Random Walks in the Representation of Reserves

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***Abstract*— Models based on Random Walks are presented in this work, to represent reserves. Their main objective is to study and guarantee the sustainability of pensions funds. The use of these models with this goal is a classical approach in the study of pensions funds.**

Keywords — *Random walk, pensions funds, ruin.*

1. Introduction

Gambler's ruin problem, which reserves behave according to a simple random walk, is presented in a lot of text books about the stochastic processes theory in relation with Markov Chains, Random Walks, Martingales and even in other contexts. Very clear approaches to this subject are Billingsley (1986, page 88) and Feller (1968, page 344) that solve the problem through the classic first step analysis in order to obtain a difference equation. This proceeding is followed in this work. In alternative, Grimmett and Stirzaker (1992, page 444) and Karlin and Taylor (1975, page 263) present also resolutions of the problem through the Martingales Theory, as an applications example of the Martingales Stopping Time Theorem.

In the next section the gambler's ruin problem is presented. In the following an approach based on the general random walk, enlarging the former one, is outlined. The work finishes with conclusions and a brief list of references.

2. Gambler's Ruin

So consider a gambler disposing of an initial capital of x euro that intends to play a sequence of games till his/her fortune reaches a value of k euro. Suppose that x and k are integer numbers that satisfy the conditions $x > 0$ and $k > x$. In each game, the gambler either wins 1 euro with probability p or loses 1 euro with probability q = 1- p. Which is the probability that the gambler ruins before attaining his/her target? That is, which is the probability of losing the x euro before accumulating wins in the amount of $k - x$ euro?

Call $X_n, n = 1, 2, \ldots$ the result of the n^{th} game. Evidently $X_1, X_2, \ldots$ are independent and identically distributed random variables with probability function:

$$P(X_n = 1) = p, P(X_n = -1) = q = 1 - p.$$

So, the reserves, that is: the fortune, of the player after the n^{th} game correspond to the simple random walk:

$$S_0 = x, S_n = S_{n-1} + X_n, n = 1, 2, \ldots .$$

It is intended to determine the gambler's ruin probability. Call this probability $\rho_k(x)$. It corresponds to the probability that $S_n = 0$ and $0 < S_i < k, i = 0, 1, \ldots, n - 1$ for $n = 1$ or $n =$ 2 or … . If $\rho_k(x)$ is conditioned to the result of the first game it is obtained, as a consequence of the Total Probability Law,

$$\rho_k(x) = p\rho_k(x + 1) + q\rho_k(x - 1), \quad 0 < x < k \qquad (2.1).$$

Considering conveniently $0 \leq x \leq k$, this difference equation is easy to solve with the support of the evident border conditions

$$\rho_k(0) = 1, \ \rho_k(k) = 0 \qquad (2.2).$$

Write (2.1) as

International Journal of Latest Trends in Finance & Economic Sciences
IJLTFES, E-ISSN: 2047-0916

$$\rho_k(x) - \rho_k(x-1) = \frac{p}{q}\left(\rho_k(x+1) - \rho_k(x)\right),$$
$$0 < x < k \qquad (2.3).$$

For $x = k - 1$ and considering (2.2),

$$\rho_k(k-2) = \rho_k(k-1)\left(1 + \frac{p}{q}\right).$$

Based on this equation, when $x = k - 2$, (2.3) becomes:

$$\rho_k(k-3) = \rho_k(k-1)\left(1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2\right).$$

Going on with this proceeding it is obtained the general expression

$$\rho_k(k-y) = \rho_k(k-1)\left(1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^{y-1}\right), 0 < y \le k \qquad (2.4).$$

Considering again (2.2), after (2.4), with $y = k$ it is obtained

$$\rho_k(k-1) = 1\Big/\left(1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^{k-1}\right) \qquad (2.5).$$

Finally, substituting (2.5) in (2.4) and performing the change of variable $y = k - x$ it is obtained the solution of the difference equation (2.1) with the border conditions (2.2):

$$\rho_k(k-1) = \begin{cases} \dfrac{1-(p/q)^{k-x}}{1-(p/q)^k}, & \text{if } p \neq \dfrac{1}{2} \\ \dfrac{k-x}{k}, & \text{if } p = \dfrac{1}{2} \end{cases} \qquad (2.6).$$

Call N_a the first passage time by a of the random walk S_n:

$$N_a = \min\{n \ge 0 : S_n = a\}.$$

In consequence it is possible to write $\rho_k(x) = P(N_0 < N_k | S_0 = x)$. And it is pertinent to take in (2.6) the limit as k converges to ∞ to evaluate $\rho(x)$, the ruin probability of a gambler infinitely ambitious.

