**MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD**
**UNIVERSITY OF MALTA, MSIDA**

**MATRICULATION CERTIFICATE EXAMINATION**
**ADVANCED LEVEL**
**SEPTEMBER 2012**

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|---|---|
| **SUBJECT:** | PURE MATHEMATICS |
| **PAPER NUMBER:** | I |
| **DATE:** | 4th SEPTEMBER 2012 |
| **TIME:** | 9.00 a.m. to 12.00 noon |

**Directions to Candidates**

*Answer* **ALL** *questions.*

*Each question carries* 10 *marks.*

*Graphical calculators are* **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Let $a$, $b$, $c$ be real numbers and suppose that $a \neq 0$. Use the method of completing the square to show that for the quadratic equation $ax^2 + bx + c = 0$ to have real roots one must have $b^2 - 4ac \geq 0$.

**[5 marks]**

(b) Show that the quadratic equation in $x$

$$(e^u + 3)x^2 + 3e^u x + (e^u - 4) = 0$$

has real and distinct roots for every real value $u$.

**[5 marks]**

2. The point $P$ has position vector $4\mathbf{i} + \mathbf{j} - \mathbf{k}$. The line $\ell_1$ passes through $P$ and is parallel to the vector $\mathbf{i} + \mathbf{k}$. The line $\ell_2$ passes through $P$ and is parallel to the vector $\mathbf{j} + \alpha\mathbf{k}$, where $\alpha$ is a positive number.
   (a) Find $\alpha$ given that $\ell_2$ makes an angle of $60°$ with $\ell_1$.
   (b) The line $\ell_3$ has equation

$$\mathbf{r} = 5\mathbf{i} - \frac{1}{2}\mathbf{j} + \beta\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}).$$

Find $\beta$ given that $\ell_3$ intersects $\ell_1$. Show that $\ell_3$ intersects $\ell_2$ as well.

**[4, 6 marks]**

3. (a) Find $2 \times 2$ matrices **A** and **B** which represent clockwise rotations through $30^o$ and $60^o$ respectively, about the origin.

   (b) Show that $\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Which single transformation is equivalent to **BA**?

   (c) Find the least natural number $n$ such that $(\mathbf{BA})^n = \mathbf{I}$, where **I** is the $2 \times 2$ identity matrix.

   (d) Find the image of the line $y = 3x + 1$ under the transformation **BA**.

**[4, 3, 1, 2 marks]**

4. (a) Express $\dfrac{8x}{9 - x^2}$ into partial fractions.

**[2 marks]**

   (b) Solve the differential equation

$$y^2(9 - x^2)\sin(y^3)\frac{dy}{dx} = 8x\,,$$

   given that $y = 0$ when $x = 0$.

**[8 marks]**

5. Express $8\sin 2x + 15\cos 2x$ in the form $R\sin(2x + \alpha)$, where $R > 0$ and $\alpha$ lies in the interval $[0, \pi/2]$. Hence find:

   (a) the greatest and least values of

$$\frac{1}{8\sin 2x + 15\cos 2x + 25}\,.$$

   (b) the values of $x$ in the range $[0, 2\pi]$, correct to to 2 decimal places, for which

$$8\sin 2x + 15\cos 2x = -10\,.$$

**[4, 2, 4 marks]**

6. (a) A function $f$ is defined by $-\sqrt{4 - x^2}$ for real values of $-2 \le x \le 0$. Define the inverse function $f^{-1}$ stating its domain and range. Sketch the graph of $f$ and $f^{-1}$.

**[6 marks]**

   (b) Given functions $f(x) = \sqrt{x}$ and $g(x) = x - 2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$ giving their domains.

**[4 marks]**

7. (a) A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model of a single roll of this die and find the probability that the outcome is less than four.

**[5 marks]**

(b) Consider $n$ people who are attending a party. We assume that every person has an equal probability of being born on every day during the year, independent of everyone else. Assuming that nobody is born on the 29$^{\text{th}}$ February and that $n \leq 365$, find the probability that each person has a distinct birthday.

**[5 marks]**

8. (a) Given that $z = 5 - 12i$ express $\sqrt{z}$ and $\dfrac{1}{\sqrt{z}}$ in the form $a + ib$, where $a$, $b$ are real numbers.

