_A + \lambda_B}{\gamma_B} \ln \frac{e^{(\gamma_B + \beta_B)t}}{e^{-\rho_B}(e^{(\gamma_B + \beta_B)t} - 1) + 1} \\
&\quad - m_B(t)p\lambda_A I(t)
\end{aligned}$$

where

$$\begin{aligned}
I(t) &= \int_0^t \frac{(1 - e^{-\rho_A})(\gamma_A + \beta_A)}{\gamma_A e^{-\rho_A}(e^{(\gamma_A + \beta_A)v} - 1) + \gamma_A} \times \\
&\quad \frac{(1 - e^{-\rho_B})(\gamma_B + \beta_B)}{\gamma_B e^{-\rho_B}(e^{(\gamma_B + \beta_B)(t-v)} - 1) + \gamma_B} dv.
\end{aligned}$$

$I(t)$  is non-negative and not bigger than

$$\frac{(\gamma_A + \beta_A)(\gamma_A + \beta_A)^t}{\gamma_A + \gamma_B}.$$

#### 4.4 Approximations

The Eq. (2) solution seems to be significantly more complex in circumstances different from those that have been mentioned. For instance, if the service times follow a LogNormal, Gama or Weibull distributions. In some cases, only the numerical solution can eventually be stained.

For appropriate values of  $t$ , the following approximations concerning the equilibrium conditions are suggested:

$$\frac{m_B(t)}{m_A(t)} \cong \frac{\lambda_A \alpha_A}{(p\lambda_A + \lambda_B) \alpha_B} \quad (4);$$

$$\frac{m_B(t)}{m_A(t)} \cong \frac{\lambda_A}{\lambda_B} \quad (5).$$

Eq. (4) seems reasonable for values of  $t$  big enough and Eq. (5) is preferred for  $t$  close to zero. For details see [7].

## 5. Observations

- Some values of the parameters  $p$  and  $\lambda_B$  have a special influence in the system behaviour. One may consider the suppression of the arch  $b$  when  $p = 0$ , of the arch  $c$  when  $p = 1$  or of the arch  $d$  for  $\lambda_B = 0$ . Under those circumstances the traffic in those arches can be neglected.

It may be admitted that the ratio  $m_B(t)/m_A(t)$  remains constant. This corresponds to the assumption that all the users of the system face identical

conditions of effort and benefit, independently of the moment they join the system. Eq. (3) supplies a natural candidate for the value of that constant:  $\lambda_A \alpha_A / (p\lambda_A + \lambda_B) \alpha_B$ . In such situation Eq. (2) should include an "excess" functions  $h(t)$ :

$$\begin{aligned}
h(t) &= m_B(t) \frac{\lambda_A \alpha_A}{(p\lambda_A + \lambda_B) \alpha_B} \int_0^t (p\lambda_A G_A(v) + \\
&\quad \lambda_B)(1 - G_B(t - v)) dv - m_A(t) \lambda_A \int_0^t (1 - \\
&\quad G_A(v)) dv.
\end{aligned}$$

The function  $h(t)$  is also interpreted in the sense of the expected value of a random variable depending on  $t$ . This approach can be generalized in a natural way to some other predefined function  $m_B(t)/m_A(t)$ .

- Assuming that the system is initially empty appears to be a strong restriction of the analysis performed. When someone meets the system already in operation and does not know when it did start, the results that have been mentioned seem to have a lesser utility. In such case, there re-evaluation or finding a estimation procedure for the initial time are determinant for practical purposes .

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