MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2013

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	9th MAY 2013
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

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1. (a) Use the substitution $u = x + \frac{1}{2}$ to find

$$\int \frac{1}{2x^2 + 2x + 5}\,\mathrm{d}x\,.$$

(b) Solve the differential equation

$$(2x^2 + 2x + 5)\frac{\mathrm{d}y}{\mathrm{d}x} = \sin y\,,$$

given that $y = \frac{\pi}{2}$ when $x = 1$.

[4, 6 marks]

2. The points A, B and C have position vectors $5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $3\mathbf{i} + 2\mathbf{j}$, respectively.

(a) Find the equation of the line ℓ_1 joining A and B.

[3 marks]

(b) Show that the point D with position vector $3\mathbf{i} - \mathbf{k}$ lies on ℓ_1, and that $\overrightarrow{CD}$ is perpendicular to ℓ_1.

[3 marks]

(c) Find the angle BAC and the area of the triangle ABC.

[4 marks]

3. A circle C passes through the point with coordinates $\left(\frac{7}{10}, -\frac{3}{10}\right)$, and cuts the x-axis twice but does not cut the y-axis. The line $y = 7x$ is a tangent to the circle C and the centre of C lies on the line $y = 3x - 5$. Find the radius and coordinates of the centre of C, and hence:
   (a) write down the *parametric* equations of the circle C in terms of a parameter t; and
   (b) find the gradient function of the circle C, leaving it in terms of the parameter t.

[7, 1, 2 marks]

4. (a) Prove that
   (i) $\cos 3A \equiv 4\cos^3 A - 3\cos A$,
   (ii) $\cos\dfrac{x}{3} + \cos\dfrac{x+2\pi}{3} + \cos\dfrac{x+4\pi}{3} \equiv 0$.

[2, 3 marks]

   (b) Hence, or otherwise, show that

$$\cos^3\frac{x}{3} + \cos^3\frac{x+2\pi}{3} + \cos^3\frac{x+4\pi}{3} \equiv \frac{3}{4}\cos x\,.$$

[5 marks]

5. (a) Find the three roots of the equation $(w+5)(w+8)(w+9) = 360$.
   (b) Let $z_0 = \sqrt{-2+6i}$, where $i = \sqrt{-1}$. Show that the solutions to the equation

$$z^2 + 6z = -11 + 6i$$

   can be expressed in the form $z = -3 \pm z_0$.
   (c) Hence, solve the equation

$$(z+1)(z+2)(z+3)^2(z+4)(z+5) = 360\,,$$

   by reducing it to the equation in (a). Give your answer in terms of z_0.

[3, 3, 4 marks]

6. (a) Evaluate $\displaystyle\int_0^{\pi/2} e^x \cos x \,\mathrm{d}x$ and $\displaystyle\int_0^{\pi/2} e^x \sin x \,\mathrm{d}x$.
   If

$$\int_0^{\pi/2} e^x (p\cos x + q\sin x)\,\mathrm{d}x = q\,,$$

   where p and q are constants, show that $p + q = 0$.
   (b) Use the substitution $x = \sin^2\theta$ to evaluate

$$\int_0^{1/2} \left(\frac{x}{1-x}\right)^{1/2} \mathrm{d}x\,.$$

[6, 4 marks]

7. A function f is defined for all real values of x by

$$f(x) = 3 - |2x - 1|.$$

   (a) Sketch the graph of $y = f(x)$, indicating the coordinates of the points where the graph crosses the coordinate axes. Hence show that the equation $f(x) = 4$ has no real roots.
   (b) State the range of f.
   (c) By finding the values of x for which $f(x) = x$, solve the inequality $f(x) < x$.

[5, 1, 4 marks]

8. An academic department offers 8 lower level courses: $\{L_1, L_2, \ldots, L_8\}$, and 10 higher level course: $\{H_1, H_2, \ldots, H_{10}\}$. A student needs to choose 4 lower level courses, and 3 higher level courses.
   (a) How many different choices are possible?
   (b) Suppose that any choice which involves one of $\{H_1 \ldots, H_5\}$ must include L_1 and any choice which involves one of $\{H_6, \ldots, H_{10}\}$ must include both L_2 and L_3. How many different choices are there?

