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Research Article

Exposition of the GRAS Method

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Abstract. The goal of this study is to provide a detailed insight on the GRAS method put forward by Junius et al. (2003) and its subsequent changes proposed by Lenzen et al. (2007) and Temurshoev et al. (2013). The GRAS method allows for balancing and updating of Input-Output (I-O) tables and Social Accounting Matrices (SAMs) with positive and negative entries. The GRAS algorithm provided by Temurshoev et al. (2013) is applied on the 2010 Macro SAM for Malta. The totals of the Rest of World account are revised subject to official publicly available data (Eurostat, 2020). The newly generated Macro SAM includes updated Rest of World account totals such that it represents more accurately the current account of the Maltese economy. Although the Rest of World account totals of the newly balanced Macro SAM conform to the latest Balance of Payments statistical developments during the time of study, the generated SAM elements may not adhere to publicly available data because of additional mathematical assumptions invoked by the GRAS method. However, the newly generated Macro SAM for Malta can be utilised by researchers, students and statisticians who are interested in the 2010 Macro SAM with an updated Rest of World account and are not influenced by the imposition of additional mathematical assumptions.

Keywords: GRAS, Macro SAM, Matrix Balancing, Minimum-Information Principle.

1 Introduction

The objective of this study is to introduce an updated version of GRAS method proposed by Temurshoev et al. (2013). This includes an application of the GRAS method to update the 2010 Macro SAM for Malta by revising the Rest of World (RoW) account. The newly updated 2010 Macro SAM should represent more accurately any current account developments within the Maltese economy during the time of study.

In his Cambridge Growth Project (Bates et al., 1963), Stone and his team developed the first bi-proportional technique to balance a matrix known as the RAS (Bates et al., 1963). The term RAS refers to the pre and post multiplication to the technical coefficients matrix **A** by two matrices termed **R** and **S** to update matrix **A**.[1] Bacharach (1970) and Omar (1967) demonstrate that the RAS procedure can also take the form of a loss minimisation problem (Günlük-Şenesen et al., 1988). The RAS balancing technique was later expanded by Günlük-Şenesen et al. (1988) to allow for updating or balancing a matrix with negative values (Günlük-Şenesen et al., 1988). A decade and a half later, Junius et al. (2003) re-discovered and formulated a theoretical alternative generalisation which they coined GRAS. The GRAS algorithm allows I-O tables with positive and negative entries to be balanced and updated (Junius et al., 2003). Lenzen et al. (2007) adjust the minimum information function of the GRAS procedure adopted by Junius et al. (2003) to obtain a more accurate result (Lenzen et al., 2007). Temurshoev et al. (2013) take into consideration the updated minimum-information principle suggested by Lenzen et al. (2007) and expand further the GRAS introduced by Junius et al. (2003) to allow for matrix balancing and updating with zeros and negative figures across entire rows and columns (Temurshoev et al., 2013).

The RAS balancing technique proposed by Stone and his team (Bates et al., 1963) adjusts an unbalanced matrix $\mathbb{X}_0$ to satisfy updated row and column totals (Bates et al., 1963).[2] The RAS method minimises information loss as long as the unbalanced matrix contains non-negative figures (Bacharach, 1970). Junius et al. (2003) note that before their GRAS formulation, there were two

[1] Matrix **A** can take the form of a non-square matrix. However, **R** and **S** are square matrices.
[2] In this article, matrix **A** and $\mathbb{X}_0$ are identical.

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ad-hoc approaches on how to apply the RAS method on the matrix $\mathbb{X}_0$ with negative values. The first ad-hoc approach was to apply the RAS on $\mathbb{X}_0$ with negative entries. However, this could potentially change the structure of the new balanced matrix $\mathbb{X}^*$ such that its elements would greatly deviate from that of the unbalanced matrix $\mathbb{X}_0$. This is because the negative integers would have a negative contribution during every iteration when performing the RAS procedure. The higher the magnitude of the negative value, the greater the deviation from the unbalanced matrix. Junius et al. (2003) further pinpoint that the second ad-hoc approach was to treat the negative elements outside the RAS method. This is done by first decomposing the matrix $\mathbb{X}_0$ in two matrices: (i) $\mathbb{P}$ with the non-negative entries of $\mathbb{X}_0$, and (ii) $\mathbb{N}$ with the absolute values of the negative entries of $\mathbb{X}_0$. Let $\mathbf{e}^\intercal$ represents a transposed summation vector, $\mathbf{u}$ represents the row summation vector and $\mathbf{v}$ represents the column summation vector of the unbalanced matrix $\mathbb{X}_0$. The column ($\tilde{\mathbf{v}}$) and row ($\tilde{\mathbf{u}}$) summation vectors of matrix $\mathbb{P}$ can be obtained via $\tilde{\mathbf{u}} = \mathbf{u} + \mathbb{N}\mathbf{e}$ and $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{e}^\intercal\mathbb{N}$.[3] The RAS algorithm is applied to the matrix $\mathbb{P}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$, which gives a target matrix $\widetilde{\mathbb{X}}$. The final target matrix is obtained by $\mathbb{X}^* = \widetilde{\mathbb{X}} - \mathbb{N}$.

