MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
MAY 2014

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	12th MAY 2014
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

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1. (a) Find constants A, B, C and D such that
   (i) $2\cos x + 3\sin x \equiv A(-\sin x + \cos x) + B(\cos x + \sin x)$;
   (ii) $1 \equiv C(x-1) + Dx$.

   (b) Solve the differential equation
$$x(x-1)(2\cos y + 3\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = \cos y + \sin y\,,$$
   given that $y = \dfrac{\pi}{2}$ when $x = 2$.

[3, 7 marks]

2. (a) Find the two possible values of α given that the vectors $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\alpha\mathbf{j}+\mathbf{k}$ make an angle of 60^o.

[4 marks]

   (b) Find the point of intersection of the line ℓ_1 with equation $\mathbf{r}_1 = 4\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k})$ and the line ℓ_2 with equation $\mathbf{r}_2 = -\mathbf{i}-(3\sqrt{6}+1)\mathbf{j}+\mu(\mathbf{i}+\beta\mathbf{j}+\mathbf{k})$, where β is the ***larger*** value of α found in part (a).

[6 marks]

3. (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y = \ln\sqrt{x^2+4}$.

[5 marks]

(b) A curve has equation $xy^3 + 2x^2y = 3$. Find the equation of the normal to the curve at the point $(1,1)$.

[5 marks]

4. (a) Solve the equation

$$2\log_7 x + 3 = 2\log_x 7\,.$$

[4 marks]

(b) The first question on an examination paper is:

Write down the quadratic equation with roots $1/\alpha$ and $1/\beta$, where α and β are non-zero real numbers.

One candidate writes

$$x^2 - \frac{1}{\alpha+\beta}x + \frac{1}{\alpha\beta} = 0$$

as an answer. Show that there are no real values of α and β for which this will give the correct answer.

[6 marks]

5. (a) Let $z_1 = 3+4i$ and $z_2 = 1+2i$, where $i = \sqrt{-1}$. Represent the complex numbers z_1z_2 and $\frac{z_1}{z_2}$ on an Argand diagram. Hence, find the area enclosed by the quadrilateral with vertices z_1z_2, $\frac{\bar{z}_1}{\bar{z}_2}$, $\frac{z_1}{z_2}$ and $\bar{z}_1\bar{z}_2$.

[5 marks]

(b) Solve the equation

$$\frac{\sqrt{5}}{2}\sec\theta - \tan\theta = 2$$

for $0^o \leq \theta \leq 360^o$, giving your answer correct to *one* decimal place.

[5 marks]

6. (a) Use integration by parts to find

$$\int x^2 e^{-3x}\, dx\,.$$

[5 marks]

(b) By using a suitable substitution show that

$$\int_{\pi/6}^{\pi/3} \frac{\sin\theta\cos\theta}{1+\sin\theta}\, d\theta = \frac{1}{2}(\sqrt{3}-1) + \ln\left(\frac{3}{2+\sqrt{3}}\right).$$

[5 marks]

7. (a) The function f is defined on the domain $x \leq 0$ by

$$f : x \mapsto \frac{x^2 - 1}{x^2 + 1}\,.$$

(i) State the range, $\operatorname{ran}(f)$, of f.
(ii) Find an expression for $f^{-1}(x)$ for $x \in \operatorname{ran}(f)$.

[2, 3 marks]

(b) Functions g and h are given by $g(x) = \sqrt{x+2}$ and $h(x) = \ln(1 - x^2)$. Find the composite function $(h \circ g)(x)$ and describe its domain assuming that the domains of g and h are the largest possible.

[2, 3 marks]

8. The table below shows the distribution by colour (green, blue and red) and size (small and large) of a collection of 20 bottles. All the other features of the bottles are exactly the same.

	Green	Blue	Red
Small	4	5	6
Large	2	0	3

These bottles are placed in a random order next to each other on a shelf.

(a) Considering the **small** bottles *only*, in how many different ways can these be placed on the shelf next to each other?
(b) Find the number of all possible different arrangements of *all* the bottles if the 15 small bottles need to be placed next to each other.
(c) What is the probability that on choosing at random one of the possible arrangements described in (b), a small green bottle is next to a large bottle of a different colour?

