MATRICULATION AND SECONDARY EDUCATION CERTIFICATE

EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL

2019 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	7th May 2019
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Attempt **ALL** questions. There are 10 questions in all.
The marks carried by each question are shown at the end of the question.
The total number of marks for all the questions in the paper is 100.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.
Unless otherwise stated, **i** and **j** are unit vectors along the x- and y- axes of a Cartesian coordinate system. Units are Newtons for force and meters for distance.

(Take $g = 10\ \text{ms}^{-2}$)

1. A uniform solid has the shape of a thick hemispherical shell of inner radius a and outer radius $2a$.

   (a) Find the centre of gravity of the solid. You can assume that the centre of gravity of a uniform hemisphere of radius a is at a distance of $3a/8$ from its centre. (7)

   (b) AB is a diameter of length $4a$ of the flat base of the solid. The solid is suspended freely from A. Find the angle which AB makes with the downward vertical. (3)

(Total: 10 marks)

2. In a Cartesian system of coordinates, three forces $-2\mathbf{i}-3\mathbf{j}$, $3\mathbf{i}+4\mathbf{j}$ and $-\mathbf{i}-\mathbf{j}$ act at the points (2,0), (0,3) and (1,1) respectively. The forces are given in Newtons, and distances in meters.

   (a) Show that these forces reduce to a couple and find the magnitude and sense of this couple. (6)

   (b) The system above is reduced to equilibrium by the addition of two further forces: $2\mathbf{i}+2\mathbf{j}$ acting at O, and $a\mathbf{i}+b\mathbf{j}$ acting at $(c,0)$. Find a, b and c. (4)

(Total: 10 marks)

3. A Cartesian system of coordinates has origin O, and has the x-axis horizontal and the y-axis vertical.

   (a) A particle is projected from O with speed V at an angle α to the horizontal. Show from first principles, that the horizontal range of the particle is $V^2 \sin 2\alpha/g$. (5)

   (b) Two particles are projected from O, one with speed 28 m s^{-1} at 45° to the horizontal, and the other with speed 35 m s^{-1} at an angle α to the horizontal. The difference between their horizontal ranges is ±15 m.

   Find the four possible values of α. **Hint**: Note that $\sin(180-\alpha)=\sin\alpha$. (5)

(Total: 10 marks)

4. A Cartesian coordinate system has origin O. Two particles A and B are moving in this system. At the instant $t = 0$ seconds, A is at (3, 0) m and moving towards O with a constant speed 4 m s^{-1}, whilst B is at (0, 5) m and moving towards O with constant speed 2 m s^{-1}.

   (a) Find in terms of t, the relative velocity and displacement of the two particles at a time $t \geq 0$. (5)

   (b) Find the time at which the particles are nearest to each other, and the minimum distance between them. (5)

(Total: 10 marks)

5. A vehicle is on the point of slipping when parked on a bend that is banked at 20^o to the horizontal.

   (a) Find the coefficient of friction between the vehicle's tyres and the surface of the road. (3)

   (b) If the vehicle were driven around this bend in a horizontal circular path of radius 60 m, find the greatest speed it could attain without slipping occurring. (7)

(Total: 10 marks)

6. A wooden stake of mass 4 kg is to be driven vertically downwards into the ground using a mallet of mass 6 kg. The speed of the mallet just prior to impact is 10 m s^{-1}. You can assume that the mallet remains in contact with the stake during and after impact.

   (a) Using conservation of momentum, find the speed with which the stake begins to enter the ground. (3)

   (b) After impact, since the mallet remains in contact with the stake, the weight of both mallet and stake assist the stake in penetrating the ground. If the ground offers a constant resistance to motion of 1000 N, find the deceleration of the stake. (4)

   (c) How far will the stake penetrate into the ground with each blow? (3)

(Total: 10 marks)

7. A and B are two points 25 cm apart on a smooth horizontal surface. A particle of mass 0.5 kg lies at A and is connected to B by a light spring of natural length 25 cm and modulus 50 N. The particle is projected towards B with speed 4 m s^{-1}.