The problem with the second ad-hoc approach was that negative entries are completely ignored. The negative values would no longer have a negative or a positive contribution during each iteration process of the RAS, which would lead to a sub-optimal result. The GRAS formulated by Junius et al. (2003) not only accounts for negative values in the unbalanced matrix $\mathbb{X}_0$, but also accepts negative values as row and column totals (Junius et al., 2003). The GRAS that shall be utilised in this paper was proposed by Temurshoev et al. (2013) who expand on the GRAS adopted by Junius et al. (2003) to allow entire column and row elements to have zeros and negative values (Junius et al., 2003). Furthermore, they utilise an updated minimum-information principle that was proposed by Lenzen et al. (2007).

The application of the GRAS was originally in the context of balancing and updating I-O tables. For instance, Günlük-Şenesen et al. (1988) compare different mechanical balancing techniques to analyse their efficacy on an I-O table for Turkey (Günlük-Şenesen et al., 1988). Furthermore, Junius et al. (2003) applied the GRAS on a hypothetical I-O table to formulate a generalised matrix balancing technique (Junius et al., 2003). The GRAS was also adopted to balance and update SAMs, such as in Thissen et al. (1998). Recent studies focus on comparative analysis between the RAS, Cross Entropy (CE), Least Squares (LS) and Linear Programming Optimisation methods to highlight their strengths, limitations, and efficacy. Robinson et al. (2001) and Lemelin et al. (2013) compare the RAS and CE methods, while Lee et al. (2014) extend the comparisons to Least Squares (LS) and Linear Programming Optimisation methods. Within the local context, studies utilising the RAS are limited to Blake et al. (2003) who balance a Maltese I-O table for 2001. To the authors' knowledge this study is the first exposition and application of the GRAS within the local context. The application of the GRAS as put forward by Temurshoev et al. (2013) is not as widely utilised as its predecessor, the RAS. As a result, this study sheds light on the GRAS and its application.

A SAM in the form of a matrix[4] represents an entire economy's circular flow of income and expenditure (Pyatt et al., 1985), which is also articulated as flexible (Round, 2003) and comprehensive (Robinson et al., 2001). The flexibility of the SAM allows for further disaggregation of activities, factors, or institutions to accommodate the scope of study. The SAM is also considered as comprehensive framework such that it captures every main economic activity, providing a static visualisation of the entire economy under study. Pyatt et al. (1985) describe the two main objectives of a SAM. The first SAM objective is that of a statistical database to obtain an organised structure of an economy. The second objective of a SAM is to allow, amongst others, the formulation of Computable General Equilibrium (CGE) models (Pyatt et al., 1985). Amongst others, CGE models can be formulated to undertake socio-economic policy analysis on matters such as poverty allectiation, monetary policy, income inequality and tourism.[5] Taffesse et al. (2004) note that a unique feature of a SAM is the high degree of consistency within the framework, such that total income and total expenditure equate (Taffesse et al., 2004).

Different SAM-types exist,[6] which application and construction depends on the scope of study. Two of the most utilised and constructed SAM-types are the Macro and Micro SAMs. The Macro SAM consists mainly of aggregate national accounts figures, with an aggregated production account. The Micro SAM further adds sectoral disaggregation to the production account, disaggregating production, factors and final demand of every institution for each sector. Therefore, the main difference between the two SAM-types is the level of sectoral disaggregation. For the context of this study, a Macro SAM shall be utilised that comprises of an aggregated production account. Numerous SAMs have been con-

[3] $\mathbf{e}$ represents a column summation vector of dimensions $n \times 1$.

[4] There are instances where the dimensions of a SAM take the form of a non-square matrix. This generally depends on the SAM disaggregation level and scope of study.

[5] Refer to Dixon et al. (2012) for a detailed discussion on CGE models.

[6] For a further discussion on the different types of SAMs, refer to Miller et al. (2009) and Cassar et al. (2013).

structured across the world.[7] Within the local context, the first reliable and coherent SAM was constructed in the doctoral dissertation by Cassar et al. (2013) that conforms to the ESA 1995. A previous attempt at constructing a SAM was done by Blake et al. (2003) but did not conform to the basic SAM structure proposed by Pyatt et al. (1976). The SAM constructed by Blake et al. (2003) was also based on an I-O table that was in turn based on two previously mechanically balanced I-O tables. During the time of this study, the latest SAM constructed for Malta was by Theuma (2020). Within the local context, the author constructed the first household extended SAM for the year 2010 (Theuma, 2020). To provide a deeper insight on the GRAS and its application, the 2010 SAM for Malta constructed in Theuma (2020) shall be updated. Therefore, this implies that the circular flow of income and expenditure of the Maltese economy shall also be updated.