In the context of the simple random walk S_n, $\rho(x) = P(N_0 < \infty | S_0 = x)$. After (2.6)

$$\rho(x) = \lim_{k\to\infty} \rho_k(x) = \begin{cases} (q/p)^x, & \text{if } p > \dfrac{1}{2} \\ 1, & \text{if } p \le \dfrac{1}{2} \end{cases} \qquad (2.7).$$

Note that $\mu = E[X_n] = 2p - 1$. It is relevant to see, after (2.7), that the ruin probability is 1 for the simple random walk at which the mean of the step is $\mu \le 0 \Leftrightarrow p \le \frac{1}{2}$.

3. The General Random Walk

Suppose that the contributions (pensions) received (paid), by time unit, for a fund may be described as a sequence of random variables $\xi_1, \xi_2, \dots$ $(\eta_1, \eta_2, \dots)$. State that $\xi_n(\eta_n)$ is the value of the contributions (pensions) received (paid) by the fund during the n^{th} time unit and so $X_n = \xi_n - \eta_n$ is the reserves variation occurred in the fund at the n^{th} time unit. Supposing that $X_1, X_2, \dots$ is a sequence of non degenerated random variables, independent and identically distributed, so the stochastic process defined recursively as:

$$\tilde{S}_0 = x, \tilde{S}_n = \tilde{S}_{n-1} + X_n, \qquad \text{with } n = 1, 2, \dots,$$

is a general random walk that represents the evolution of the fund reserves, since the initial level x till the value $\tilde{S}_n$ after n time units.

It is intended to study the game reserves exhaustion probability, that is the fund ruin. For x and k real numbers fulfilling $x > 0$ and $k > x$, it is considered first the evaluation of $\rho_k(x)$, the probability that the fund reserves decrease from an initial value x to a value in $(-\infty, 0]$ before reaching a value in $[k, +\infty)$. Then, by passing the limit, as in the former section, it is considered the evaluation of $\rho(x)$, the eventual fund ruin probability, admitting so that the random walk, that represents its reserves, evolves with no restrictions at the right of 0.

The method exposed is recognized in the stochastic processes literature as Wald's Approximation. The expositions of Grimmett and Stirzaker (1992, page 407) and Cox and Miller (1965, page 55), about this subject are closely followed in this work. It will be considered the process $S_n = \tilde{S}_n - x$, that is, the random walk

$$S_0 = 0, S_n = S_{n-1} + X_n, n = 1, 2, \dots .$$

instead of the $\tilde{S}_n$ process.

So, when evaluating $\rho_k(x)$, what in fact is being considered is the probability that the process S_n is visiting the set $(-\infty,-x]$ before visiting the set $[k-x,+\infty)$. And when evaluating $\rho(x)$ what is being considered is only the probability that the process S_n goes down from the initial value 0 till a level lesser or equal than – x.

Begin considering the non-null value θ for which the X_1 moments generator function assumes the value 1. It is assumed that such a θ exists, that is, θ satisfies

$$E\left[e^{\theta X_1}\right] = 1, \theta \neq 0 \qquad (3.1).$$

Define the process:

$$M_n = e^{\theta S_n}, n = 0, 1, 2, \dots .$$

It is obvious that $E[|M_n|] < \infty$ and that, after (3.1)

$$\begin{aligned} E[M_{n+1}|X_1, X_2, \dots, X_n] &= E\left[e^{\theta(S_n+X_{n+1})}|X_1, X_2, \dots, X_n\right] \\ &= e^{\theta S_n}E\left[e^{\theta X_{n+1}}|X_1, X_2, \dots, X_n\right] \\ &= M_n. \end{aligned}$$

So, the process M_n is a Martingale in relation to the sequence of random variables $X_1, X_2, \dots$. Consider now N the S_n first passage time to outside the interval $(-x, k-x)$:

$$N = \min\{n \geq 0: S_n \leq -x \text{ or } S_n \geq k-x\}.$$

It is easy to check that the random variable N is a stopping time – or a Markov time – for which the following conditions are fulfilled:

$$E[N] < \infty \text{ and}$$
$$E[|M_{n+1} - M_n||X_1, X_2, \dots, X_n] \leq 2e^{|\theta|a}, \text{for } n < N$$

and $a = -x$ or $a = k - x$.