[2, 8 marks]

9. (a) Prove that if $a + b + c = 0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

   (b) Hence, or otherwise, express in partial fractions

$$\frac{2x^2 - 8x - 1}{(x-2)^3 + (x+1)^3 + (1-2x)^3}.$$

[4, 6 marks]

10. The function f is given by $f(x) = \ln(x+2)$, for $x > -2$, and the function g is given by $g(x) = 2 - 3e^x$.
   (a) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same pair of axes, marking clearly any asymptotes and where the curves cross the coordinate axes.
   (b) The matrix $\mathbf{A}$ represents a clockwise rotation by $\pi/2$ radians about the origin and the matrix $\mathbf{B}$ represents an enlargement by a scale factor of 3 (that is, the unit square is mapped onto the square with vertices $(0,0)$, $(3,0)$, $(3,3)$ and $(0,3)$). Write down the matrices $\mathbf{A}$ and $\mathbf{B}$, and show that $\mathbf{AB} = \mathbf{BA}$.
   (c) The transformations represented by the matrices $\mathbf{A}$ and $\mathbf{B}$ are applied in succession to the curve of $y = f(x)$. Find the image of the curve $y = f(x)$ and hence write it in terms of the function g.

[4, 2, 4 marks]

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MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2013

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	10th MAY 2013
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Solve the differential equation
$$x(x+2)\frac{\mathrm{d}y}{\mathrm{d}x}+2y=x^2(x+2)^2\cos x^3\,,$$
given that $y=0$ when $x=\sqrt[3]{\pi}$.

[7 marks]

(b) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-10\frac{\mathrm{d}y}{\mathrm{d}x}+34y=34e^{10x}\,,$$
given that $y=-1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=6$ when $x=0$.

[8 marks]

2. The equation of the plane Π_1 is $x-2y+z=4$. The points A, B and C have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $2\mathbf{i}-3\mathbf{j}-\mathbf{k}$ and $5\mathbf{i}-\mathbf{j}$, respectively.

(a) Find the equation of the plane Π_2 that passes through A and B, and is perpendicular to Π_1. Find the equation of the line ℓ_1 where the two planes meet.

[4, 4 marks]

(b) Find the equation of the plane Π_3 that passes through C and is perpendicular to both Π_1 and Π_2.

[3 marks]

(c) Find the distance from C to Π_1 and the volume of the tetrahedron with vertices at the origin and the points A, B and C.

[4 marks]

3. (a) Two fair 6-sided dice are rolled. Each one of the 36 possible outcomes are assumed to be equally likely.
   (i) Find the probability that doubles are rolled.
   (ii) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.
   (iii) Find the probability that at least one die roll is a 6.
   (iv) Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

   (b) John is to enter a special kind of chess tournament, in which he is to play one game with each of *three* opponents. He gets to choose the order in which he plays his opponents, knowing the probability of a win against each opponent. John wins the tournament if he wins *two games in a row*, and he wants to maximize the probability of winning the tournament. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.

[8, 7 marks]

4. (a) Show that the equation $\ln(1+x^3)=3$ has a solution between 2 and 3.
   (i) Use the Newton-Raphson method to find an approximate value of this solution, taking 3 as a first approximation. Do *two* iterations and give your working to *four* decimal places.
   (ii) Solve the equation and compare the answer to the result obtained from the Newton-Raphson method.

   (b) Evaluate the integral $\displaystyle\int_0^{0.4} \ln(1+x^3)\,\mathrm{d}x$
   (i) by Simpson's Rule with an interval width of $h=0.1$, and
   (ii) by integrating the *first two terms* of the series expansion of $\ln(1+x^3)$.

   Give your answers to *four* decimal places.

