MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2024 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	11^{th} May 2024
TIME:	09:00 a.m. to 12:05 p.m.

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions is 100.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. Four forces $(-3\mathbf{i}-5\mathbf{j})$, $4\mathbf{j}$, $\mathbf{i}$, and $(2\mathbf{i}+\mathbf{j})$, in units of newtons, act at the points with Cartesian coordinates $(0,-2)$, $(1,0)$, $(-3,0)$, and $(-1,4)$, respectively, in units of metres.

   (a) Show that these forces reduce to a couple, and find the magnitude and direction of its moment. (5)

   (b) Two forces are now added to the system as follows: $f\,\mathbf{i}$ at (x, y) and $-f\,\mathbf{i}$ at $(x, y+1)$. Find the value of f in newtons so that the system is in equilibrium. (5)

   (**Total: 10 marks**)

2. A particle of mass 2 kg is moving uniformly along the straight line given by the vector equation

$$\boldsymbol{r}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

   in units of metres, for the time t in the range $0 \leqslant t \leqslant 2$ in seconds. At $t = 2$, a constant force $\boldsymbol{F} = (3\mathbf{i} - 5\mathbf{j})\,\mathrm{N}$ is applied to the particle.

   (a) Find:

      (i) the vector equation of the path followed by the particle for $t \geqslant 2$; (4)

      (ii) the kinetic energies at times $t = 0$, $t = 2$, and $t = 4$; (2)

      (iii) the work done by the force as a function of time, starting at $t = 0$. (2)

   (b) Explain the relation between the work done and the value of the kinetic energy at $t = 0$, 2, and 4. (2)

   (**Total: 10 marks**)

3. A particle P of mass m moves in a vertical circle with centre O along the smooth inner surface of a hollow sphere of internal radius 3 m and the same centre O. The particle is projected from the lowest point A of the sphere with a horizontal velocity of $v\,\mathrm{m\,s^{-1}}$. As the particle moves, the line OP makes an angle θ with the downward vertical.

   (a) If $v = 9$, find the speed of the particle and the normal reaction force in terms of θ and m. (5)

   (b) Hence determine the value of θ at which the particle leaves the inner surface of the sphere. (1)

   (c) Find the minimum values of v if the particle is to reach:

      (i) that point with $\theta = 90°$; (2)

      (ii) the highest point of the sphere. (2)

(**Total: 10 marks**)

4. A golf ball is struck at point A on level ground with speed $50\,\mathrm{m\,s^{-1}}$ at an angle of elevation of 35° directly towards a hole which is 240 m away. Neglect air resistance and consider the ball as a particle.

   (a) Find the distance from A at which the ball hits the ground. (4)

   (b) Assuming the ball does not rebound upon hitting the ground and moves horizontally and uniformly towards the hole, find how long it takes for the ball to reach the hole from the moment it is struck. (3)

   (c) The next player hits the ball from the same initial point but adjusts the initial angle of elevation to 37°. Does the ball enter the hole? (3)

(**Total: 10 marks**)

5. A tugboat of mass 50 tons exerts a constant horizontal engine force of 130 kN to pull a ship of mass 200 tons, starting from rest. There are constant resistance forces on the tugboat and ship of 1 kN and 4 kN, respectively. [1 ton equals 1000 kg.]

   (a) Find the common acceleration of the ship and tugboat. (3)

   (b) The tugboat pulls the ship with a cable that makes an angle of 30° to the horizontal. Find the tension in the cable. (3)

   (c) At one point, the cable breaks. Assuming that the resistances to motion and the engine force do not change, find the acceleration of the tugboat and the deceleration of the ship at this instant. (4)

(**Total: 10 marks**)

6. A child at the edge of a pier throws a stone at a height of 1 m above sea water, with a velocity of $(\sqrt{5}\mathbf{i}+5\mathbf{j})\ \mathrm{m\,s^{-1}}$, where **i** and **j** are unit vectors aligned in the horizontal and vertical directions respectively. Neglecting air resistance, find:

   (a) the maximum height above the pier attained by the stone; (2)

   (b) the total time of flight of the stone until it hits the water; (4)

   (c) the angle that the stone makes with the horizontal when it hits the water. (4)

   (**Total: 10 marks**)

7. Three smooth spheres A, B, and C of the same radius have masses of 4 kg, 2 kg, and 4 kg respectively. They move in a straight line on a smooth horizontal plane. Spheres A and B collide directly while moving in opposite directions at the same speed of $9\ \mathrm{m\,s^{-1}}$, while C is at rest.

