MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2021 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	5th June 2021
TIME:	16:00 to 19:05

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions in the paper is 100.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x-, y- and z-axis of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$)

1. Three light uniform rods AB, BC and CA are smoothly jointed together at A, B, C to form a light framework. Angle A is $60°$, angle B is $30°$, whilst angle C is $90°$. The framework is freely suspended from C and carries weights of P N and 100 N at A and B respectively. With these two weights, the rod AB is horizontal.
   (a) Find the weight P.
   [5 marks]
   (b) Show that the magnitude of the force in BC is 200 N. Find the force in AB and state whether it is in tension or compression.
   [5 marks]

2. ABC is an equilateral triangle having vertices $A(0,0)$, $B(2,0)$ and $C((1,\sqrt{3})$. D and E are the midpoints of AB and BC respectively. A force of 2 N acts along $\overrightarrow{AC}$ and forces $\mathbf{P}$ and $\mathbf{Q}$ act along $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively.
   (a) Express the three forces in the system in $\mathbf{i}, \mathbf{j}$ notation.
   [3 marks]
   (b) Find the magnitude and sense of $\mathbf{P}$ and $\mathbf{Q}$ if the system reduces to a couple.
   [4 marks]
   (c) By taking moments, or otherwise, find the magnitude and sense of $\mathbf{P}$ and $\mathbf{Q}$ if the system reduces to a single force along $\overrightarrow{DE}$.
   [3 marks]

3. A body is formed by removing a hemisphere from a uniform metal solid cylinder. The cylinder has radius $2a$ and length $4a$. The hemisphere has radius a and its plane face lies in one of the circular faces of the cylinder. The resulting solid is symmetrical about its central axis.
   (a) Find, in terms of a, the distance of the centre of mass of the body from the circular face of the solid that remains complete.
   [7 marks]
   (b) If the solid is suspended from a point on the circumference of this circular face, find the angle which this face makes with the downward vertical.
   [3 marks]

4. A lift travels a distance of 22 m from rest at the basement to rest at the top floor. Initially the lift moves with constant acceleration $a\,\mathrm{m\,s^{-2}}$ for a distance of 5 m; it continues with constant speed $v\,\mathrm{m\,s^{-1}}$ for 14 m; it is then brought to rest at the top floor by a constant retardation $d\,\mathrm{m\,s^{-2}}$.
   (a) Sketch a velocity-time graph and write down the times taken by the lift to cover the three stages of the journey in terms of v only.
   [4 marks]
   (b) If the total time that the lift is moving is 6 s, calculate the values of v, a and d.
   [4 marks]
   (c) A child of mass 30 kg is standing on the floor of the lift. Find the reaction between the child and the floor in the last stage of the journey.
   [2 marks]

5. A uniform ladder of mass 20 kg and length 3 m rests against a smooth vertical wall with the bottom of the ladder on smooth horizontal ground and attached by means of a light inextensible rope, 1 m long, to the base of the wall.
   (a) Find the tension in the rope.
   [5 marks]
   (b) If the breaking strain of the rope is 250 N, find how far up the ladder a man of 80 kg can safely ascend.
   [5 marks]

6. A locomotive of mass 15000 kg, working at the rate of 220 kW, pulls a train of mass 35000 kg up a straight track which makes an angle of $\sin^{-1}(1/50)$ with the horizontal.
   (a) Given that the acceleration is $0.23\,\mathrm{m\,s^{-2}}$ when the speed is $10\,\mathrm{m\,s^{-1}}$, find the magnitude of the resistance to motion at this speed.
   [5 marks]
   (b) Assuming this resistance is proportional to the speed, find the greatest speed of the train up the slope if the rate of working is unchanged.
   [5 marks]

7. A solid wooden cube of mass m and side l at rest on a smooth horizontal table is hit by a bullet of mass $m/5$ moving with speed v. The bullet enters the cube at the centre of one vertical face and emerges with speed $4v/5$ **relative** to the block at the centre of the opposite face.
   (a) Find the velocity of the cube.
   [2 marks]
   (b) If the force exerted by the bullet on the cube is constant, draw a velocity-time graph for the cube and for the bullet relative to the cube.
   [4 marks]
   (c) Find the time the bullet takes to pass through the cube.
   [2 marks]
   (d) Find the distance moved by the cube in this time.
   [2 marks]

