MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION EXAMINATION
ADVANCED LEVEL

MAY 2013

SUBJECT:	APPLIED MATHEMATICS
PAPER NUMBER:	I
DATE:	15th May 2013
TIME:	9.00 a.m. to 12.00 noon

Directions to candidates

Attempt all questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10$ ms^{-2}).

1. A rectangle ABCD has sides AB $= 3a$ and BC $= 4a$. Forces of magnitude $7W, 6W, 10W$, $13W$ and $15W$ act along the lines BA, BC, DC, DA and AC respectively, in the directions indicated by the order of the letters.

   (i) Find the resultant of this system, and the distance from A at which its line of action cuts AD.

   (ii) An extra force, P, is now added at D so that the system reduces to a couple. Find the value of P and the magnitude of the resulting couple.

[6, 4 marks]

2. Two particles, A of mass $2m$ and B of mass m, move on a smooth horizontal table in opposite directions with speeds $5u$ and $3u$ respectively. The particles collide directly.

   (i) Find their velocities after the collision in terms of u and the coefficient of restitution e.

   (ii) Show that the magnitude of the impulse exerted by B on A is $\frac{16}{3}mu(1+e)$.

   (iii) Find the value of e if the speed of B after the collision is $3u$.

   (iv) Whilst moving at this speed, B collides and coalesces with a stationary particle C of mass $5m$. Find the velocity of the resulting particle.

[4, 2, 1, 3 marks]

3. A ball was thrown from a balcony above a horizontal lawn. The velocity of projection was 10 ms^{-1} at an angle of elevation α, where $\tan\alpha = 3/4$. The ball moved freely under gravity and took 3 s to reach the lawn from the instant when it was thrown. Find:

   (i) the vertical height above the lawn from which the ball was thrown;

   (ii) the horizontal distance between the point of projection and the point A at which the ball hit the lawn;

   (iii) the angle between the direction of the velocity and the horizontal at the instant when the ball reached A.

[5, 2, 3 marks]

4. A particle of weight W rests in limiting equilibrium on the inner rough surface of a fixed rough hollow sphere of internal radius a. The coefficient of friction between the particle and the sphere is $3/4$.

   (i) Draw a diagram showing the forces acting on the particle.

   (ii) Find the normal reaction and the frictional force acting on the particle.

   (iii) Find the depth of the particle below the centre of the sphere.

[3, 6, 1 marks]

5. A uniform solid right circular cone has its top removed by cutting the cone by a plane parallel to the base, leaving a truncated cone of height h, the radii of its ends being r and $4r$.

   (i) Find the distance of the centroid of the truncated cone from its broader end.

   (ii) If the truncated cone is suspended freely from a point on the rim of the broader end, and $h = r$, find the angle which the axis of the cone makes with the vertical.

[7, 3 marks]

6. The frictional resistance to the motion of a car of mass 1000 kg is kv Newtons, where v is its speed in ms^{-1} and k is a constant. The car ascends a hill of inclination α, where $\sin\alpha = 0.1$.

   (i) Given that the car has a steady speed of 8 ms^{-1} up this hill when the power exerted by the engine is 9.76 kW, show that k is numerically equal to 27.5 Nsm^{-1}.

   (ii) Find the steady speed at which the car ascends the hill if the power exerted by the engine is 12.8 kW.

   (iii) Find the acceleration of the car when it is travelling at 10 ms^{-1} and the power exerted by the engine is 14.8 kW.

[4, 3, 3 marks]

7. At 10.00 a.m., a battleship, whose maximum speed is 30 km/hr, sights a submarine which is moving due North at 10 km/hr. When first sighted, the submarine is 15 km North-East of the battleship. Find:

   (i) the direction in which the battleship must be steered in order to intercept the submarine as quickly as possible;

   (ii) the time at which they meet;

   (iii) their distance apart at 10.15 a.m.

[7, 1, 2 marks]

8. A uniform rod AB of weight W and length $2a$ rests in a vertical plane with the end A in contact with a smooth vertical wall, the end B being at a lower level than A. The rod is inclined at 60° to the vertical and is held in equilibrium by a light string attached to B and to a point C vertically above A.