   (a) Show that the subsequent motion is simple harmonic. (6)

   (b) Find the periodic time and the amplitude of these oscillations. (4)

(Total: 10 marks)

8. A framework consists of three identical uniform rods AB, BC and CA smoothly jointed together at A, B and C. The framework, which is suspended by a string from the mid-point of AB, lies in a vertical plane with AB horizontal and with C below AB. The weight of each rod is W, and the system is in equilibrium. Find:

   (a) the tension in the string; (2)

   (b) the magnitude of the reaction at C; (4)

   (c) the magnitude of the reaction at B. (4)

(Total: 10 marks)

9. A car of mass 800 kg ascends a hill inclined at $\sin^{-1}(1/100)$ to the horizontal against a constant resistance of 420 N. Its engine is working at a constant rate of 15 kW. Find:

   (a) the acceleration of the car up the hill when travelling with a speed of 10 m s^{-1}; (8)

   (b) the maximum speed of the car up the hill. (2)

(Total: 10 marks)

10. A uniform beam AB is supported at an angle θ to the horizontal by a light string attached to end B, and with end A resting on rough horizontal ground. The beam and the string lie in the same vertical plane, with the string at right angles to the beam. The system is in limiting equilibrium.

    Show that the coefficient of friction, μ, between the beam and the ground is given by

$$\mu = \frac{\cos\theta \sin\theta}{2 - \cos^2\theta}.$$

(Total: 10 marks)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE

EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL

2019 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	11th May 2019
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are **not** allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

In this paper, **i**, **j** and **k** are unit vectors along the x-, y- and z-axes of a Cartesian coordinate system.

(Take $g = 10\ \text{ms}^{-2}$)

1. A light beam, of length $2a$, is clamped horizontally at one end, and carries a load W at its midpoint and another load W at its free end. This cantilever system is in equilibrium.

   (a) Find the force and couple exerted by the clamp. (2)

   (b) Find expressions for the shearing force and bending moment at any point along the beam. (9)

   (c) Draw a sketch of the shearing force and bending moment along the beam. (4)

(Total: 15 marks)

2. The points P, Q, R and S have position vectors:

$$\mathbf{i}-\mathbf{j}-\mathbf{k}, \quad \mathbf{i}+\mathbf{j}+\mathbf{k}, \quad \mathbf{i}+\mathbf{j}, \quad 2\mathbf{i}+\mathbf{j}+\mathbf{k} \quad \text{respectively.}$$

(a) Show that the straight line with vector equation $\mathbf{r}=\mathbf{i}+t\mathbf{j}+t\mathbf{k}$ passes through the points P and Q. (2)

(b) Find the vector equation of the line passing through R and S. (4)

(c) Show that PQ and RS are inclined at 60° to each other. (3)

(d) Show that the vector $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ is perpendicular to both PQ and RS. By finding the scalar product of $\mathbf{a}$ with $\overrightarrow{PR}$, or otherwise, find the shortest distance between PQ and RS. (6)

(Total: 15 marks)

3. A system is made up of three variable forces which depend on a real parameter t. The three forces are:

$\mathbf{F}_1 = 2\cos t\,\mathbf{i}$ acting at the point with position vector $\mathbf{0}$,

$\mathbf{F}_2 = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ acting at $\mathbf{i}+\mathbf{j}$, and

$\mathbf{F}_3 = 3\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ acting at $-3\mathbf{i}+2\mathbf{j}$.

(a) If the system is equivalent to a single force $\mathbf{F}$ at the origin O, and a couple $\mathbf{G}$, find the values of $\mathbf{F}$ and $\mathbf{G}$ in terms of the parameter t. (8)

(b) Find the equation of the line of action of the resultant, and show that this line passes through a fixed point which is independent of t. (7)

(Total: 15 marks)

4. Two uniform smooth spheres, each of mass m and radius a, collide when moving on a horizontal plane. Before impact, the spheres are moving in the same direction with speeds $2u$ and u along parallel lines which are at a distance $6a/5$ apart. The coefficient of restitution between the spheres is $1/2$.