8. A particle is projected from an origin O with a velocity of $(4\mathbf{i}+11\mathbf{j})\,\mathrm{m\,s^{-1}}$. The particle passes through a point P which has position vector $(8\mathbf{i}+b\mathbf{j})\,\mathrm{m}$.
   (a) Find the time taken for the particle to reach P from O, and hence find the value of b.
   [4 marks]
   (b) Find the velocity of the particle at P, and the direction in which the particle is moving at this instant.
   [3 marks]
   (c) Find the Cartesian equation of the path of the particle.
   [3 marks]

9. A particle of mass m moves in a horizontal plane under the action of a variable force $\mathbf{F}$ so that the position vector of the particle at time t is given by $\mathbf{r} = 4\cos(\omega t)\mathbf{i} + 3\sin(\omega t)\mathbf{j}$, where ω is a constant.
   (a) Find the velocity and acceleration vectors of the particle at time t.
   [2 marks]
   (b) Show that the greatest magnitude of $\mathbf{F}$ is $4m\omega^2$ and state the time at which this occurs for the first time.
   [3 marks]
   (c) If the force $\mathbf{F}$ ceases to act at time $t = \dfrac{\pi}{3\omega}$, find the position and velocity vectors of the particle at this instant. Hence find its position vector when $t = \dfrac{4\pi}{3\omega}$.
   [5 marks]

10. A smooth pulley is fixed above a horizontal table. Particles P, of mass $4m$, and Q, of mass $2m$, are attached to the ends of a light inextensible string which passes over the pulley, P being at rest on the table and Q hanging freely. A particle R, of mass m, falls vertically from rest through a height h, strikes Q, adheres to it and sets the whole system in motion. Assume that the combined particle does not reach the table before the system first comes to instantaneous rest. Show,
    (a) using the impulse-momentum relation, that the particles will start to move with speed $\sqrt{2gh}/7$,

    [5 marks]

    (b) using the conservation of energy principle, that the height that P will rise above the table is $h/7$.

    [5 marks]

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ADVANCED MATRICULATION LEVEL
2021 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	8th June 2021
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions and each question carries 15 marks.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x-, y- and z-axis of a Cartesian coordinate system.

(Take $g = 10\, m s^{-2}$)

1. The forces $\mathbf{F}_1 = (\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ N and $\mathbf{F}_2 = (-\mathbf{i}+2\mathbf{k})$ N act on a rigid body at the points whose position vectors relative to the origin O are $\mathbf{r}_1 = (\mathbf{j}+\mathbf{k})$ m and $\mathbf{r}_2 = (4\mathbf{i}+\mathbf{j}-\mathbf{k})$ m respectively. A third force $\mathbf{F}_3 = (P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})$ N acting at the point with position vector $\mathbf{r}_3 = (\alpha\mathbf{i}+\beta\mathbf{j}+5\mathbf{k})$ m, is then added so that $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$, and the sum of the moments of the three forces about O is equal to $(-7\mathbf{i}+9\mathbf{j}-6\mathbf{k})$ Nm.
   (a) Find $\mathbf{F}_3$ and $\mathbf{r}_3$.

   [9 marks]

   (b) A fourth force $\mathbf{F}_4$, acting at the point $3\mathbf{i}+3\mathbf{j}+\mathbf{k}$, is then added to the above system so that the sum of the moments of the four forces about O is zero. Find a possible value of $\mathbf{F}_4$, and state with reasons whether the system is in equilibrium.

   [6 marks]

2. A light rigid beam of length 6 m is supported at two points 1 m from each end and carries a uniformly distributed load of 400 $\mathrm{N m^{-1}}$ over its whole length. This structural system is in equilibrium.
   (a) Find the normal reactions at the supports of the beam.

   [2 marks]

   (b) Find the shearing force and bending moment at any point along the beam.

   [9 marks]

   (c) Draw a sketch of the shearing force and bending moment, and deduce where the bending moment is largest.

   [4 marks]

3. (a) An inextensible string of length $8r/5$ is attached at one end to a point on the rim of a uniform solid hemisphere and at the other end to a point on a smooth vertical wall. The hemisphere is of weight W and radius r and rests with its curved surface against the wall. A particle is attached to the lowest point of the rim and, in equilibrium, is in line with the string. Find the weight of the particle.

[7 marks]

(b) It is found that if the particle is removed the hemisphere can rest in the same position as before provided that the string is replaced by a longer one attached to a higher point on the wall. Show that the new string is inclined to the wall at an angle $\tan^{-1}(1/24)$.

[8 marks]

[**Note**: The centre of gravity of a uniform solid hemisphere of radius r is at a distance $3r/8$ from its plane face.]