   (a) If the coefficient of restitution between the spheres A and B is $\frac{2}{3}$, determine the speeds of A and B after the collision. (6)

   (b) After the collision, sphere B moves on to hit sphere C directly. As a result, sphere B comes to rest. Determine the coefficient of restitution between spheres B and C. (4)

   (**Total: 10 marks**)

8. A light spring AB of natural length 20 cm and modulus of elasticity 200 N is placed along a rough slope inclined at an angle of $\tan^{-1}\frac{3}{4}$ to the horizontal, along a line of greatest slope, with its lower end A fixed to the plane. A block of mass 3 kg is placed at the upper end B of the spring, without being attached to it. The spring is compressed by a distance of 8 cm and the block is released so that it moves up the slope.

   (a) Find the elastic potential energy in the spring at its maximum compression. (3)

   (b) If the coefficient of friction between the slope and the block is $\frac{1}{12}$, calculate the distance moved by the block along the slope from its point of release until it comes momentarily at rest. (7)

   (**Total: 10 marks**)

9. Two ships, the "Paola" P and the "Qormi" Q, leave Birzebbuga B at 08:00 hours. Ship P moves along the vector $\mathbf{i}+\sqrt{3}\mathbf{j}$ with a constant speed of $3\ \mathrm{km\,h^{-1}}$, while ship Q moves with a constant velocity $-3\sqrt{3}\mathbf{j}\ \mathrm{km\,h^{-1}}$ relative to ship P. Find:

   (a) the velocities of P and Q relative to B, in vector form; (3)

   (b) the time taken until the two ships are 18 km apart; (2)

   (c) the distance and direction of the ships relative to B when they are 18 km apart. (The direction should be quoted in terms of the angle the ships make with the **i** vector.) (5)

   (**Total: 10 marks**)

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10. A smooth particle of mass 0.2 kg is attached to a fixed point on a smooth flat table with an inextensible cord of length 1.5 m. The particle is then made to rotate uniformly in a horizontal circle on the table with a period of 3 s. Find:

   (a) the frequency of the motion; (2)

   (b) the angular speed of the particle; (2)

   (c) the centripetal acceleration of the particle; (2)

   (d) the tension in the chord; (2)

   (e) the linear speed of the particle. (2)

(**Total: 10 marks**)

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2024 FIRST SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	13th May 2024
TIME:	4:00 p.m. to 7:05 p.m.

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions in all.
Each question carries 15 marks.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\text{m}\,\text{s}^{-2}$.)

1. A uniform horizontal beam ABC has length $5a$ and mass 30 kg. It is simply supported at points A and B, where $AB = 3a$.
   (a) Find the forces acting on the beam at A and B. (2)
   (b) Find the shear force and bending moment as a function of x, the distance along the beam from A. (5)
   (c) Draw the diagram of the shear force and bending moments along the beam. (4)
   (d) The beam is shifted by an amount a towards A. At which part of the beam is the difference in shear force between the two configurations at a maximum? (4)

   (Total: 15 marks)

2. A cube of side length 25 cm consists of parallel squares $ABCD$ and $EFGH$ joined by edges AE, BF, CG and DH. The corners A, B, and C have respective position vectors

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 21 \\ 17 \\ 3 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 12 \\ 29 \\ 23 \end{pmatrix}.$$

   (a) Find the position vector of D. (2)
   (b) Using the cross product, or otherwise, find the vector $\overrightarrow{AE}$, where AE is perpendicular to AB and AD. (2)
   (c) Deduce the position vectors of the points E, F, G, and H. (4)
   (d) Three forces of equal magnitude, 10 N, act along the sides $\overrightarrow{AC}$, $\overrightarrow{GB}$, and $\overrightarrow{CH}$. Find the equivalent system of force and torque acting at E. (7)

   (Total: 15 marks)

3. A uniform solid cylinder A has base diameter a and height a. A second cylinder B has base diameter a and height $3a$. They are made of the same material. A is placed with its base on a rough plane which is inclined at an angle θ to the horizontal, while B is placed on top of A so that their faces in contact coincide. The coefficient of friction of A with the plane is $1/3$ and that of B with A is $1/2$.

   (a) Show that, in general, a mass placed on the plane is in limiting equilibrium when the coefficient of friction $\mu = \tan\theta$. (4)

   (b) Find the angle θ at which:

      (i) B is about to slide off A; (2)

      (ii) B is about to topple off A; (3)

      (iii) A and B together are about to slide down the plane; (2)

      (iv) A and B together are about to topple over. (3)

   (c) Hence decide which of these possibilities happens first as the plane's inclination is increased from zero. (1)

   (**Total: 15 marks**)

4. A framed picture is modelled as a uniform square lamina $ABCD$ of mass 3 kg and side length 1 m. The picture is hung from a smooth peg P by a 1.2 m thin copper wire APB with $AP = 0.8$ m. Find:

   (a) the angles of the triangle APB; (3)

   (b) the angle that AB makes with the horizontal; (5)

   (c) the tension in the wires AP and BP; (4)

   (d) the tension in the wires *if* the picture is righted so that $AP = BP$. (3)

   (**Total: 15 marks**)

5. Alice is playing at archery by trying to hit bullseye at a distance of 50 m. The arrow is to be considered as a particle that is projected from a height of 1.5 m above level ground with a speed of $50\,\mathrm{m\,s^{-1}}$ and the target is a vertical disc of diameter 50 cm with its centre at a height of 1.28 m above ground.