   (i) Show that the forces on the rod are concurrent.

   (ii) Draw a diagram showing the forces acting on the rod, and the point of intersection of these forces. Show that CA $= a$.

   (iii) Using Lami's theorem, or otherwise, find the tension in the string and the reaction at the wall.

[2, 4, 4 marks]

9. One end of a light elastic string of natural length l and modulus of elasticity $3mg$ is attached to a fixed point O. To the other end, is attached a particle P of mass m. The particle is projected vertically downwards from O with a speed $\sqrt{3gl}$. The particle has speed v when it is at a depth x below O.

   (i) Using the principle of conservation of energy, obtain an expression for the speed, v, in terms of g, x, l, for the range $x > l$.

   (ii) Show that the greatest depth below O reached by P is $8l/3$.

   (iii) Find the maximum speed of P.

[6, 2, 2 marks]

10. A particle of mass 4 kg moves under the action of a force $\mathbf{F}$. At time t, the momentum of the particle is $8\cos t\mathbf{i} - 12\sin t\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are mutually perpendicular unit vectors in the x-y plane.

    (i) Find the velocity of the particle in terms of t.

    (ii) Find $\mathbf{F}$ in terms of t.

    (iii) Given that when $t = 0$, the position vector of the particle is $\mathbf{i} - \mathbf{j}$, find its position vector and its distance from the origin when $t = \pi/2$.

    (iv) By integrating the scalar product of $\mathbf{F}$ and the velocity with respect to t, or otherwise, find the work done by the force $\mathbf{F}$ in the time range $0 \leq t \leq \pi/2$.

[1, 2, 4, 3 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION EXAMINATION
ADVANCED LEVEL

MAY 2013

SUBJECT:	APPLIED MATHEMATICS
PAPER NUMBER:	II
DATE:	18th May 2013
TIME:	4.00 p.m. to 7.00 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10$ ms^{-2}).

1. A light, uniform beam AB of length $2a$ has midpoint C. The beam rests horizontally on two supports at its ends A and B, and carries a load $4W$, which is *uniformly distributed* between the points A and C. This structural system is in equilibrium.

   (i) Find the reactions at the supports A and B.

   (ii) Find the shearing force and bending moment at any point along the beam.

   (iii) Draw a sketch of the shearing force and bending moment.

   (iv) Find the maximum bending moment and the point on the beam where it occurs.

[2, 8, 4, 1 marks]

2. A force $\mathbf{i}+\mathbf{j}-\mathbf{k}$ acts through the origin together with a couple of moment $\mathbf{i}+\mathbf{j}+2\mathbf{k}$.

   (i) Prove that the vector moment of this couple is perpendicular to the force, and hence show that the couple and force are coplanar.

   (ii) Show that the couple and force are equivalent to a single force and find a vector equation for its line of action.

[4, 11 marks]

3. A smooth uniform sphere P is stationary on a rectangular horizontal table OABC.

   Sphere P is struck by an identical sphere Q which is moving on the table with velocity $(16\mathbf{i}+13\mathbf{j})u$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along OA and OC respectively.

   After impact, spheres P and Q have velocities in the directions of the vectors $3\mathbf{i}+4\mathbf{j}$ and $7\mathbf{i}+\mathbf{j}$ respectively.

   (i) Using scalar products, or otherwise, express the three vectors given above in the form
   $$\alpha(3\mathbf{i}+4\mathbf{j})+\beta(4\mathbf{i}-3\mathbf{j}),$$
   where α and β are constants to be determined.

   (ii) Show that the line of centres has the direction of the vector $3\mathbf{i}+4\mathbf{j}$.

   (iii) Prove that the coefficient of restitution is $\frac{1}{2}$.

[4, 2, 9 marks]

4. A particle of mass m rests at the highest point of the outer surface of a smooth cylinder of radius a which is fixed with its axis horizontal. The particle is slightly disturbed from rest so that it begins to travel in a vertical circle.