2 Data and Methodology

In order to update the SAM via the GRAS, the Maltese 2010 Macro SAM was obtained from the post-graduate dissertation published by Theuma (2020). The 2010 Macro SAM for Malta was readily available and adheres by the latest European System of National and Regional Accounting (ESA 2010) framework and the second revision of the Statistical Classification of Economic Activities in the European Community (NACE). The Macro SAM is encompassed of five institutional accounts of which three are domestic, two factor accounts, and an aggregated production account. The three domestic institutional accounts are namely the household, government and enterprises institutions. The factor account is split between labour and other value added. The production account of the SAM is aggregated to a single cell and hence explains its name Macro SAM. The updated BoP data that shall be utilised to update the Macro SAM was obtained directly from Eurostat (Eurostat, 2020). More specifically, the current account total of the Macro SAM was revised to match the latest BoP statistical developments. To update and balance the Macro SAM, MATLAB shall be utilised which is a proprietary multi-paradigm programming language and numerical computing environment developed by MathWorks. The applied MATLAB code for GRAS is publicly available by Temurshoev (2020).

The RAS algorithm is known as a biproportional technique (Lahr et al., 2004). The main idea behind biproportional technique algorithms is to transform an initial matrix $\mathbb{X}_0$ to a target matrix $\mathbb{X}^*$ of the same dimensions. The RAS procedure is an iterative algorithm where the rows and columns of matrix $\mathbb{X}_0$ are updated using proportions that are based on known row and columns sums of $\mathbb{X}^*$. Let $\langle\zeta\rangle$ be a square matrix with the vector ζ on its diagonal and zeros elsewhere, and let $\mathbf{e}$ be a summation vector consisting of ones with appropriate dimensions. The RAS iteratively adjusts the column ($\mathbf{v}$) and row ($\mathbf{u}$) sums of an initial matrix $\mathbb{X}_0$ to approximate a new matrix $\mathbb{X}^*$ with given target row and column sums. The scaling for the row and column sums is given in steps 2 and 3. An example of the RAS procedure is presented in the Appendix section of this study. The RAS iterative steps are described in Lahr et al. (2004) as follows.

Step 1 (Initialization). Set $p = 0$, where p refers to the number of iterations. Let $\mathbb{X}_0^0 = \mathbb{X}_0$.

Step 2 (Row Scaling). Let $p = p + 1$, and $\mathbb{R}^p = \langle\mathbb{X}^*\mathbf{e}\rangle\langle\mathbb{X}_0^{p-1}\mathbf{e}\rangle^{-1}$ and $\mathbb{X}_0^{p-\frac{1}{2}} = \mathbb{R}^{(p)}\mathbb{X}_0^{p-1}$. Where $\mathbb{X}_0^{p-1}$ is the updated matrix $\mathbb{X}_0$ after $p-1$ iterations and $\mathbb{X}_0^{p-\frac{1}{2}}$ represents the matrix $\mathbb{X}_0^{p-1}$ after computing row scaling.

Step 3 (Column Scaling). Next let $\mathbb{S}^p = \langle\mathbf{e}^\intercal\mathbb{X}^*\rangle\langle\mathbf{e}^\intercal\mathbb{X}_0^{p-\frac{1}{2}}\rangle^{-1}$ and $\mathbb{X}_0^p = \mathbb{X}_0^{p-\frac{1}{2}}\mathbb{S}^p$.

The above presented steps describe a single full iteration of the RAS bi-proportional algorithm, which are repeated until a desirable matrix is achieved. Bacharach (1970) put forward the notion that a solution for this algorithm always exists. In fact, the author demonstrated that this simple bi-proportional algorithm can be derived from minimizing a minimum information function

$$f(\mathbb{X}^*, \mathbb{X}_0) = \sum_{i,j} x^*_{ij} \ln\left(\frac{x^*_{ij}}{ex_{0ij}}\right), \text{[8]} \tag{1}$$

where e denotes the irrational exponential constant. Equation 1 is subject to the constraints $\mathbf{u}$ and $\mathbf{v}$ of known row and column totals

$$\sum_j x^*_{ij} = u_i \quad \text{and} \quad \sum_i x^*_{ij} = v_j.$$

Unfortunately, the RAS is erratic when negative values are present in the initial matrix $\mathbb{X}_0$. Applying the RAS method on $\mathbb{X}_0$ with negative values could lead to $\mathbb{X}^*$ that is not comparable to the initial matrix $\mathbb{X}_0$. Building on previous work, Junius et al. (2003) provided a minimization information loss problem, similar to the one solved by Bacharach (1970), but they account for the negative values in matrix $\mathbb{X}_0$. This method is referred to as the GRAS. The argument lies in minimizing

$$f(\mathbb{X}^*, \mathbb{X}_0) = \sum_{i,j} |x^*_{ij}| \ln\left(\frac{x^*_{ij}}{x_{0ij}}\right) \tag{2}$$

[7] Refer to Theuma (2020) for an overview of SAMs constructed internationally.

