MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION EXAMINATION
ADVANCED LEVEL

SEPTEMBER 2013

SUBJECT:	**APPLIED MATHEMATICS**
PAPER NUMBER:	I
DATE:	**3rd September 2013**
TIME:	**9.00 a.m. to 12.00 noon**

Directions to candidates

Attempt all questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10$ ms^{-2}).

1. A particle moving in a horizontal straight line with uniform acceleration passes points A, B, C, D which are at distances a, b, c, d respectively from its initial position. The particle takes n seconds to cover *each* of the distances AB, BC and CD.

   (i) Show that $d - a = 3(c - b)$.

   (ii) If the velocity of the particle when at D is three times its velocity at A, show that $5b = 3a + 2c$.

[6, 4 marks]

2. A system of forces acting in a Cartesian plane with origin O and with perpendicular axes Ox and Oy consists of:

   a force $10P$ along Ox,
   a force $-9P$ along Oy,
   a force $13P$ along $\overrightarrow{OA}$, where A is the point $(12a, 5a)$,
   a force $20P$ along $\overrightarrow{AB}$, where B is the point $(8a, 8a)$.

   (i) Find the magnitude, direction and equation of the line of action of the resultant of this system.

   (ii) A clockwise coplanar couple of magnitude $240Pa$ is added to the system. Find the magnitude, direction and equation of the line of action of the resultant of the new system.

[7, 3 marks]

3. A *non-uniform* straight beam ABCD of length 3 m rests in a horizontal position on supports at B and C, where AB = BC = CD.

   When a mass of 1 kg is placed at D, the beam just tilts.

   When a mass of 1 kg is placed at A, the pressures on the supports are equal.

   Find the mass of the beam and the position of its centre of mass.

[10 marks]

4. Two particles are projected simultaneously from two points A and B on level ground at a distance of 150 m apart.

   The first particle is projected vertically upwards from A with an initial speed of u ms^{-1}.

   The second particle is projected from B towards A with an angle of projection α.

   If the particles collide when they are both at their greatest height above the level of AB, show that

$$\tan\alpha = \frac{u^2}{150g}.$$

[10 marks]

5. A car of mass 1200 kg tows another car of mass 800 kg on level ground, the frictional resistances being 120 N and 80 N respectively. The tow rope breaks if the tension in it exceds 2000N.

   (i) Find the maximum acceleration possible.

   (ii) Find the maximum power the towing car can use at the instant when the speed is 10 km/hr.

[4, 6 marks]

6. A uniform rod AB, of length $2l$ and weight W, is in equilibrium with the end A on a rough horizontal floor and the end B against a smooth vertical wall. The rod makes an angle $\tan^{-1} 2$ with the horizontal, and is in a vertical plane perpendicular to the wall.

   (i) Find the least possible value of μ, the coefficient of friction between the floor and the rod.

   (ii) If $\mu = 5/16$, find the distance from A of the highest point of the rod at which a particle of weight W can be attached without disturbing equilibrium.

[5, 5 marks]

7. A circle is given by the equation $\mathbf{r} = (1 - \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$, and a straight line is given by $\mathbf{r} = \lambda(\sqrt{3}\mathbf{i} + \mathbf{j})$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the direction of the x- and y-axis of a Cartesian coordinate system with origin O.

   (i) Draw a diagram of the circle and the line on the same axes. Verify that they intersect at O and at the point S where $\theta = 2\pi/3$.

   (ii) Two particles P and Q start simultaneously from O and arrive simultaneously at S. Particle P moves at constant speed u along the shorter circular path from O to S. Particle Q starts from rest at O and moves to S with uniform acceleration f along the straight path. Find f in terms of u.

[4, 2; 4 marks]

8. A light elastic spring has modulus of elasticity $2mg$ and natural length $2a$. A particle P of mass $4m$ is attached to the midpoint of the spring whose ends are attached to two fixed points, A and B, distant $8a$ apart in a vertical line. A is at a level higher than B. The system is in equilibrium.

   (i) Find the length of AP at equilibrium.

   (ii) The particle is displaced slightly downwards, so that it starts making vertical oscillations along AB. Show that the motion is simple harmonic and that its periodic time is $2\pi\sqrt{a/g}$.