[8, 7 marks]

5. Consider the curve with equation $y=f(x)$, where $f(x)=\dfrac{(2x-1)(x-3)}{x^2-1}$.
   (a) Show that the curve has no stationary points.
   (b) Show also that the curve has a point of inflexion between $x=0.2$ and $x=0.3$.
   (c) Find the equations of all the linear asymptotes.
   (d) Sketch the graph of the curve, indicating clearly where the curve cuts the coordinate axes and the horizontal asymptote.
   (e) Sketch the curve whose equation is $y=\dfrac{2}{f(x)}$.

[3, 3, 3, 3, 3 marks]

6. (a) The polar equations of the curves C_1 and C_2 are given by

$$r = \sin\left(3\theta + \frac{\pi}{4}\right) \quad \text{and} \quad r = \cos\left(3\theta + \frac{\pi}{4}\right),$$

respectively. Sketch the two curves on the same set of axes for $0 \leq \theta \leq \pi$. Hence, or otherwise:

(i) deduce the values of θ at which the two curves intersect; and

(ii) show that the area which is common to the two curves is equal to $\dfrac{\pi - 2}{8}$ square units.

[5, 2, 4 marks]

(b) Solve the following equation for x:

$$\begin{vmatrix} \cosh x & 1 & \sinh x \\ 1 & \sinh x \cosh x & 1 \\ -\cosh x & 1 & -\sinh x \end{vmatrix} = -4^{x+1}.$$

[4 marks]

7. (a) Given that $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \cos^2 x \, \mathrm{d}x$, show that for $n \geq 2$,

$$I_n = \frac{1}{4(n+1)} \left(2\left(\frac{\pi}{2}\right)^{n+1} - (n+1)n(n-1)I_{n-2} \right).$$

The curve of $y = x^2 \cos x$, for $0 \leq x \leq \frac{\pi}{2}$, is rotated by 2π radians about the x-axis to form a solid. Find the volume of the solid generated.

[7, 4 marks]

(b) Find the points on the curve with equation $4x^2 + 9y^2 = 36$ which are closest to the point with coordinates $(1, 0)$.

[4 marks]

8. Let $\mathbf{A} = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$.

(a) Solve the equation

$$(\mathbf{A} - 3\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{I}$ is the 3×3 identity matrix.

(b) Show that $(\mathbf{A} - \mathbf{I})(\mathbf{A} - 3\mathbf{I}) = \mathbf{0}$. Hence, or otherwise, find $\mathbf{A}^{-1}$.

(c) Find the image of the plane with Cartesian equation $x - y + 2z = 4$ under the transformation $\mathbf{A}$. Comment on your answer.

[5, 5, 5 marks]

9. (a) Let $P(x, y)$ be the point on an Argand diagram representing the complex number $z = x + iy$ and satisfying
$$\operatorname{Im}\left(\frac{z+i}{(1+i)(z-i)}\right) = 0\,.$$
Express the equation of the locus of $P(x, y)$ in Cartesian form and sketch the locus.

[7 marks]

(b) Let $Q(x, y)$ be the point on the same diagram representing the complex number $z = x + iy$ and satisfying the following two relations simultaneously
$$\arg(z+i) - \arg(z-i) = \frac{\pi}{4}\,,$$
$$|z-i| > |z+i|\,.$$
Use your result of (a) or otherwise, to describe the locus of $Q(x, y)$.

[8 marks]

10. Prove the following statements by the principle of mathematical induction.

(a) For every integer $n \geq 1$, the sum of the squares of the first $2n$ positive integers is given by the formula
$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}\,.$$

[6 marks]

(b) Given that
$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{1 + 2x_n} \text{ for every integer } n \geq 1\,,$$
then $x_n < 4$ for all $n \geq 1$.

[4 marks]

(c) If $\sin x \neq 0$ then
$$\cos x \cdot \cos 2x \cdot \cos 4x \cdot \cdots \cdot \cos 2^{n-1} x = \frac{\sin 2^n x}{2^n \sin x}\,,$$
for every integer $n \geq 1$.

[5 marks]