[8] The value x_{0ij} can be negative. However, x^*_{0ij} will also be negative, implying that $\ln\left(\frac{x_{0ij}}{x^*_{0ij}}\right)$ can be calculated.

subject to the constraints $\mathbf{u}$ and $\mathbf{v}$ of known row and column totals

$$\sum_j x^*_{ij} = u_i \quad \text{and} \quad \sum_i x^*_{ij} = v_j$$

Define $z_{ij} = \frac{x^*_{ij}}{x_{0ij}} > 0$ if $x_{0ij} > 0$ and $z_{ij} = 0$ if $x_{0ij} = 0$. Then the problem in equation 2 can be re-written as

$$f(\mathbb{Z}, \mathbb{X}_0) = \sum_{i,j} |x_{0ij}| z_{ij} \ln(z_{ij}) \tag{3}$$

with the Lagrange function

$$L(Z, \lambda, \tau) = \sum_{(i,j)\in P} x_{0ij} z_{ij} \ln(z_{ij}) - \sum_{(i,j)\in N} x_{0ij} z_{ij} \ln(z_{ij}) + \sum_i \lambda_i \left(u_i - \sum_j x_{0ij} z_{ij}\right) + \sum_j \tau_j \left(v_j - \sum_i x_{0ij} z_{ij}\right)$$

where P are the pair of indices (i, j) for which $x_{0ij} \geq 0$ and N the set of pairs of indices (i, j) for which $x_{0ij} < 0$. From the Lagrange equation the following theorem follows.

Theorem 1. *Let $Z = \{z_{ij}\}$, $\lambda = \{\lambda_1, \ldots, \lambda_m\}$ and $\tau = \{\tau_1, \ldots, \tau_n\}$. Then*

$$z_{ij} = \frac{r_i s_j}{e} \quad \text{if } x_{0ij} \geq 0 \tag{4}$$

$$z_{ij} = \frac{1}{r_i s_j e} \quad \text{if } x_{0ij} < 0 \tag{5}$$

where $r_i = e^{\lambda_i}$ and $s_j = e^{\tau_j}$.

Proof. Consider the optimality condition

$$\frac{\partial L(Z, \lambda, \tau)}{\partial z_{ij}} = 0.$$

For $x_{0ij} \geq 0$ we have

$$x_{0ij} \ln z_{ij} + x_{0ij} - \lambda_i x_{0ij} - \tau_j x_{0ij} = 0,$$

this is equivalent to

$$\ln(z_{ij}) = \lambda_i + \tau_j - 1 \implies z_{ij} = e^{\lambda_i} e^{\tau_j} e^{-1}.$$

For $x_{0ij} < 0$ we have

$$-x_{0ij} \ln z_{ij} - x_{0ij} - \lambda_i x_{0ij} - \tau_j x_{0ij} = 0,$$

this is equivalent to

$$\ln(z_{ij}) = -\lambda_i - \tau_j - 1 \implies z_{ij} = e^{-\lambda_i} e^{-\tau_j} e^{-1}. \quad \square$$

Corollary 2. *For the target matrix $\mathbb{X}^*$ it follows that*

$$x^*_{ij} = \frac{r_i x_{0ij} s_j}{e} \geq 0 \quad \text{if } x_{0ij} \geq 0 \tag{6}$$

and

$$x^*_{ij} = \frac{x_{0ij}}{r_i s_j e} < 0 \quad \text{if } x_{0ij} < 0 \tag{7}$$

Theorem 3. *The diagonal matrices $\mathbb{S}$ and $\mathbb{R}$ are the solution of the system of non-linear equations :*

$$(\mathbb{RPS} - \mathbb{R}^{-1}\mathbb{NS}^{-1})\boldsymbol{e} = \boldsymbol{u}^\star \tag{8}$$

$$\boldsymbol{e}^\intercal(\mathbb{RPS} - \mathbb{R}^{-1}\mathbb{NS}^{-1}) = \boldsymbol{v}^\star \tag{9}$$

where

$$p_{ij} = \begin{cases} x_{0ij} & x_{0ij} \geq 0 \\ 0 & x_{0ij} = 0 \end{cases}$$

$$n_{ij} = \begin{cases} -x_{0ij} & x_{0ij} < 0 \\ 0 & x_{0ij} \geq 0 \end{cases}$$

$\boldsymbol{u}^\star = e\boldsymbol{u}$ and $\boldsymbol{v}^\star = e\boldsymbol{v}$.

An algorithm similar to the RAS can be formulated for the GRAS. It is described as follows:

Step 1 (Initialization). Start from a given initial matrix $\mathbb{R}(0)$.

Step 2 Use equation 8 to calculate the matrix $\mathbb{S}(1)$.

Step 3 Use equation 9 to calculate the matrix $\mathbb{R}(1)$ using the matrix $\mathbb{S}(1)$ obtained in the previous step.

