MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION EXAMINATION
ADVANCED LEVEL

MAY 2014

SUBJECT:	**APPLIED MATHEMATICS**
PAPER NUMBER:	I
DATE:	**17th May 2014**
TIME:	**9.00 a.m. to 12.00 noon**

Directions to candidates

Attempt all questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

Unless otherwise stated, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x, y and z axes of a Cartesian coordinate system with origin O.

(Take $g = 10$ ms^{-2}).

1. A regular pentagon OBCDE, of side 2 m, has vertex O at the origin of a Cartesian coordinate system, with side OB in the direction of the positive x-axis, and with C lying in the first quadrant. Forces of 5 N act along OB, BC and OD in the direction implied by the order of the letters. Find:

   (i) the resultant of this system in $\mathbf{i}, \mathbf{j}$ notation;

   (ii) the distance from O at which its line of action cuts OB;

   (iii) the equation of the line of action of the resultant.

[5, 3, 2 marks]

2. A uniform ladder of mass 30 kg is placed with its base on a rough horizontal floor and its top against a smooth vertical wall, with the ladder making an angle of 60° with the floor.

   (i) Find the magnitude of the *total* horizontal force that is acting at the base of the ladder if the system is in equilibrium.

   (ii) The coefficient of friction between the ladder and the floor is 1/4. What is the magnitude of the *extra* horizontal force, apart from friction, which must be applied at the base of the ladder,

   a) if the ladder is on the point of slipping outwards;

   b) if the ladder is on the point of slipping inwards?

[6; 2, 2 marks]

3. A child's spinning top is made of a solid uniform cylinder of radius 1 cm and height 2 cm, at the base of which is attached the base of a uniform solid cone of equal density, with base radius 1 cm and height d cm. The top has an axis of symmetry which is common to both the cylinder and the cone.

   (i) Find in terms of d the distance of the centroid of the top from its circular face.

   (ii) The top lies in equilibrium on a horizontal plane with a generator of the cylinder in contact with the plane. Find the maximum value of d for this to be possible.

   Hint: It can be assumed that the centre of gravity of a uniform cone of height h is at a distance of $h/4$ from the centre of the base. **[7, 3 marks]**

4. A girl throws a stone from a height of 1.5 m above level ground with an initial speed of 10 ms^{-1} at an angle of elevation α above the horizontal. The origin, O, is at the point where the girl is standing on the ground.

   (i) Find the velocity and displacement of the stone as a function of t.

   (ii) Find the Cartesian equation of the path followed by the stone in terms of $\tan\alpha$.

   (iii) While following this path, the stone hits a bottle standing on a wall 4 m high and 5 m from the girl. Find the value of α, if it is known that the horizontal velocity of the stone at this point is larger than 6 ms^{-1}.

   (iv) Using this value of α, find the time taken by the stone to hit the bottle, and the direction in which the stone is moving when it hits the bottle.

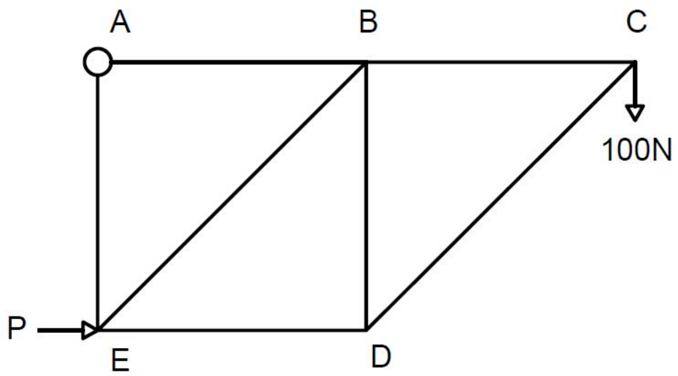
[4, 1, 3, 2 marks]

5. Three identical spheres A, B and C, each of mas m, lie in a straight line on a smooth horizontal surface with B between A and C. Spheres A and B are projected directly towards each other with speeds $3u$ and $2u$ respectively, and C is projected directly away from A and B with speed $2u$.

   If the coefficient of restitution between any two spheres is e, show that B will only collide with C provided $e > 3/5$.

[10 marks]

6. The light framework in the diagram is smoothly hinged at A and is held with AE vertical by a horizontal force P at E. The rods AB, BC, BD, ED, EA are all equal, and ABC is horizontal. The framework carries a load of 100 N at C.



   Find the force P, the reaction at A and the forces in the rods ED, BD and BC.