   (i) Find the vertical distance travelled by the particle before it leaves the surface of the cylinder.

   (ii) After leaving the cylinder, how far does the particle fall while travelling a distance a horizontally?

[10, 5 marks]

5. A particle P of mass m lies on a smooth horizontal table and is attached by two light elastic strings, of lengths $3a$, $2a$ and moduli λ, 2λ to two fixed points S, T respectively on the table. The length of ST is $7a$.

   (i) Show that when the particle is in equilibrium, SP $= 9a/2$.

   (ii) The particle is held at rest at the point in the line ST where SP $= 5a$, and then released. Show that the subsequent motion of the particle is simple harmonic of period $\pi\sqrt{(3ma/\lambda)}$.

   (iii) Find the maximum speed of the particle during this motion.

[5, 9, 1 marks]

6. A go-kart of mass 100 kg moves along a straight road. At time t seconds, when its velocity is v ms^{-1}, its engine exerts a force of $\dfrac{4000}{v}$ Newtons, and is subject to a resistive force of $10v$ Newtons.

   (i) Using Newton's Second Law, show that $\dfrac{dv}{dt} = \dfrac{400 - v^2}{10v}$.

   (ii) Solve this differential equation assuming that the go-kart starts from rest.

   (iii) How long does the go-kart take to reach half its maximum speed?

[4, 9, 2 marks]

7. A particle is describing a plane curve, and at a time t is at a point with polar coordinates (r, θ) referred to a pole O.

   The particle, P, of mass m, is travelling along the curve $r(1+\cos\theta) = 2a$ on a smooth horizontal plane. The radius vector OP rotates with constant angular speed ω.

   (i) Show that the speed of particle P is $\omega\sqrt{r^3/a}$.

   (ii) Find the resultant horizontal force acting on the particle P when $\theta = \pi/2$.

   (iii) Draw a diagram showing the force in (ii) in conjunction with the pole, the initial line, the radius vector, and a small relevant section of the path.

   Hint: First write down the radial and transverse components of the velocity and acceleration of a general particle in plane polar coordinates.

[6, 7, 2 marks]

8. (a) A uniform rod has mass m and length $2a$. Show that the moment of inertia of the rod about an axis perpendicular to it at a distance x from its midpoint is

$$\frac{1}{3}m(a^2 + 3x^2).$$

(b) A uniform rod AB, of mass m and length $6b$, is smoothly pivoted to a fixed point at a distance of $2b$ from one end, and is free to swing in a vertical plane. The rod is held in a horizontal position, and released from rest.

(i) Prove that after it has rotated through an angle θ, its angular velocity is

$$\sqrt{\frac{g\sin\theta}{2b}}.$$

(ii) Find the reaction on the pivot when the rod is vertical.

[4; 5, 6 marks]

9. A uniform square lamina, of mass M and side $2a$, has moment of inertia about a side equal to $4Ma^2/3$. The lamina is smoothly hinged along a horizontal side, and hangs vertically. A bullet of mass m, moving horizontally with speed v, strikes the lamina perpendicularly at the midpoint of the lower horizontal side, and adheres to the lamina.

(i) Show that the whole system starts to rotate with an angular speed given by

$$a\omega = \frac{3mv}{2(M + 3m)}.$$

(ii) Find in terms of M, m, g, and a, the least value of v^2 which would cause the whole system to make complete revolutions.

[6, 9 marks]

10. A uniform solid cylinder has mass m and radius a. A thin cylindical ring also has mass m and radius a. The solid cylinder and the ring roll from rest, without slipping, down lines of greatest slope of a *rough* plane inclined at an angle α to the horizontal. It can be assumed that during the motion, the axes of the cylinder and the ring are always horizontal.

(i) Write down the moments of inertia of the cylinder and the ring about their axes.

(ii) Show that the solid cylinder covers a distance $\frac{1}{12}gt^2\sin\alpha$ further than the ring in time t.

(iii) Find the minimum value of the coefficient of friction between the plane and the rolling bodies.

[2, 10, 3 marks]