MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2012

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	9th MAY 2012
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. Solve the differential equation

$$(y+1)\sin x \frac{\mathrm{d}y}{\mathrm{d}x} = (y^2+1)\tan^2 x\,,$$

given that $y = 0$ when $x = \pi$.

[10 marks]

2. The point P has position vector $\alpha\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The line ℓ_1 passes through P and is parallel to the vector $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. The line ℓ_2 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} - \mathbf{k})$.
(a) Find α, given that the lines ℓ_1 and ℓ_2 intersect.

[4 marks]

(b) Find the point of intersection of the lines, and the acute angle between them.

[3 marks]

(c) Find β and γ given that the vector $2\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$ is perpendicular to both lines.

[3 marks]

3. (a) A circle has centre $C(5,8)$ and radius $\sqrt{29}$.
   (i) Find the equation of this circle.
   (ii) The circle cuts the y-axis at R and Q, where Q is above R. Find the coordinates of Q and R, and find the equation of the tangent at Q.

[6 marks]

(b) The points A and B have coordinates $(1,4)$ and $(4,7)$ respectively. The point $P(x,y)$ is such that its distance from A is twice its distance from B. Find the equation of the locus of P and show that this locus is a circle with the same centre as the circle in (a).

[4 marks]

4. Prove that $\cos 3x = 4\cos^3 x - 3\cos x$.
Hence, or otherwise, find the general solution of the following equation.

$$(4\cos^2 x - 3)(4\cos^2 3x - 3)(4\cos^2 9x - 3) = 1\,.$$

[4, 6 marks]

5. The n^{th} term of an arithmetic progression is k times the m^{th} term. Show that the sum of the first N terms can be expressed as

$$S_N = \frac{Nd}{2(1-k)}\Big[2k(m-1) - 2(n-1) + (1-k)(N-1)\Big]\,,$$

where d is the common difference. Hence, or otherwise, find the sum of the first hundred terms of an arithmetic progression $a_1, a_2, a_3, \ldots$, given that $a_{35} = a_{21} + 42 = 2a_{15}$.

[5, 5 marks]

6. (a) A function f is defined by $x^2 - 3x + 2$ for real values of $x \leq 1.5$. Define the inverse function f^{-1} stating its domain and range.

[5 marks]

(b) Determine algebraically whether the functions $f(x) = \sqrt{x}$, $x \geq 0$, and $g(x) = x^2$, are inverses of each other.

[5 marks]

7. (a) Ninety students, including Peter and Paul, are to be split into three classes of equal size, and this is to be done at random. What is the probability that Peter and Paul end up in the same class.

[4 marks]

(b) How many arrangements can be made of the letters in the word **MISSISSIPPI**? Find the probabilities that one such arrangement picked at random has
   (i) all the four **S**'s next to each other;
   (ii) the two **P**'s separated from each other.

[2, 2, 2 marks]

8. (a) Resolve into partial fractions:

$$\frac{13-x}{6(2x^2+11x+5)}\,.$$

[5 marks]

(b) Solve the following equation:

$$\sqrt{x+\sqrt{4x+\sqrt{16x+\sqrt{64x+5}}}}-\sqrt{x}=1\,.$$

[5 marks]

9. (a) Differentiate with respect to x:
   (i) x^2e^{9x},
   (ii) $\dfrac{2\sin\sqrt{x}}{(2x-1)^4}$.

[5 marks]

(b) A curve is given implicitly by $3y^2+4xy+x^2=4$. Find the gradient in terms of x and y. The line ℓ is the tangent to the curve at the point $(2,0)$. Find the equation of ℓ. Find also the angle between the line ℓ and the line with equation $4x-y+7=0$, giving your answer to the nearest degree.

[5 marks]

10. (a) Use integration by parts to find

$$\int e^{2x}\cos 2x\ \mathrm{d}x\,.$$

Hence, or otherwise, find

$$\int e^{2x}\sin^2 x\ \mathrm{d}x\,.$$

[6 marks]

(b) Use the substitution $t=\sqrt{1-x^2}$ to find

$$\int \frac{x^5}{\sqrt{1-x^2}}\ \mathrm{d}x\,,$$

giving your answer in terms of x.

[4 marks]

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MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2012

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	9th MAY 2012
TIME:	4.00 p.m. to 7.00 p.m.

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = e^{\cos x - \sin x}\sin x\,,$$

given that $y = -\frac{1}{e}$ when $x = \frac{\pi}{2}$.

[7 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 4x - 5\,,$$

given that $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$ when $x = 0$.
[When finding the particular integral, use $Px^2 + Qx$ as a trial solution.]

[8 marks]

2. The points A, B and C have position vectors $3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, $\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, respectively.

(a) Find the equation of the line ℓ_1 passing through B and C, and the position vector of the point D on the line ℓ_1 such that $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{BC}$.

(b) Find the area of the triangle ABC.

(c) Find the equation of the plane Π_1 passing through A, B and C.

(d) Find the equation of the plane Π_2 that passes through B and C and is perpendicular to Π_1, and the distance from A to the plane Π_2.

[4, 3, 3, 5 marks]

3. (a) Assume that the events A, B, C, D are independent and that $P[C \cap D] > 0$. Find the probabilities $P[A|C \cap D]$ and $P[B|C \cap D]$ in terms of $P[A]$ and $P[B]$, and hence show that

$$P[A \cup B|C \cap D] = P[A \cup B]\,.$$

(b) A fair 4-sided die is rolled twice and we assume that all sixteen possible outcomes are equally likely. Let X and Y be the result of the 1st and the 2nd roll, respectively. Determine the conditional probability $P[A|B]$, where

$$A = \{\text{maximum of } X \text{ and } Y = m\} \quad \text{and} \quad B = \{\text{minimum of } X \text{ and } Y = 2\}\,,$$

and m takes each of the values 1, 2, 3, 4.