[10 marks]

7. A smooth bead of mass 0.1 kg is threaded on a light inextensible string which is of length 0.7 m. The string has one end attached to a fixed point A and the other end to a fixed point B at a distance of 0.5 m vertically below A. The bead moves in a horizontal circle about the line AB with a constant angular speed of ω rad s^{-1}, and with the string taut.

   If the bead is at a point C on the string with AC = 0.4 m, find the value of ω, and the tension in the string.

   Hint: It can be assumed that the tension is the same in both parts of the string.

[10 marks]

8. A uniform solid hemisphere of mass m and radius a rests with its curved surface in contact with a rough horizontal floor and a rough vertical wall. The coefficient of friction is μ at both points of contact. At equilibrium, the plane face of the hemisphere makes an angle θ with the floor. It can be assumed that equilibrium is limiting at both points of contact, and that all the forces in the system lie in one vertical plane which is normal to the floor, the wall and the plane face of the hemisphere.

   (i) Draw a diagram showing the forces acting in this system.

   (ii) Show that the normal reaction of the floor on the hemisphere is $\dfrac{mg}{1+\mu^2}$.

   (iii) Find an expression for $\sin\theta$ in terms of μ. In this part, you can assume that the centroid of the hemisphere is at a distance of $3a/8$ from its centre.

[3, 4, 3 marks]

9. A particle of weight W is attached by two light inextensible strings, each of length a, to two fixed points distant a apart in a horizontal line.

   (i) Write down the tension in either string.

   (ii) One of the strings is now replaced by an elastic string of the same natural length, and it is found that in the new position of equilibrium, this string has stretched to a length of $5a/4$.

   Using Lami's Theorem, or otherwise, find, in terms of W, the tensions in the two strings and the modulus of elasticity of the elastic string.

[2; 7, 1 marks]

10. A particle of mass $4m$ is attached to the midpoint of a light spring of modulus $2mg$ whose ends are attached to two fixed points distant $8a$ apart in a vertical line. The natural length of the spring is $2a$.

   (i) When the system is in equilibrium, find the position of the particle below the upper fixed point.

   (ii) The particle is then disturbed slightly from rest in a vertical direction. Show that the particle performs simple harmonic motion with periodic time $2\pi\sqrt{a/g}$.

[4, 6 marks]

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MATRICULATION EXAMINATION
ADVANCED LEVEL

MAY 2014

SUBJECT:	APPLIED MATHEMATICS
PAPER NUMBER:	II
DATE:	20th May 2014
TIME:	4.00 p.m. to 7.00 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

Unless otherwise stated, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x, y and z axes of a Cartesian coordinate system with origin O.

(Take $g = 10$ ms^{-2}).

1. ABCD is a light uniform beam of length 12 m, with AB = 1 m, BC = 8 m, and CD = 3 m. The beam rests horizontally on smooth supports at B and C, and carries a uniformly distributed load of magnitude 2 Nm^{-1} over its whole length. This structural system is in equilibrium.

   (i) Find the normal reactions at the supports B and C of the beam.

   (ii) Find the shearing force and bending moment at any point along the beam.

   (iii) Draw a sketch of the shearing force and bending moment.

   (iv) Find where the bending moment is largest.

[2; 3, 5; 2, 2; 1 marks]

2. Three variable forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act at points with position vectors $\mathbf{0}$, $\mathbf{i}+\mathbf{j}$, $-3\mathbf{i}+2\mathbf{j}$ respectively, and at time t,

$$\mathbf{F}_1 = 2\cos t\mathbf{i}, \quad \mathbf{F}_2 = \cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad \mathbf{F}_3 = 3\sin t\mathbf{i} + \cos t\mathbf{j}.$$

(i) If the system is reduced to a single force $\mathbf{F}$ at O and a couple $\mathbf{G}$, find the values of $\mathbf{F}$ and $\mathbf{G}$.

(ii) If the system is reduced to a single force at some point Q on the y-axis, show that Q is independent of t, and find the vector equation of the line of action of the resultant.

[10, 5 marks]

3. The position vector of a particle of mass m at time t is

$$\mathbf{r} = a\sin t\mathbf{i} + a\cos t\mathbf{j} + at\mathbf{k}.$$

(i) Draw a sketch of the path of this particle.

(ii) Find the velocity and acceleration of the particle at time t.

(iii) Find the angle between the velocities of the particle at $t = 0$ and $t = \pi/2$.

(iv) Show that the kinetic energy of the particle and the magnitude of the force acting on it are both constant.

(v) Find the magnitude of the moment of the force about the origin at $t = \pi/4$.