Step 4 The algorithm continues this way by finding $\mathbb{R}(0) \to \mathbb{S}(1) \to \mathbb{R}(1) \to \mathbb{S}(2) \to \mathbb{R}(2) \to etc.$

Step 5 It reaches its solution of diagonal matrices $\mathbb{R}$ and $\mathbb{S}$ if for an arbitrary $\epsilon > 0$,

$$\left\|(\mathbb{RPS})\mathbf{e} - (\mathbb{R}^{-1}\mathbb{NS}^{-1})\mathbf{e} - \mathbf{u}^\star\right\| < \epsilon \left\|\mathbf{u}^\star\right\|$$
$$\left\|\mathbf{e}^\intercal(\mathbb{RPS}) - \mathbf{e}^\intercal(\mathbb{R}^{-1}\mathbb{NS}^{-1}) - \mathbf{v}^\star\right\| < \epsilon \left\|\mathbf{v}^\star\right\|$$

For the initial matrix $\mathbb{R}(0)$, Junius et al. (2003) suggest $\langle \mathbf{e} \rangle$. For motivation behind this suggestion one can check Stone (1961), Toh (1998) and van der Linden et al. (2000).
In their work, Lenzen et al. (2007) outlined a problem with the construction of Junius et al. (2003). The authors outline an issue that occurs when starting with an initial matrix $\mathbb{X}_0$ already satisfying row and columns sums. In such a case, the initial estimate should be optimal solution but by construction, the algorithm of Junius et al. (2003) generates a suboptimal solution. To solve this issue, Lenzen et al. (2007) provided a new target function by including an irrational exponent e:

$$f(\mathbb{Z}, \mathbb{X}_0) = \sum_{i,j} |x_{0ij}| z_{ij} \ln\left(\frac{z_{ij}}{e}\right), \tag{10}$$

where $z_{ij} = \frac{x^*_{ij}}{x_{0ij}}$. The Lagrangean function is formulated by

$$L(Z,\lambda,\tau) = \sum_{(i,j)\in P} x_{0ij} z_{ij} \ln\left(\frac{z_{ij}}{e}\right) - \sum_{(i,j)\in N} x_{0ij} z_{ij} \ln\left(\frac{z_{ij}}{e}\right) + \sum_i \lambda_i \left(u_i - \sum_j x_{0ij} z_{ij}\right) + \sum_j \tau_j \left(v_j - \sum_i x_{0ij} z_{ij}\right).$$

Theorem 4. *Let $Z = \{z_{ij}\}$, $\lambda = \{\lambda_1, \dots, \lambda_m\}$ and $\tau = \{\tau_1, \dots, \tau_n\}$. Then*

$$z_{ij} = r_i s_j \quad \text{if } x_{0ij} \geq 0 \tag{11}$$

$$z_{ij} = \frac{1}{r_i s_j} \quad \text{if } x_{0ij} < 0 \tag{12}$$

where $r_i = e^{\lambda_i}$ and $s_j = e^{\tau_j}$.

Proof. Consider the optimality condition

$$\frac{\partial L(Z,\lambda,\tau)}{\partial z_{ij}} = 0.$$

For $x_{0ij} \geq 0$ we have

$$x_{0ij} \ln\left(\frac{z_{ij}}{e}\right) + x_{0ij} - \lambda_i x_{0ij} - \tau_j x_{0ij} = 0,$$

this is equivalent to

$$\ln(z_{ij}) = \lambda_i + \tau_j \implies z_{ij} = e^{\lambda_i} e^{\tau_j}.$$

For $x_{0ij} < 0$ we have

$$-x_{0ij} \ln\left(\frac{z_{ij}}{e}\right) - x_{0ij} - \lambda_i x_{0ij} - \tau_j x_{0ij} = 0,$$

this is equivalent to

$$\ln(z_{ij}) = -\lambda_i - \tau_j \implies z_{ij} = e^{-\lambda_i} e^{-\tau_j}.$$

□

Corollary 5. *For the target matrix $\mathbb{X}^*$ it follows that*

$$x^*_{ij} = r_i x_{0ij} s_j \geq 0 \quad \text{if } x_{0ij} \geq 0 \tag{13}$$

and

$$x^*_{ij} = \frac{x_{0ij}}{r_i s_j} < 0 \quad \text{if } x_{0ij} < 0 \tag{14}$$

Theorem 6. *The diagonal matrices $\mathbb{S}$ and $\mathbb{R}$ are the solution of the system of non-linear equations:*

$$(\mathbb{RPS} - \mathbb{R}^{-1}\mathbb{NS}^{-1})\boldsymbol{e} = \boldsymbol{u} \tag{15}$$

and

$$\boldsymbol{e}^\intercal (\mathbb{RPS} - \mathbb{R}^{-1}\mathbb{NS}^{-1}) = \boldsymbol{v} \tag{16}$$

where

$$p_{ij} = \begin{cases} x_{0ij} & x_{0ij} \geq 0 \\ 0 & x_{0ij} = 0 \end{cases}$$

$$n_{ij} = \begin{cases} -x_{0ij} & x_{0ij} < 0 \\ 0 & x_{0ij} \geq 0 \end{cases}$$

The multipliers r_i and s_j are derived from the solution of the quadratic equations

$$p_i(s) r_i^2 - u_i r_i - n_i(s) = 0 \tag{17}$$

$$p_j(r) s_j^2 - v_j s_j - n_j(r) = 0 \tag{18}$$

where

$$p_i(s) = \sum_j p_{ij} s_j, \quad p_j(r) = \sum_j p_{ij} r_i \tag{19}$$

$$n_i(s) = \sum_j \frac{n_{ij}}{s_j}, \quad n_j(s) = \sum_i \frac{n_{ij}}{s_i}$$

The solutions are given by

$$r_i = \frac{u_i + \sqrt{u_i^2 + 4p_i(s) n_i(s)}}{2p_i(s)} \tag{20}$$

and

$$s_j = \frac{v_j + \sqrt{v_j^2 + 4p_j(r) n_j(r)}}{2p_j(r)} \tag{21}$$

The algorithm for this updated GRAS is given by

Step 1 (Initialization). Start from a given initial matrix $\mathbb{R}(0) = \langle \mathbf{e} \rangle$.