[5, 5 marks]

9. A Cartesian system has origin O and unit vectors $\mathbf{i}$ and $\mathbf{j}$ along the x- and y- axes.

   At noon, an aircraft A is at a point with position vector $5\mathbf{i} + \mathbf{j}$ and is moving with constant velocity $-\mathbf{i} + 3\mathbf{j}$.

   A second plane B is simultaneously at the point with position vector $3\mathbf{i} - 3\mathbf{j}$ and is moving with constant velocity $2\mathbf{i} + 5\mathbf{j}$.

   (i) Show that at noon, A and B are equidistant from any point on the line with vector equation $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j})$.

   (ii) If $t = 0$ at noon, find the value of t when A and B are closest together.

[4, 6 marks]

10. A light elastic string has natural length a and modulus λ. One end of this string is fastened to a fixed point A, and a mass m is attached to the other end. The particle is released from rest at A, and first comes to rest when it has fallen a distance $3a$.

   (i) Using conservation of energy, or otherwise, show that $\lambda = \frac{3}{2}mg$.

   (ii) Show that at the lowest point of its path, the acceleration of the particle is $2g$ upwards.

   (iii) Find in terms of g and a the speed of the particle when the magnitude of the acceleration is $g/2$.

[3, 3, 4 marks]

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MATRICULATION EXAMINATION
ADVANCED LEVEL

SEPTEMBER 2013

SUBJECT:	**APPLIED MATHEMATICS**
PAPER NUMBER:	II
DATE:	**4th September 2013**
TIME:	**9.00 a.m. to 12.00 noon**

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

(Take $g = 10$ ms^{-2}).

1. A straight uniform beam has weight W and length l. The beam is clamped horizontally at one end, and carries a particle, also of weight W, at a distance of $3l/4$ from the clamped end. This structural system is in equilibrium.

   (i) Find the reaction and couple at the clamped end of the beam.

   (ii) Find the shearing force and bending moment at any point along the beam.

   (iii) Draw a sketch of the shearing force and bending moment.

   (iv) Find the maximum bending moment and the point on the beam where it occurs.

[2, 8, 4, 1 marks]

2. The three concurrent forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ are in equilibrium when acting at the points whose position vectors are $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ respectively. These vector quantities are given as follows:

$$\mathbf{F}_1 = 5\mathbf{i} + 6\mathbf{j} \qquad \mathbf{r}_1 = c\mathbf{i} + \mathbf{j}$$
$$\mathbf{F}_2 = a\mathbf{i} - 4\mathbf{j} \qquad \mathbf{r}_2 = 2\mathbf{i} - \mathbf{j}$$
$$\mathbf{F}_3 = -6\mathbf{i} + b\mathbf{j} \qquad \mathbf{r}_3 = 3\mathbf{i} + 2\mathbf{j},$$

where $\mathbf{i}$, $\mathbf{j}$ are unit vectors along the x- and y-axes of a Cartesian coordinate system.

(i) Calculate the values of the constants a, b and c and the position of concurrence of the lines of action of the three forces.

(ii) The force $\mathbf{F}_3$ is now reversed whilst $\mathbf{F}_1, \mathbf{F}_2, \mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ remain unchanged, and a clockwise couple in the $\mathbf{i}, \mathbf{j}$ plane, of magnitude 21 units, is introduced into the system. Find the resultant of this system and the vector equation of its line of action.

[10, 5 marks]

3. A black ball is stationary on a rectangular table OABC. It is then struck by a white ball of equal mass and equal radius with velocity $u(-2\mathbf{i} + 11\mathbf{j})$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along OA and OC respectively. After impact the black and white balls have velocities *parallel* to the vectors $-3\mathbf{i} + 4\mathbf{j}$ and $2\mathbf{i} + 4\mathbf{j}$ respectively.

(i) Using conservation of momentum, find the velocities of the balls after impact.

(ii) Write down the direction of the line of centres for this impact.

(iii) By resolving velocities along the line of centres, show that the coefficient of restitution between the two balls is 1/2.

[7, 2, 6 marks]

4. A car of mass 1000 kg accelerates from rest on level ground with a power of 8000 W. The resistance to motion is $2000v$, where v is the speed of the car in ms^{-1}.

