L-Università ta' Malta

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL

2020 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	25^{th} April 2020
TIME:	9:00 a.m. to 12:05 p.m.

Directions to candidates

Attempt **ALL** questions. There are 10 questions in all.
The marks carried by each question are shown at the end of the question.
The total number of marks for all the questions in the paper is 100.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.
Unless otherwise stated, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x- and y- axes of a Cartesian coordinate system. Units are Newtons for force and metres for distance.

(Take $g = 10\ \text{ms}^{-2}$)

1. A regular hexagon ABCDEF of side $2a$ lies in a Cartesian coordinate system, with A at the origin, AC along the x-axis, and AF along the y-axis.

   Three forces acting along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ have a resultant which acts along $\overrightarrow{DF}$.

   When a couple $4Pa$ in the sense CBA is added to the plane of the hexagon, the resultant acts along $\overrightarrow{CA}$.

   Find the magnitudes of the **THREE** forces in terms of P.

(Total: 10 marks)

2. (a) A particle is projected with speed u at an angle α above the horizontal. Show from first principles, that the horizontal range of the particle is $u^2 \sin 2\alpha/g$, and that the maximum height reached is $u^2 \sin^2 \alpha/2g$. (5)

   (b) The maximum horizontal range a particle can achieve with an initial speed u is R. If a particle projected with speed u has horizontal range $3R/5$, calculate the two possible angles of projection. Show that the difference in the maximum heights attained with these angles is $2R/5$. (5)

(Total: 10 marks)

3. A uniform lamina is in the form of a square ABCD of side 2 m. E is a point on AB such that EB = x metres, where $x < 2$, and the portion EBC is removed.

   (a) Find the distance of the centroid from the side AD. (5)

   (b) Show that if the lamina is placed in a vertical plane with AE on a rough horizontal surface, it will topple if $x > 3 - \sqrt{3}$. (2)

   (c) If $x = 1.5$, and the weight of the lamina is W, find the least force which must be applied to the lamina at C to prevent it from toppling. (3)

(Total: 10 marks)

4. Two frigates A and B keep observations on a foreign warship C.

   To A, which is moving at 10 knots on a course 030°, the ship C appears to be travelling in a direction 120°.

   To B, whose speed is 12 knots on a course 150°, C appears to be travelling due East.

   Find the true velocity of the warship.

(Total: 10 marks)

5. AB is a light inextensible string of length $2l$ having a particle of mass m attached at A, and another particle of mass m attached at B. The system lies on a smooth horizontal table, with B initially at a point C distant l from A. The particle at the end B is projected across the table with speed u perpendicular to AC.

   Find the velocity of each particle after the impulse, and the magnitude of the impulsive tension.

(Total: 10 marks)

6. A car travelling at 28 ms^{-1} has no tendency to slip on a circular track of radius 200 m banked at an angle θ to the horizontal.

   (a) Find θ. (4)

   (b) When the speed is decreased to 22 ms^{-1} the car is just on the point of slipping down the track. Find the coefficient of friction between the car and the track. (6)

(Total: 10 marks)

7. A framework consists of four identical light rods AB, BC, CD and DA which are joined together to form a square ABCD. The square shape is maintained by means of a fifth rod BD, which is a diagonal of the square. All the rods are smoothly jointed together.

   The framework, which lies in a vertical plane with AB horizontal and with AB below CD, is smoothly pivoted at A, and carries a weight W at B. The system is held in equilbrium by a horizontal force P acting along $\overrightarrow{CD}$.

   (a) By taking moments about A, find P in terms of W. (2)

   (b) Find the force in the rods, indicating whether they are in tension or compression.

   (8)

(Total: 10 marks)

8. A car of mass 840 kg moving on a straight level road has its engine working at 70 kW against a constant total resistance of 2100N.

   (a) Find the acceleration of the car in ms^{-2} if its speed is 90 km/hr. (5)

   (b) Find the maximum speed in km/hr at which this car could move up an incline of $\sin^{-1}(1/10)$ against the same resistance with the engine working at the same rate.