4. (a) Two particles P and Q are moving with constant velocity $\mathbf{v}_1 = 5\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{v}_2 = 3\mathbf{i}+4\mathbf{j}-3\mathbf{k}$, respectively. Find the velocity vector of P relative to Q.

[1 mark]

(b) At time $t = 0$ the particle P is at the point whose position vector is $\mathbf{r}_1 = -4\mathbf{i}+7\mathbf{j}-6\mathbf{k}$. If P collides with Q when $t = 5$, find the position vector of Q at $t = 0$.

[7 marks]

(c) The velocity of P relative to a third moving particle R is in the direction $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and the velocity Q relative to R is in the direction $2\mathbf{i}+3\mathbf{j}-6\mathbf{k}$. Find the magnitude of the velocity of R.

[7 marks]

5. A smooth sphere A of mass m, another smooth sphere B of identical size but of mass $2m$, move towards each other with velocities $\mathbf{i}+2\mathbf{j}$ and $-\mathbf{i}+3\mathbf{j}$ respectively. They collide when their line of centres is parallel to $\mathbf{i}-\mathbf{j}$. The coefficient of restitution is 0.5.

Find the velocities of A and B after impact.

[15 marks]

6. (a) Two particles of mass $2m$ and m are connected by a light inextensible string over a pulley of mass $2m$ which may be regarded as a uniform circular disc, free to rotate about a fixed axis through its centre. The system is released from rest when the particles are at an equal height. If the string does not slip on the pulley, prove by using conservation of energy, that the speed of the particles when each has moved a vertical distance h is $\sqrt{\frac{gh}{2}}$.

[6 marks]

(b) Find the acceleration of the particles and the tension in each part of the string.

[9 marks]

7. A bead of weight W which can slide onto a smooth circular hoop of radius r, fixed in a vertical plane, is attached to the highest point of the hoop by an elastic string of natural length r and modulus of elasticity λ.
   (a) Find the elastic potential energy stored in the string when it makes an angle θ with the downward vertical.

   [3 marks]

   (b) By taking the tangent line at the highest point of the hoop as a datum for zero potential energy, show that the potential energy $V(\theta)$ of the system is

$$V(\theta) = \frac{1}{2}\lambda r(2\cos\theta - 1)^2 - 2rW\cos^2\theta\,.$$

   [4 marks]

   (c) By solving the equation $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$, find the values of θ for which the system is in equilibrium.

   [4 marks]

   (d) Show that when $W = 3\lambda/4$ there is only one position of equilibrium and by finding $\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}$ determine whether it is a stable or unstable position.

   [4 marks]

8. A parachutist of mass m falls freely until his parachute opens. When the parachute opens, he experiences an upward resistance kv where v is his speed and k is a positive constant.
   (a) Prove that, after time t from the opening of his parachute,

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv\,.$$

   [2 marks]

   (b) Prove also that, irrespective of his speed when he opens his parachute, his speed approaches a limiting value $\frac{mg}{k}$, provided he falls for sufficiently long time.

   [5 marks]

   (c) The parachutist falls from rest freely under gravity for a time $\frac{m}{2k}$ and then opens his parachute. Prove that the total distance he has fallen when his velocity is $\frac{3mg}{4k}$ is

$$\frac{m^2g}{8k^2}(8\ln 2 - 1).$$

   [8 marks]

9. A particle is moving in a straight line and at time t its distance from a fixed point O in the line is x m and its speed is v m s^{-1}.

   (a) Show that $\dfrac{d^2x}{dt^2} = v\dfrac{dv}{dx}$.

   [1 mark]

   (b) If the speed v and the distance x are related by the equation

   $$v^2 = 128 + 32x - 16x^2,$$

   show that the motion is simple harmonic about the point $x = 1$.

   [6 marks]

   (c) Find the period of oscillation and the amplitude.

   [4 marks]

   (d) Find the maximum velocity and maximum acceleration of the particle.

   [4 marks]

10. (a) Prove by integration that the moment of inertia I of a uniform solid circular cylinder of mass M, radius r and length $2l$ about a diameter AB of one of its ends is given by

   $$I = \frac{M}{12}\left(3r^2 + 16l^2\right).$$

   [6 marks]

   (b) The cylinder can rotate freely about a fixed horizontal axis which coincides with AB.

   (i) Find the period of small oscillations.

   [4 marks]

   (ii) Find the ratio of r to l for which the period is least, given that the volume of the cylinder remains constant.

   [5 marks]