(i) Using Newton's Second Law of Motion, set up a differential equation relating the speed v of the car to t, the time elapsed in seconds.

(ii) Integrate this differential equation to find v in terms of t.

(iii) Find the speed of the car as $t \to \infty$.

[3, 9, 3 marks]

5. A particle P of mass m is held on the surface of a fixed smooth solid sphere centre O and radius a at a point such that OP makes an angle of $\pi/4$ with the upward vertical. The particle is released at time $t = 0$. At a subsequent time $t > 0$, the particle is still in contact with the sphere, and OP makes an angle θ with the upward vertical.

   (i) Using conservation of energy, or otherwise, obtain in terms of g, a and θ, the velocity of the particle while it is still in contact with the sphere.

   (ii) Find the normal reaction on the particle under these conditions.

   (iii) Deduce the value of $\cos\theta$ when the particle leaves the sphere.

   (iv) Find the horizontal and vertical components of the velocity at this instant.

[6, 6, 1, 2 marks]

6. Two uniform rods AB and BC of equal weight W but of lengths a and $2a$ are freely jointed together at B. The rods stand in a vertical plane with their ends A and C on rough horizontal ground, such that angle ABC $= 90^{\mathrm{o}}$. One of the rods is in limiting equilibrium. It can be assumed that the coefficient of friction, μ, is the same for both rods.

   (i) Draw a diagram showing the forces in the system as a whole, and in each rod separately.

   (ii) Find the least value of μ for equilibrium to be possible.

   (iii) Find the reaction at the hinge.

[4, 8, 3 marks]

7. A uniform solid right circular cone has height h and semi-vertical angle α.

   (i) Show by integration that the centre of mass of the cone is at a distance $3h/4$ from its vertex. You can assume that the volume of the cone is $\dfrac{1}{3}\pi h^3 \tan^2\alpha$.

   (ii) A plane parallel to the base and at a distance $h/2$ from the vertex divides the cone into a smaller cone of height $h/2$ and a frustum also of height $h/2$. Show that the centre of mass of the frustum is at a distance $11h/56$ from its larger plane end.

   (iii) The frustum is placed with its curved surface in contact with a horizontal table. Show that equilibrium is only possible if $45\cos^2\alpha \geq 28$.

[6, 7, 2 marks]

8. A square frame is made up of four uniform rods, each of mass m and length $2l$, rigidly connected together to form a square. The frame is free to rotate in its own vertical plane about a smooth fixed horizontal axis passing through one corner of the frame and perpendicular to it. The frame is released from rest when the diagonal through the axis is horizontal.

   (i) Show that the moment of inertia of the frame about this axis is $40ml^2/3$.

   (ii) Show that when the frame has turned through an angle θ, the angular speed is given by

   $$\dot{\theta}^2 = \frac{3\sqrt{2}g\sin\theta}{5l},$$

   and find the angular acceleration $\ddot{\theta}$.

   (iii) Find the reaction on the pivot initially, when $\theta = 0^\circ$.

[5, 5, 5 marks]

9. (a) Find from first principles, the moment of inertia of a uniform circular disc of mass m and radius a about an axis perpendicular to the disc and passing through its centre.

   (b) A mass M rests on a smooth horizontal table. It is connected by a light inelastic string which passes over a rough pulley at the edge of the table to a mass m hanging freely. The pulley is a uniform circular disc of mass m and radius a.

   Show that the acceleration of the mass m is $\dfrac{2mg}{2M+3m}$, and find the tension in the vertical portion of the string.

[5, 10 marks]

10. A uniform square lamina ABCD has mass m and side a. It can rotate freely about a smooth fixed horizontal axis passing through AD. Its moment of inertia about this axis is $ma^2/3$.

    The lamina is in equilibrium with BC below AD, when a small impulse J is applied to the lamina at the midpoint of BC. The impulse is normal to the plane of the lamina.

    (i) Find in terms of J, m and a, the angular velocity of the lamina immediately after the application of the impulse.

    (ii) Find J if the lamina just manages to complete whole revolutions about AD.

    (iii) If the lamina is hanging in equilibrium, and disturbed slightly, find the periodic time of the resulting oscillations.

[4, 6, 5 marks]