(a) Find the velocity of the two spheres after impact. (11)

(b) Show that the angle between their paths is approximately 27°. (4)

(Total: 15 marks)

5. A particle of mass m is free to move in the $\mathbf{i},\mathbf{j}$ plane under the influence of a time-dependent force $mue^{-t}\mathbf{j}$, where t is the time. Initially, at $t=0$, the particle has position vector $\mathbf{r}=\mathbf{0}$ and velocity vector $\mathbf{v}=v\mathbf{i}$. The parameters m, u and v are constants.

   (a) Using Newton's Second Law, set up a differential equation for the motion of this particle. (2)

   (b) By integrating twice with respect to t, obtain relations for $\mathbf{v}$ and $\mathbf{r}$ in terms of t, m, u and v. (8)

   (c) Show that when $t=\ln 2$, the velocity is $\mathbf{v}=v\mathbf{i}+u/2\mathbf{j}$. (2)

   (d) Show that the path of the particle is asymptotic to $\mathbf{r}=vt\mathbf{i}+u(t-1)\mathbf{j}$. Find the Cartesian equation of the asymptote. (3)

(Total: 15 marks)

6. Two particles of mass m and $2m$ are attached to the ends of a light elastic string of natural length $2a$ and modulus of elasticity $4mg$. The particles are held at a distance $3a$ apart on a smooth horizontal plane and are then released.

   (a) Using Newton's Second Law of Motion for each particle, or otherwise, show that the centre of mass of the system remains at rest in the subsequent motion. (7)

   (b) Find how far each particle travels before they collide. (3)

   (c) Using conservation of energy, or otherwise, find the speed of the particles immediately before impact. (5)

(Total: 15 marks)

7. A hemispherical bowl of radius r is fixed with its circular rim uppermost, and lying in a horizontal plane. A uniform rod of length l and weight W rests over the rim of the bowl, and has one end in contact with the inner surface of the bowl. All contacts are smooth and the rod is inclined at an angle θ to the horizontal.

   (a) Draw a diagram showing carefully the forces acting in the system. (5)

   (b) Using the fact that the forces are concurrent, or otherwise, show that

   $$4r\cos 2\theta = l\cos\theta.$$ (5)

   (c) Using Lami's Theorem, or otherwise, find the reactions on the rod in terms of W and θ. (5)

(Total: 15 marks)

8. (a) A uniform solid sphere has mass m and radius a. Using integration, show that the moment of inertia of the sphere about a diameter is $\frac{2}{5}ma^2$. (5)

   (b) A compund pendulum consists of a uniform solid sphere of mass m and radius a attached to a pivot by means of a uniform rod of mass $m/4$ and length $2a$. The centre of the sphere is at a distance $3a$ from the pivot. The pendulum can swing freely in a vertical plane about a smooth fixed horizontal axis passing through the pivot and perpendicular to the rod.

      (i) Find the moment of inertia of the pendulum about this axis. (4)

      (ii) Show that the equivalent simple pendulum for small oscillations about this axis is less than $3a$ by approximately 0.17%. (6)

(Total: 15 marks)

9. A uniform rod AB of mass m and length $2a$ is suspended from a smooth fixed pivot at its end A. The rod is hanging at rest, when a particle of mass m, moving horizontally with speed u, collides with it at a point at a distance x below A. The particle adheres to the rod.

   (a) Prove that immediately after impact, the rod starts to move with angular velocity $3ux/(3x^2+4a^2)$. (6)

   (b) Find the value of x for which this angular velocity is maximum. (4)

   (c) Using this value for x, find the value of u for which the rod just reaches the horizontal position in the subsequent motion. (5)

(Total: 15 marks)

10. (a) A solid uniform cylinder has mass m and radius a, whilst a thin cylindrical ring also has mass m and radius a. Write down the moments of inertia of these bodies about their own axes. (2)

    (b) The solid cylinder and the ring roll from rest without slipping down lines of greatest slope of a plane inclined at an angle α to the horizontal.

       (i) Show that the solid cylinder covers a distance $\frac{1}{12}gt^2\sin\alpha$ further than the ring at time t. (9)

       (ii) Find the minimum value of the coefficient of friction between the plane and the rolling bodies. (4)

(Total: 15 marks)