Step t For $t \in \{1, 2, ..., N\}$. Find $s_j(t)$ and $r_i(t)$ using equations 20 and 21.

Step N Stop when $s_j(N) - s_j(N-1) < \epsilon$ for every j and a sufficiently small ϵ. Calculate the value of x^*_{ij} using equations 13 and 14, $r_i(N)$ and $s_j(N)$.

The two methods described above suffice provided that matrix $\mathbb{P}$ satisfying $\mathbb{X}_0 = \mathbb{P} - \mathbb{N}$ does not have an entire rows of 0. In practice this might not always be the case. To solve this, an improvement was made by Temurshoev et al. (2013). Define

$$r_i = \begin{cases} \frac{u_i + \sqrt{u_i^2 + 4p_i(s) n_i(s)}}{2p_i(s)} & p_i(s) > 0 \\ -\frac{n_i(s)}{u_i} & p_i(s) = 0 \end{cases} \tag{22}$$

$$s_j = \begin{cases} \frac{v_j + \sqrt{v_j^2 + 4p_j(r) n_j(r)}}{2p_j(r)} & p_i(s) > 0 \\ -\frac{n_j(r)}{v_j} & p_j(r) = 0 \end{cases} \tag{23}$$

This modification suffices to solve the issue of the matrix $\mathbb{P}$ having entire rows and columns of zeros. Furthermore, it is evident that the values of r_i and s_j are positive. The iterative process is identical to the algorithm used in the work of Lenzen et al. (2007).

3 Results and Discussion

Table 1 presents the original Macro SAM constructed by Theuma (2020). This is utilised to generate the matrix $\mathbb{X}_0$ with updated RoW totals. The Macro SAM depicted in Table 1 takes the form of a 9×9 dimensional matrix, which provides a static picture of the entire circular flow of income and expenditure of the Maltese economy for the reference year of 2010. Table 1 contains a negative entry of -136.78 million euro, which reflects a situation of dissavings coming from the Enterprises Institution.[9] From Table 1 it can be observed that there are no rows and columns comprised of only zeros or negative values. From the same table, the rows and columns which contain a negative value also have at least one positive entry. This implies that the GRAS algorithm proposed by Junius et al. (2003) can also be applied to update the Maltese Macro SAM for the reference year of 2010. We utilise the GRAS algorithm proposed by Temurshoev et al. (2013) to make use of the latest GRAS modification, which includes the revised minimum information criteria by Lenzen et al. (2007). The RoW totals of the original Macro SAM in Table 1 are updated to match those in Table 2. Therefore, as depicted in Table 4, the RoW totals were revised upwards to 24,888.04 million euro (Eurostat, 2020) to better represent the current account and simultaneously conform to the latest BoP statistical developments. With the exception of the RoW Account, the other Macro SAM totals found within Table 2 remain unchanged.

The publicly available MATLAB code (Temurshoev, 2020) proposed by Temurshoev et al. (2013) with a threshold level of $\varepsilon = 10^{-6}$ generates an updated 2010 Macro SAM for Malta in Table 3 within 34 iterations. The row and column totals of every SAM account in Table 3 conform exactly to those set in Table 2. The GRAS algorithm generated the upper left block matrix in Table 3 with dimensions 9×9, subject to the revised RoW totals. The changes between Tables 1 and 3 are clearly listed in Table 4. There are several important findings that deserve attention from the newly generated Macro SAM in Table 3. The first point to notice from Table 3 is that the elements of the Labour (L) and the Other Value Added (K) accounts remain the same when compared to Table 1. This happened for two reasons. The first reason is because the totals of the Labour (L) and Other Value Added (K) accounts remain unchanged.[10] Secondly, the Labour (L) and Other Value Added (K) accounts constitute only of one element. The entries of the Labour (L) and Other Value Added (L) accounts of 2,846.27 and 2,960.51 million euro, respectively, represent the reward-payments for the utilisation of factors of production by productive activities, which are generally encompassed of land, labour and capital. Since there are no differences between the Factor Account generated by the GRAS procedure and the one constructed in Theuma (2020), a high degree of consistency is retained with respect to the Labour (K) and the Other Value Added (K) Accounts.