(c) In a survey, 85% of the employees say they favour a certain company policy. Previous experience indicates that 20% of those who do not favour the policy say that they do, out of fear of reprisal. What is the probability that an employee picked at random really does favour the company policy? It is reasonable to assume that all who favour say so.

[5, 5, 5 marks]

4. (Note: All angles have to be taken in radians throughout this question.)

(a) Show that the equation $3\sin(e^x) = 2$ has a solution between -1 and 0.

(i) Use the Newton-Raphson method to find an approximate value of this solution, taking 0 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

(ii) Solve the equation and compare the answer to the result obtained from the Newton-Raphson method.

[9 marks]

(b) Evaluate the integral $\displaystyle\int_0^1 \sin(e^x)\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to *four* decimal places.

[6 marks]

5. A curve has equation $y = \dfrac{8x(x-2)}{(4x-1)^2}$.

(a) Find the coordinates of the turning point of the curve.

(b) Sketch the curve, marking clearly any intercepts and asymptotes.

(c) Deduce the range of y from your graph and find the range of values of x for which $y > \frac{1}{2}$.

(d) Find the number of solutions of the equation $\dfrac{8x(x-2)}{(4x-1)^2} = \sin x$ in the region $-\pi \le x \le 3\pi$.

[4, 6, 3, 2 marks]

6. The curve C and the line ℓ have polar equations $r = 9+2\cos\theta$ and $r = 5\sec\theta$, respectively.
   (a) Sketch the curve and the line on the same axes.
   (b) Find the polar coordinates of the points of intersection of the line and the curve.
   (c) Find the area enclosed by the curve, giving your answer in terms of π.
   (d) The line ℓ divides the curve into two regions. Find the area of the smaller region, giving your answer to one decimal place.

[5, 3, 4, 3 marks]

7. Prove that $\displaystyle\int_0^{\pi/3} \tan\theta \, \mathrm{d}\theta = \ln 2$.
   (a) Given $I_n = \displaystyle\int_0^{\pi/3} \tan^n\theta \, \mathrm{d}\theta$, show that $I_n = \dfrac{\sqrt{3}^{n-1}}{n-1} - I_{n-2}$. Deduce that $I_5 = \dfrac{3}{4} + \ln 2$ and $I_6 = \dfrac{27\sqrt{3} - 5\pi}{15}$.
   (b) A curve is given by $y = \tan^4 x$. The region bounded by this curve and the x-axis from $x = 0$ to $x = \dfrac{\pi}{3}$ is rotated through one complete revolution about the x-axis to form a solid of revolution. Find the volume of this solid.

[2, 9, 4 marks]

8. (a) Show that the following system of equations either has a unique solution or is inconsistent.

$$\begin{aligned} x + y - 3z &= 1 \\ x - 4y + 2z &= 2 \\ kx + 2y - z &= 3 \end{aligned}$$

   (i) For what value of k is the system inconsistent?
   (ii) Solve the equations when $k = 3/2$.

[6 marks]

   (b) The *trace* of a square matrix X, denoted by $\operatorname{tr} X$, is defined to be the sum of the entries on the leading diagonal (the diagonal from the upper left to the lower right) of X. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   Show that:
   (i) $\det A^2 = (\det A)^2$ and $\operatorname{tr} A^2 = (\operatorname{tr} A)^2 - 2\det A$,
   (ii) $A^2 - (\operatorname{tr} A)A + (\det A)I = 0$, where I is the 2×2-identity matrix.
   Hence, or otherwise, find A when

$$A^2 = \begin{pmatrix} 93 & 64 \\ 176 & 125 \end{pmatrix}.$$

[9 marks]

9. (a) Express the following in the form $a + ib$, where a and b are real numbers.

$$\frac{\left(\frac{5\sqrt{3}}{2} + i\frac{5}{2}\right)^6}{\left(\frac{\sqrt{5}}{2} + i\frac{\sqrt{5}}{2}\right)^3}$$

[5 marks]

(b) Let

$$\begin{aligned} Z_n &= e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \cdots + e^{i(2n-1)\theta}\,, \\ S_n &= \sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta\,, \\ C_n &= \cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta\,. \end{aligned}$$

(i) Show that

$$e^{2i\theta} Z_n - Z_n = e^{i(2n+1)\theta} - e^{i\theta}\,.$$

(ii) Hence, prove that

$$C_n = \frac{\sin n\theta \cos n\theta}{\sin\theta} \quad \text{and} \quad S_n = \frac{\sin^2 n\theta}{\sin\theta}\,.$$

(iii) Deduce that

$$\int_0^{\pi/2} \frac{\sin 2n\theta}{\sin\theta}\, \mathrm{d}\theta = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{2n-1}\right).$$

[10 marks]

10. Prove the following statements by mathematical induction.

(a) For every positive integer n, $n(n+1)$ is even (i.e. divisible by 2).

[5 marks]

(b) For every positive integer $n \geq 2$, $n^3 - n$ is a multiple of 6.

[5 marks]

(c) The number of non-empty subsets of a set of n elements is $2^n - 1$.

[5 marks]

[Use (a) to help you in the inductive step of (b).]