**[5 marks]**

(b) Show that $z = 2 - 5i$ is a root of the equation

$$z^3 - 7z^2 + 41z - 87 = 0\,.$$

Find the other two roots.

**[5 marks]**

9. (a) Differentiate $y = e^{12x}\sqrt{6x - 1}$ and simplify your answer in the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{9e^{12x}(bx + c)}{\sqrt{6x - 1}}$, where $b$ and $c$ are integers to be determined.

(b) A curve is given parametrically by $x = \dfrac{t^3}{t + 1}$ and $y = \dfrac{t - 1}{t + 1}$. Find:

(i) the gradient in terms of $t$;
(ii) the equation of the tangent at the point where $t = 1$;
(iii) the distance of the point $(2, 3)$ from this line.

**[4, 3, 2, 1 marks]**

10. Use the suggested substitutions to find

(i) $\displaystyle\int \frac{\sin x \cos^3 x}{1 + \cos^2 x}\,\mathrm{d}x$ [substitution: $1 + \cos^2 x = t$],

(ii) $\displaystyle\int x \sin\sqrt{x}\,\mathrm{d}x$ [substitution: $\sqrt{x} = t$],

giving your answers in terms of $x$.

**[4, 6 marks]**

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MATRICULATION CERTIFICATE EXAMINATION
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SEPTEMBER 2012

| | |
|---|---|
| SUBJECT: | PURE MATHEMATICS |
| PAPER NUMBER: | II |
| DATE: | 5th SEPTEMBER 2012 |
| TIME: | 9.00 a.m. to 12.00 noon |

**Directions to Candidates**

*Answer* **SEVEN** *questions. Each question carries* 15 *marks.*

*Graphical calculators are* **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Find $\displaystyle\int \frac{2\ln x}{x\left[(\ln x)^2+1\right]}\,\mathrm{d}x$. [Hint: use the substitution $u = (\ln x)^2 + 1$.]
   (b) Solve the differential equation
$$x\left((\ln x)^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\ln x = 6x^2,$$
   given that $y = 9$ when $x = 1$.
   (c) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 9y = 5\cos 2x,$$
   given that $y = -\frac{5}{13}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$ when $x = 0$.

**[2, 6, 7 marks]**

2. The plane $\Pi_1$ has equation $2x - 3y + z = 26$. The point $A$ has position vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
   (a) Find the equation of the line $\ell_1$ passing through $A$ that is perpendicular to the plane $\Pi_1$. Find the point $B$ where $\ell_1$ intersects $\Pi_1$.
   (b) Show that
$$\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$$
   is the vector equation of a line $\ell_2$ in the plane $\Pi_1$ that passes through $B$.
   (c) Let $C$ and $D$ be the two points on the line $\ell_2$ that are a distance of $4\sqrt{5}$ units from $A$. Find the position vectors of $C$ and $D$. Find the area of the triangles $ABC$ and $ABD$.

**[4, 3, 8 marks]**

3. A curve has equation $y = \dfrac{x^2 - 5x + 7}{x - 3}$.

   (a) Find the range of $y$ for real $x$. Hence, or otherwise, find the coordinates of the turning points.

   **[6 marks]**

   (b) Find the equations of the asymptotes and sketch the curve.

   **[7 marks]**

   (c) Show that

   $$\frac{x^2 - 5x + 7}{x - 3} = \frac{1}{x^2 + 1}$$

   has no real solution.

   **[2 marks]**

4. (Note: throughout this question, all angles have to be taken in radians.)

   (a) Show that the equation $\ln(\cos x) + 1 = 0$ has a solution between 1 and 1.5.

   (i) Use the Newton-Raphson method to find an approximate value of this solution, taking 1.25 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

   (ii) Solve the equation and compare the answer to the result you obtained from the Newton-Raphson method.

   **[9 marks]**

   (b) Evaluate the integral $\int_{-1}^{1} \ln(\cos x)\, dx$ by Simpson's Rule with an interval width of $h = 0.5$. Give your answers to *four* decimal places.

   **[6 marks]**

5. Prove the following statements by mathematical induction.

   (a) For every positive integer $n$,

   $$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}.$$

   **[4 marks]**

   (b) For every non-negative integer $n$, $3^n \geq 2^n$.