The second important finding is the significant change in the aggregate Intermediate Consumption element found within the production account as depicted in Table 4. At sectoral level, the aggregate output for every sector would still remain the same because total output for every sector would be treated as control totals. However, the interindustry transactions would be subject to change following the GRAS procedure. This in turn directly influences researchers, statisticians or students who intend to analyse the production structure of Malta at a sectoral level. More specifically, the amount of input purchases required for the production processes to produce goods and services would change. One reason why the Aggregate Intermediate Consumption element changed drastically is due to the fact that it has a large magnitude which absorbs the RoW revisions since the Labour (L) and Other Value Added (K) account elements remained unchanged. This suggests that the GRAS-generated Macro SAM in this study should be utilised with caution, depending on the scope of study. For instance, studies that utilise the GRAS-generated SAM to estimate Demand-Driven Leontief multipliers could end up with different estimates when compared to a non-mathematically balanced SAM. This is because the interindustry transactions in the latter SAM would be assumed to make more economic sense without imposing further mathematical assumptions.

From Table 3, it can be observed that after updating the totals of the RoW Account in Table 2 and performing the GRAS-algorithm, imports and exports of goods and services amount to 9,403 and 10,673 million euro respectively. However, from official publicly available data (Eurostat, 2020), one can observe that imports and exports of goods and services should amount to around 10,115 and 10,154 million euro respectively. Therefore, a limitation of the GRAS is that although the row and column totals of the RoW Account were updated, the elements within the Macro SAM are mathematically generated according to the GRAS algorithm suggested

[9] Theuma (2020) pinpoints that this figure was obtained as a balancing entry of the Capital Account. However, this figure had been estimated after already having obtained reliable estimates for the other SAM Accounts

[10] Recall that the totals of the RoW Account were the ones subject to change in this study.

by Temurshoev et al. (2013). Since these elements are generated by the GRAS-algorithm, they cannot be equal to official published statistical figures as they may lose economic sense. One way to solve this issue is to keep some entries fixed before running the GRAS. However, this would produce a suboptimal solution when compared to the scenario of keeping no elements fixed. The GRAS would have less entries on which to balance the Macro SAM and as a result would increase the magnitude to which the remaining elements would change, producing results that could make no economic sense.

Another important finding worth discussing is that the highest discrepancies come from the RoW Account, as visualised in Table 4. A reason behind this is that the RoW account already had elements with substantial magnitudes. In fact, when comparing Tables 1 and 3, it can be seen that throughout the Macro SAMs, the elements with significant flow entries experience a larger difference as seen from Table 4. Similarly, the elements with lower magnitudes experience lower changes as observed from Table 4. However, the magnitude of changes by the GRAS balancing process depends on how much the SAM totals change and whether several entries will remain fixed. In other words, the magnitude of the elements in the Macro SAM when performing the GRAS algorithm will change based on the magnitude of the revised totals.

4 Conclusion

The main aim of this study is to provide a detailed exposition of the GRAS algorithm. An advantage of the GRAS is its versatility to be applied when negative entries are present in the initial matrix. This naturally puts the GRAS method as a possible tool to use when balancing a matrix with negative values.

In this study, the GRAS was applied to update the 2010 Macro SAM for Malta. This was done by updating the totals of the RoW account and balance it accordingly. The GRAS was important because of the negative value that is present in the initial Macro SAM provided for the reference year of 2020 (Theuma, 2020). The final result obtained represents more accurately the current account of the Maltese economy while simultaneously reflecting the latest BoP statistical developments.

In general, there are various ways of constructing a SAM. The Results section presents that our SAM exhibits notable changes after performing the GRAS when compared to the SAM constructed by Theuma (2020). The Factor account did not result any changes after performing the GRAS, which ensures a high degree of consistency with the initial Macro SAM. However, every element of the RoW account exhibited notable changes after performing the GRAS. Also, the Intermediate Consumption aggregate changed after performing the GRAS, which suggests that the purchases required for every sector to produce goods and services changed. The SAM generated in the study can be utilised by researchers, statisticians and students depending on the scope of their research. This is because the elements of the generated SAM elements do not compare to publicly available data. This is important because the GRAS applied in this study was carried out without keeping any elements fixed in the initial Macro SAM. It is possible to keep some values fixed in the initial Macro SAM by using a similar procedure as described by Junius et al. (2003). However, one should keep in mind that the consequence of keeping fixed entries would result in a suboptimal solution. For future studies, one can opt to use several methods similar to the GRAS, including the KRAS, Linear Programming Optimisation, CE and LS to analyse and compare different generated SAMs. The main aim of the study was to explore the GRAS method. To this end, no entries were kept fixed in order to obtain an optimal GRAS solution.

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Appendix

Example 7. *Set $p = 0$ and*

$$\mathbb{X}_0^0 = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

We need to find a matrix $\mathbb{X}^*$ with target row sums

$$\begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

and target column sums

$$\begin{bmatrix} 9 & 9 \end{bmatrix}$$

Set $p = 1$ and start the row scaling

$$\mathbb{R}^1 = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Let

$$\mathbb{X}_0^{p-\frac{1}{2}} = \mathbb{X}_0^{\frac{1}{2}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$$

From the above, the rows sums of matrix $\mathbb{X}_0^{\frac{1}{2}}$ satisfy the required target row sums. However, the column sums of matrix $\mathbb{X}_0^{\frac{1}{2}}$ do not satisfy the required column sums. Next we adjust the columns sums with respect to the target columns sums.

$$\mathbb{S}^1 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

Let

$$\mathbb{X}_0^1 = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & 3 \end{bmatrix}$$

The matrix $\mathbb{X}_0^1 = \mathbb{X}^*$ satisfies the target row and column sums. In this case, the process stops here. However, if the matrix $\mathbb{X}_0^1$ does not satisfy the target row and column sums, set $p = 2$ and repeat the process above.