   (5)

(Total: 10 marks)

9. AP and PB are two identical elastic strings of natural length 1 m and modulus of elasticity λ. A particle of mass m is attached to the two strings at P. The system is placed on a smooth horizontal table, with A and B fixed 4 m apart, and with P midway between A and B. The mass at P is then pulled through a small distance x towards B and released from rest.

   Using Newton's Second Law, show that the subsequent motion is simple harmonic, and find its periodic time. **(Total: 10 marks)**

10. Two identical uniform rods AB and BC, each of length $2a$ and weight W, are smoothly pivoted at B. The rods lie in a vertical plane, with their ends A and C resting on rough horizontal ground. The coefficient of friction at both A and C is $1/2$. The rods are on the point of slipping when $\angle BAC = \theta$. Find:

    (a) the normal reaction of the floor on the rods; (3)

    (b) the reaction at the hinge B; (4)

    (c) the value of θ. (3)

(Total: 10 marks)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE

EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL

2020 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	27^{th} April 2020
TIME:	4:00 p.m. to 7:05 p.m.

Directions to candidates

Answer **SEVEN** questions. In all there are 10 questions each carrying 15 marks.

Graphical calculators are **not** allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

In this paper, **i**, **j** and **k** are unit vectors along the x-, y- and z-axes of a Cartesian coordinate system.

(Take $g = 10\ \text{ms}^{-2}$)

1. ABCD is a uniform, light beam, of length $4a$, with AB=CD=a, and BC =$2a$. The beam is simply supported at its ends A and D, and carries a load of $2W$ distributed uniformly on BC. The beam is in equilibrium.

   (a) Find the reactions at the supports. (2)

   (b) Find expressions for the shearing force and bending moment at any point along the beam. (9)

   (c) Draw a sketch of the shearing force and bending moment along the beam. (4)

(Total: 15 marks)

2. The vertices of a tetrahedron ABCD have position vectors **a** , **b** , **c** , **d** respectively, where

$$\mathbf{a} = 6\mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = 7\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = 7\mathbf{i} + \mathbf{k}, \quad \mathbf{d} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Forces of magnitude 60 and $6\sqrt{13}$ units act along $\overrightarrow{CB}$ and $\overrightarrow{CD}$ respectively. A third force acts at A. If the system reduces to a couple, find:

(a) the force at A; (8)

(b) the magnitude of the couple, and a unit vector along its axis. (7)

(Total: 15 marks)

3. (a) Masses $5m$, $3m$ and $2m$ are situated at points with position vectors:

$2\mathbf{i} + 2\mathbf{k}$, $\quad -8\mathbf{i} - 12\mathbf{j} + 5\mathbf{k}$, $\quad 3\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ respectively.

Find the position vector of the centroid of these point masses. (4)

(b) A particle with position vector $2\mathbf{i}$ is kept in equilibrium by the following five forces:

$\mathbf{P} = 7\mathbf{i} - 15\mathbf{j} + \mathbf{k}$, $\quad \mathbf{Q} = 5\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$,

$\mathbf{R}_1$ acting along the line $2\mathbf{i} + a(3\mathbf{i} + 4\mathbf{k})$,

$\mathbf{R}_2$ acting along the line $2\mathbf{i} + b(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$,

$\mathbf{R}_3$ acting along the line $2\mathbf{i} + c(-3\mathbf{i} + 4\mathbf{j})$.

Find the magnitudes of $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$. (11)

(Total: 15 marks)

4. A smooth uniform sphere A, moving with speed u, collides with an identical sphere B at rest, the direction of motion just before impact being inclined at an angle α to the line of centres. The coefficient of restitution between the spheres is e.

(a) Find the magnitude and direction of the velocities of A and B after the impact in terms of u, α and e. (9)

Given that $\tan^2\alpha = 8/27$ and $e = 2/3$, show that:

(b) the speed of A is halved by the impact; (3)

(c) the direction of motion of A is turned through an angle $\tan^{-1}(2\sqrt{6}/5)$. (3)

(Total: 15 marks)

5. A particle A of mass m is held on the surface of a fixed smooth solid sphere centre O and radius a at a point P, such that OP makes an acute angle θ_0 with the upward vertical, where $\cos\theta_0 = 3/4$. The particle is released from P, and has velocity v when it is at a position P_1 on the sphere, where OP_1 makes an angle θ with the upward vertical.