   (a) Find:

      (i) the angle to the horizontal ($< 45°$) that Alice should shoot at to hit the centre of the disc; (4)

      (ii) the time taken for the arrow to hit the target. (2)

   (b) Find the difference in the angles of projection ($< 45°$) in order that the arrow is within 20 cm of bullseye. (5)

   (c) The target is now moved a distance h further away from the archer. By keeping the same initial angle of projection as in part (a)(i), the arrow now hits the lowest point of the disc. What is the value of h? (4)

   (**Total: 15 marks**)

6. A space centre is interested in studying how to divert a meteoroid A which is moving along a collision course with Earth. A is modelled as a smooth sphere of mass 30 000 kg moving with a speed of 700 km h^{-1}. A 500 kg smooth spherical mass B is to impact A along the line joining their centres of mass perpendicularly to the direction of motion of A and the collision is required to divert the direction of motion of A by an angle of 10°. The coefficient of restitution between A and B is $\frac{1}{3}$. Find:

   (a) the speeds of A and B perpendicular to the line joining their centres of mass after the collision; (2)

   (b) the speed of A parallel to the line joining their centres of mass after the collision; (2)

   (c) the initial speed of B; (9)

   (d) the final speed of B. (2)

   (**Total: 15 marks**)

7. A light spring balance consists of a light spring AB of natural length 25 cm and modulus of elasticity 64 N, hanging vertically from the fixed point A, and a hook attached at B. A mass of 1 kg is placed on the hook. The damping force is given by $k\dot{x}$, where x is the displacement of B from the equilibrium position, $k = 32\,\mathrm{N\,s\,m^{-1}}$, and the dot represents the derivative with respect to time.

   (a) Show that x is governed by the differential equation
   $$\ddot{x} + 32\dot{x} + 256x = 0.$$ (7)

   (b) If the mass is released from rest when the spring is unstretched, solve the above differential equation. (8)

   (**Total: 15 marks**)

8. A thin uniform annulus of mass $m = 50$ g has inner and outer radii equal to $r_1 = 1$ cm and $r_2 = 1.5$ cm, respectively. It spins about a diagonal on a smooth pivot with a period of 0.2 s.

   (a) Show that the moment of inertia of an annulus about an axis perpendicular to its plane and passing through its centre is
   $$I_z = \tfrac{1}{2}m(r_1^2 + r_2^2).$$ (8)

   (b) Using symmetry, or otherwise, deduce an expression for the moment of inertia about a diameter of the annulus. (4)

   (c) Find the kinetic energy of the annulus. (3)

   (**Total: 15 marks**)

9. A uniform disc of mass 1.5 kg and radius $30\sqrt{2}$ cm is smoothly pivoted along a diagonal so that it rests in equilibrium in a horizontal plane. A particle P of mass 0.5 kg is dropped from a height of 2 m onto the disc hitting it at a distance of $5\sqrt{10}$ cm from the axis. The particle sticks to the disc and the combined system S starts rotating. The moment of inertia of a disc of mass m and radius r about a diameter is $\frac{1}{4}mr^2$.

   (a) Find the velocity with which P hits the disc. (2)

   (b) Obtain an expression for the combined moment of inertia of the particle and disc. (3)

   (c) What is the angular velocity of S just after impact? (4)

   (d) Show that, in the absence of resistive forces, the system keeps on rotating. (4)

   (e) Determine the minimum angular velocity of the system during its motion. (2)

   (**Total: 15 marks**)

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10. A thin uniform rod AB of mass m and length $6l$ has two identical uniform solid spheres of mass $\frac{5}{2}m$ and radius l attached at each end of the rod, with their centres aligned with the axis of the rod. The rod is freely pivoted at a point O on it whose distance from the centre of mass G of the system is x.

(a) Express the moment of inertia of the system in terms of m and l:

(i) about G; (6)

(ii) about O. (3)

(b) Show that the frequency of small oscillations about O, in a vertical plane, diminishes to zero as x is reduced to zero. (4)

(c) What is the periodic time for small oscillations when the system is pivoted from the centre of one of the spheres and $l = 108.6\,\text{cm}$? (2)

(**Total: 15 marks**)