[3, 2, 3, 4, 3 marks]

4. A particle of mass m is projected from the highest point of a fixed smooth sphere of radius a. In the subsequent motion, the particle slides down the outside surface of the sphere, and leaves the surface of the sphere with speed $2\sqrt{ga/5}$. Find:

(i) the vertical distance travelled by the particle while it is in contact with the sphere;

(ii) the initial speed of projection;

(iii) the speed of the particle when it is level with the horizontal diameter of the sphere.

[8, 5, 2 marks]

5. ABC is a light rigid straight rod with AB = BC. Three particles each of mass m are attached to the rod at A, B and C respectively. Initially, the system lies at rest on a smooth horizontal table, when an impulse J is applied to C in the plane of the table at an angle of 45° to BC produced. Find in terms of J and m:

   (i) the velocities of the three particles immediately after the impulse is applied;

   (ii) the impulsive tensions in AB and BC;

   (iii) the kinetic energy generated by the impulse.

[6, 6, 3 marks]

6. A raindrop of mass m is initially at rest at a high altitude. It falls vertically downwards under the influence of gravity, g, and of a resistance kv, where v is the velocity at time t, and k is a constant.

   (i) Using Newton's Second Law, obtain a first order differential equation for $\dfrac{dv}{dt}$ in terms of m, g, k and v.

   (ii) Solve this differential equation to obtain an expression for v in terms of the other variables.

   (iii) Find the largest velocity that can be attained by the particle.

   (iv) Find the time taken by the particle to attain half its maximum velocity.

[2, 9, 2, 2 marks]

7. A uniform rod AB of mass m and length $2l$ is smoothly jointed at A to a fixed point. A light elastic string of natural length l and modulus of elasticity mg connects the end B to a point C distant $2l$ vertically above A. The angle BAC is denoted by 2θ, whilst the total potential energy (elastic plus gravitational energy) is denoted by V.

   (i) Obtain an expression for V in terms of m, g, l and θ.

   (ii) Find $\dfrac{dV}{d\theta}$.

   (iii) By setting this derivative equal to zero, verify that $\theta = \pi/2$ and $\theta = \sin^{-1}(1/3)$ are positions of equilibrium.

   (iv) Find $\dfrac{d^2V}{d\theta^2}$. For which of the two angles in (iii) is the second derivative positive? This angle gives the position of stable equilibrium of the system.

Hints: $\dfrac{d}{d\theta}(\cos 2\theta) = -2\sin 2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$, and $\dfrac{d}{d\theta}(\sin 2\theta) = 2\cos 2\theta$.

[8, 2, 2, 3 marks]

8. A uniform disc has mass m, radius a and centre O. Q is a point on the disc, where $OQ = x$.

   (i) Show by integration that the moment of inertia of the disc about an axis passing through O and perpendicular to the disc is $\frac{1}{2}ma^2$.

   (ii) Using the parallel axes theorem, find the moment of inertia about an axis perpendicular to the disc, and passing through Q.

   (iii) The disc can rotate freely about a smooth fixed horizontal axis passing through Q and normal to the disc. By taking moments about Q, or otherwise, find the period of small oscillations of the disc about its equilibrium position.

   (iv) Show that this period is minimum when $x = a/\sqrt{2}$ and find the minimum period.

[4, 1, 7, 3 marks]

9. A trapdoor has the form of a uniform square lamina, of mass m and side $2a$. Its moment of inertia about a side is equal to $4ma^2/3$. The trapdoor can rotate about a fixed horizontal axis which contains one of its sides. Initially the trapdoor is held at rest in a horizontal position. When released from rest, the trapdoor swings downwards and attains an angular velocity of $\sqrt{g/4a}$ when it first reaches the vertical position. It can be assumed that the frictional couple acting at the axis is constant.

   (i) Using work-energy principles, find the magnitude of the frictional couple.

   (ii) If θ is the angle between the initial position of the trapdoor and its position where it is again at instantaneous rest, show that $\sin\theta = \dfrac{5\theta}{3\pi}$.

   (iii) Find the vertical component of the reaction at the hinge when the trapdoor first reaches the vertical position.

[5, 5, 5 marks]

10. A uniform rod, AB, of mass m and length $2a$, is free to rotate about a smooth fixed horizontal axis passing through A and perpendicular to the rod.

    The rod is held horizontally, and then set in motion with initial angular speed ω, so that B starts moving downwards. When the rod becomes vertical, B collides with a stationary particle of mass m, which adheres to it.

    In the subsequent motion, the rod comes to instantaneous rest in the horizontal position. Show that

$$\omega = \sqrt{\frac{33g}{2a}}.$$

[15 marks]