   (a) Show that $v^2 = \frac{1}{2}ga(3-4\cos\theta)$. (4)

   (b) Find in terms of θ the normal reaction on the particle at P_1. (4)

   (c) Show that the particle leaves the surface when OA makes an angle $\pi/3$ with the upward vertical. (2)

   (d) After it leaves the sphere, find the vertical distance through which the particle falls whilst travelling a horizontal distance a. (5)

(Total: 15 marks)

6. (a) Show that, if k and ω are positive constants, the roots of the equation $\lambda^2+k\lambda+\omega^2 = 0$ are distinct and both negative when $k > 2\omega$. (2)

   (b) A particle of mass m moves under the action of a force $\omega^2|x|$ per unit mass directed towards the origin and a resisting force kv per unit mass, where x and v are the displacement from the origin and the speed respectively after time t. Show that the differential equation satisfied by x and t is

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0.$$

   (2)

   (c) The particle starts from rest at $x = a$. Show that:

$$(\lambda_2 - \lambda_1)x = a(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t})$$

   where $-\lambda_1$ and $-\lambda_2$ are the roots of the quadratic equation given in part (a). (9)

   (d) Show that the particle does **not** pass through the origin. (2)

(Total: 15 marks)

7. Four equal uniform heavy rods are smoothly jointed together to form a rhombus ABCD. A and C are connected by a light elastic string whose natural length is half the length of each rod, and whose modulus of elasticity is twice the weight of a rod. The system is suspended smoothly from A.

   (a) Find the potential energy, Π, of the system in terms of $\theta \equiv \angle BAC$. (9)

   (b) Find $\dfrac{d\Pi}{d\theta}$. By putting it equal to zero, find the **TWO** positions of equilibrium of the system. (3)

   (c) Find $\dfrac{d^2\Pi}{d\theta^2}$ at the positions of equilibrium. Find the position of stable equilibrium, given that the sign of the second derivative has to be positive for stable equilibrium. (3)

(Total: 15 marks)

8. A uniform rod AB, of mass $6m$ and length $2a$, has a particle of mass m attached at B. The system is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. The rod is released from rest when AB is horizontal.

   (a) Show that in the subsequent motion, $3a\dot{\theta}^2 = 4g\cos\theta$, where θ is the angle between AB and the downward vertical. (4)

   (b) Find the angular acceleration $\ddot{\theta}$ of the rod. (4)

   (c) Find the horizontal and vertical components of the force exerted by the rod on the axis of rotation when $\theta = \pi/4$. (7)

(Total: 15 marks)

9. A uniform circular disc of mass m, radius a and centre O lies at rest on a smooth horizontal table. The disc is given a horizontal blow at a point A of its circumference, such that the initial velocity of A is of magnitude V at an angle of 45° to AO.

   (a) Find the angular velocity of the disc in terms of V, and show that the magnitude of the blow is $mV\sqrt{5}/3$. (4)

   (b) Find the velocity of the centre O. (4)

   (c) Show that the kinetic energy of the disc after impact is $mV^2/3$. (4)

   (d) Find the distance travelled by the centre O while the disc makes one revolution. (3)

(Total: 15 marks)

10. (a) Show that the moment of inertia of a uniform rod of mass m and length l about an axis through the centre of the rod and inclined at an angle θ to the rod is $\frac{1}{12} m\, l^2 \sin^2 \theta$. (3)

(b) A uniform wire of mass $3m$ and length $3l$, is bent to form three sides AB, BC, CD of a square. Using the parallel axes theorem, find the moment of inertia of this wire about an axis passing through point B and the midpoint of CD. (6)

(c) A system of particles is placed along the wire as follows:

(i) a particle of mass $\frac{m}{6}$ is placed at each of the two points A and D;

(ii) a particle of mass $\frac{m}{3}$ is placed at each of the two points B and C;

(iii) a particle of mass $\frac{2m}{3}$ is placed at each of the midpoints of the three sides.

Find the moment of inertia of these particles about the given axis in part (b). Show that this is equal to the moment of inertia found in part (b). (6)

(Total: 15 marks)