In relation with this subject see Grimmett and Stirzaker (1992, page 467). Under these conditions, it is possible to apply the Martingales Stopping Time Theorem and, in consequence:

$$E[M_n] = E[M_0] = 1 \qquad (3.2).$$

Also

$$E[M_n] = E\left[e^{\theta S_N}|S_N \leq -x\right]P(S_N \leq -x) + E\left[e^{\theta S_N}|S_N \geq k-x\right]P(S_N \geq k-x) \ (3.3).$$

Performing the approximations

$$E\left[e^{\theta S_N}|S_N \leq -x\right] \cong e^{-\theta x}$$

and

$$E\left[e^{\theta S_N}|S_N \geq k-x\right] \cong e^{\theta(k-x)},$$

and considering that $P(S_N \leq -x) = \rho_k(x) = 1 - P(S_N \geq k-x)$, after (3.2) and (3.3), it is obtained

$$\rho_k(x) \cong \frac{1 - e^{\theta(k-x)}}{e^{-\theta x} - e^{\theta(k-x)}}, \text{when } E[X_1] \neq 0 \quad (3.4).$$

This is the Classic Approximation for the Ruin Probability in the conditions stated in (3.1). Note that to admit a non-null solution θ for the equation $E\left[e^{\theta X_1}\right] = 1$ implies in fact to assume that $E[X_1] \neq 0$.

Out of these considerations is the situation for which the only solution of the equation $E\left[e^{\theta X_1}\right] = 1$ is precisely $\theta = 0$; it means, situation at which $E[X_1] = 0$. This case may be dealt through the following passage to the limit:

$$\rho_k(x) \cong \lim_{\theta \to 0} \frac{1 - e^{\theta(k-x)}}{e^{-\theta x} - e^{\theta(k-x)}} = \frac{k-x}{k}, \quad \text{when } E[X_1] = 0 \qquad (3.5).$$

As for $\rho(x)$, the probability that the process S_n decreases eventually from the initial value 0 to a level lesser or equal than – x, is also got from (3.4), now for a different passage to the limit:

$$\rho(x) \cong \lim_{k \to \infty} \frac{1 - e^{\theta(k-x)}}{e^{-\theta x} - e^{\theta(k-x)}} = e^{\theta x}, \quad \text{if } \theta < 0 \Leftrightarrow E[X_1] > 0 \qquad (3.6).$$

As it may be deduced from the former section results about the simple random walk, it is legitimate to accept $\rho(x) = 1$ when $\theta \geq 0 \Leftrightarrow E[X_1] \leq 0$.

4. Example

Suppose that $X_1, X_2, \dots$ constitute a sequence of independent random variables with normal distribution with mean μ and standard deviation σ. That is, admit that X_n, the fund reserves variation at the n^{th} time unit, has normal distribution with those parameters. In this case, the moments generator function is:

$$E\left[e^{\theta X_1}\right] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{\theta x - \frac{(x-\mu)^2}{2\sigma^2}} dx = e^{\theta\mu + \frac{\theta^2\sigma^2}{2}}.$$

The solution of the equation (3.1) is given by

$$\theta = \frac{-2\mu}{\sigma^2}, \mu \neq 0.$$

The ruin probability $\rho_k(x)$ is obtained substituting this result in (3.4):

$$\rho_k(x) \cong \frac{1 - e^{-\frac{2\mu(k-x)}{\sigma^2}}}{e^{\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu(k-x)}{\sigma^2}}}, \text{when } \mu \neq 0 \quad (4.1).$$

It is evident that this particularization does not influence the approximation to $\rho_k(x)$ when $\mu = 0$ that, as it was seen, is given by (3.5). After (3.6),

$$\rho(x) \cong e^{-\frac{2\mu x}{\sigma^2}}, \qquad \text{when } \mu > 0 \qquad (4.2).$$

5. Conclusions

The simple and general random walks are classic stochastic processes broadly studied. They do not appear only as reserves evolution models. Are also used to build more complex systems and as analysis instruments, in a theoretical feature, of other kind of systems.

In the approach presented some methodologies applied in the study of this kind of processes are highlighted: Difference Equations and Martingales Theory.

It is important to note that, with this approach the reserves systems are treated as if they were physical systems. It is not obvious that the direct application of these principles to financial reserves funds, ignoring their own valuation and devaluation dynamics as time goes by, is legitimate.

Those models, and the consequent stability systems appreciation done with basis on the evaluation of the probability of the reserves exhaustion or ruin, seem valid only in scenarios where constant prices are considered. The integration of factors associated to the time depreciation process of the value of the money in the modulation of financial reserves, although complicating eventually the mathematical models involved, seems important.

Acknowledgments

The authors would like to thank to Professor João Figueira in particular his permission to use the results of Figueira (2003).

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