   **[5 marks]**

   (c) For every positive integer $n \geq 2$,

   $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}$$

   and hence.

   $$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < 1\,.$$

   **[6 marks]**

6. (a) Show that for two sets $A$ and $B$ we have:
   (i) $A' = (A' \cap B) \cup (A' \cap B')$,
   (ii) $B' = (A \cap B') \cup (A' \cap B')$,
   (iii) $(A \cap B)' = (A' \cap B) \cup (A' \cap B') \cup (A \cap B')$.

   Consider rolling a fair six-sided die. Let $A$ be the set of outcomes where the roll is an odd number. Let $B$ be the set of outcomes where the roll is less than 4. Calculate the sets on both sides of the equality in part (iii), and verify that the equality holds.

   **[6 marks]**

   (b) We draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that:
   (i) The 7 cards include exactly 3 aces.
   (ii) The 7 cards include exactly 2 kings.
   (iii) The 7 cards include exactly 3 aces, or exactly 2 kings, or both.

   **[9 marks]**

7. (a) Use De Moivre's Theorem to show that:

$$(1 + \sqrt{3}i)^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right),$$

$$(1 - i)^n = \sqrt{2^n}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right).$$

   If $1 + \sqrt{3}i$ is a root of the equation

$$z^7 + \frac{(1+i)^{10}}{z^6} + 2\sqrt{2}(1-i)\sqrt{z} = a + ib,$$

   find the values of the real constants $a$ and $b$.

   **[8 marks]**

   (b) Find the points of intersection of the loci given by the equations $|z - 2 - i| = 4$ and $|z - 5 - 3i| = |z - 1 + i|$.

   **[7 marks]**

8. The curve $S$ and the circle $C$ have polar equations $r = 3 + 2\cos\theta$ and $r = 3$ respectively.
   (a) Sketch the curve and the circle on the same axes and find the polar coordinates of their points of intersection.

   **[6 marks]**

   (b) Find the area of overlap of the curve and circle.

   **[6 marks]**

   (c) A line passing through the pole meets the curve $S$ at $A$ and $B$. Show that whatever the gradient of this line, $AB$ has a fixed length of 6 units.

   **[3 marks]**

9. (a) Use the definitions of the hyperbolic functions to prove the identity

$$\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y\,.$$

**[4 marks]**

(b) Express $13\cosh x + 5\sinh x$ in the form $R\cosh(x+k)$, where $R$ is positive and $k$ is in logarithmic form. Hence sketch the graph of $y = 13\cosh x + 5\sinh x$ and find the range of values of $p$ for which $13\cosh x + 5\sinh x = p$ has two real distinct roots.

**[6 marks]**

(c) Show that

$$\int_0^1 \frac{1}{13\cosh x + 5\sinh x}\,\mathrm{d}x = \int_0^1 \frac{e^x}{9e^{2x}+4}\,\mathrm{d}x\,.$$

By using the substitution $u = e^x$, or otherwise, find the value of this integral.

**[5 marks]**

10. (a) Let $\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{u}_2 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{u}_3 = \mathbf{i} - \mathbf{k}$. By letting $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, solve simultaneously for $\mathbf{x}$ the following linear equations

$$\mathbf{x}\cdot\mathbf{u}_1 = 1\,,$$
$$\mathbf{x}\cdot\mathbf{u}_2 = -1\,,$$
$$\mathbf{x}\cdot\mathbf{u}_3 = 5\,.$$

**[7 marks]**

(b) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a $2\times 2$ matrix, where $a$, $b$, $c$ and $d$ are real numbers. We say that $A$ *admits an eigenvector* if there exists a unit vector $\mathbf{u}$ and a real number $\lambda$ such that $A\mathbf{u} = \lambda\mathbf{u}$.

(i) Show that $A$ admits an eigenvector precisely when

$$(\operatorname{tr} A)^2 - 4\det A \geq 0\,,$$

where $\operatorname{tr} A$ is the trace of $A$, i.e. the sum of the entries on the leading diagonal (the diagonal from the upper left to the lower right) of $A$.

(ii) Deduce that if $b = c$ then $A$ admits an eigenvector.

(iii) Find an example of a $2\times 2$ matrix that does not admit an eigenvector.

**[8 marks]**