Table 1: Original 2010 Macro SAM for Malta

MacroSAM in Euro Millions (000's)	P	H	F	G	C	E	L	K	T	Tot
Production Activities (P)	3,880.95	3,020.91	0	1,247.73	872.117	8,577.18	0	0	0	17,598.89
Households (H)	0	0	189.79	863.99	0	1,738.25	2,846.27	677.9	0	6,316.21
Enterprises (F)	0	991.14	0	220.08	0	3,982.49	0	2,030.78	0	7,224.54
Government (G)	0	577.48	59.05	0	0	1,123.48	0	251.83	792.74	2,804.55
Capital (C)	0	193.88	-136.78	427.19	0	1,073.94	0	0	0	1,558.18
RoW Imports (E)	7,748.25	1,054.81	7,112.47	32.80	577.46	692.68	0	0	0	17,218.46
Compensation of Employees (L)	2,846.27	0	0	0	0	0	0	0	0	2,846.27
Other Value Added (K)	2,960.51	0	0	0	0	0	0	0	0	2,960.51
Net Taxes on Products and Production (T)	162.908	478	0	12.79	108.64	30.4	0	0	0	792.70
Total (Tot)	17,598.91	6,316.21	7,224.54	2,804.55	1,558.18	17,218.46	2,846.27	2,960.51	792.70	59,320.36

Source: (Theuma, 2020).

Table 2: Updated Macro SAM Row and Column Totals

MacroSAM in Euro Millions (000's)	P	H	F	G	C	E	L	K	T	Tot
Production Activities (P)										17,598.89
Households (H)										6,316.21
Enterprises (F)										7,224.54
Government (G)										2,804.55
Capital (C)										1,558.18
RoW Imports (E)										24,888.04
Compensation of Employees (L)										2,846.27
Other Value Added (K)										2,960.51
Net Taxes on Products and Production (T)										792.70
Total (Tot)	17,598.91	6,316.21	7,224.54	2,804.55	1,558.18	24,888.04	2,846.27	2,960.51	792.70	66,989.91

Source: Authors' own calculations.

Table 3: Updated GRAS 2010 Macro SAM for Malta

MacroSAM in Euro Millions (000's)	P	H	F	G	C	E	L	K	T	Tot
Production Activities (P)	2,288.57	2,638.78	0	1,431.04	566.64	10,673.9	0	0	0	17,598.89
Households (H)	0	0	39.34	656.78	0	2,087.12	2,846.27	686.68	0	6,316.21
Enterprises (F)	0	416.78	0	162.56	0	4,646.33	0	1,998.82	0	7,224.54
Government (G)	0	269.43	13.19	0	0	1,454.27	0	275.01	792.70	2,804.55
Capital (C)	0	103.44	-535.28	400.35	0	1,590	0	0	0	1,558.18
RoW Imports (E)	9,403.64	2,386.51	7707.28	130.26	912.04	4,348.29	0	0	0	24,888.04
Compensation of Employees (L)	2,846.27	0	0	0	0	0	0	0	0	2,846.27
Other Value Added (K)	2,960.51	0	0	0	0	0	0	0	0	2,960.51
Net Taxes on Products and Production (T)	99.91	501.27	0	23.56	79.5	88.45	0	0	0	792.70
Total (Tot)	17,598.91	6,316.21	7,224.54	2,804.55	1,558.18	24,888.04	2,846.27	2,960.51	792.70	66,989.91

Source: Authors' own calculations.

Table 4: Differences Between the Initial and the New GRAS 2010 Macro SAM for Malta

MacroSAM in Euro Millions (000's)	P	H	F	G	C	E	L	K	T	Tot
Production Activities (P)	-1,592	-382	0	183	-305	2,097	0	0	0	0
Households (H)	0	0	-150	-207	0	349	0	9	0	0
Enterprises (F)	0	-574	0	-58	0	664	0	-32	0	0
Government (G)	0	-308	-46	0	0	331	0	23	0	0
Capital (C)	0	-90	-398	-27	0	516	0	0	0	0
RoW Imports (E)	1,655	1,332	595	97	335	3,656	0	0	0	7,670
Compensation of Employees (L)	0	0	0	0	0	0	0	0	0	0
Other Value Added (K)	0	0	0	0	0	0	0	0	0	0
Net Taxes on Products and Production (T)	-63	23	0	11	-29	58	0	0	0	0
Total (Tot)	0	0	0	0	0	7,670	0	0	0	7,670

Source: Authors' own calculations.