[2, 3, 5 marks]

9. (a) Find the equation of the circle $\mathscr{C}_1$ with centre at $(-2, -4)$ having the x-axis as one of its tangents.
(b) Show that the equation of the locus $\mathscr{C}_2$ of the point which is equidistant from the line ℓ_1 with equation $y = x + 4$ and the point with coordinates $(-1, -1)$ is given by

$$x^2 + y^2 + 2xy - 4x + 12y - 12 = 0\,.$$

(c) Find the coordinates of the points of intersection of $\mathscr{C}_1$ and $\mathscr{C}_2$, and deduce that they lie on the line ℓ_2 with equation $x = -2$.

[3, 3, 4 marks]

10. (a) Find the sum of the whole numbers that are divisible by 7 and lie between 100 and 200.

[3 marks]

(b) Write the recurring decimal

$$0.158158158\ldots$$

as a sum of a geometric progression. Hence write the recurring decimal as a rational number.

[3 marks]

(c) Given that

$$(1-2x)^5(2+x)^6 \equiv a + bx + cx^2 + dx^3 + \ldots,$$

find the values of the constants a, b, c and d.

[4 marks]

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MATRICULATION CERTIFICATE EXAMINATION
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MAY 2014

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	13th MAY 2014
TIME:	9.00 a.m. to 12.00 noon

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Solve the differential equation
$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r\cot\theta = \cos\theta\,,$$
given that $r = 1$ when $\theta = \dfrac{\pi}{2}$.

[7 marks]

(b) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 12\sin 2x\,,$$
given that $y = -3\pi$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 5$ when $x = \pi$.
[When finding the particular integral, use $Px\cos 2x + Qx\sin 2x$ as a trial solution.]

[8 marks]

2. The points A, B and C have position vectors $4\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $9\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $6\mathbf{i}-\mathbf{j}+2\mathbf{k}$, respectively.
(a) Find the equation of the plane Π_1 that passes through A, B and C.

[3 marks]

(b) The point D has position vector $5\mathbf{i}-3\mathbf{j}+14\mathbf{k}$. Find the distance from D to the plane Π_1, and the volume of the tetrahedron with vertices at A, B, C and D.

[4 marks]

(c) Find the equation of the line ℓ that passes through D and is perpendicular to Π_1. Find the point of intersection of ℓ and Π_1.

[4 marks]

(d) The plane Π_2 passes through A, B and D. Find the angle made by the planes Π_1 and Π_2.

[4 marks]

3. (a) Show that the equation $e^{1/x} = x$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 2 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

[7 marks]

(b) (i) Express the length of the portion of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ as an integral.

(ii) Estimate the integral by Simpson's Rule with an interval width of $h = \dfrac{\pi}{6}$. Give your answers to *four* decimal places.

[8 marks]

4. Let the function f_1 be given by $f_1(x) \equiv \dfrac{x^2+5x+6}{2x^2-11x+5}$, and consider the curve having equation $y = f_1(x)$.

(a) Find the equations of all the linear asymptotes of the curve.

[3 marks]

(b) Determine the coordinates of the stationary points on the curve (giving your answer correct to *four* decimal places).

[3 marks]

(c) Sketch the curve $y = f_1(x)$, indicating clearly where the curve cuts the coordinate axes and the horizontal asymptote.

[3 marks]

(d) State the range of values of $f_1(x)$.

[2 marks]

(e) If the function f_2 is given by $f_2(x) \equiv -\dfrac{1}{2}|x-2|$, deduce graphically that the equation $f_1(x) = f_2(x)$ has no real roots.

[4 marks]

5. The curve $\mathscr{C}_1$ has polar equation $r = 1 + \dfrac{2}{5}\sin 5\theta$.

(a) Find the minimum and maximum values of r and the corresponding values of θ in the range $[-\pi/2, \pi/2]$.

[4 marks]

(b) Hence, or otherwise, sketch the curve $\mathscr{C}_1$ for $0 \le \theta \le 2\pi$.

[4 marks]

The curve $\mathscr{C}_2$ has polar equation $r = \dfrac{3}{5}$, for values of θ between 0 and 2π.

(c) Deduce the polar coordinates of all the points of intersection of $\mathscr{C}_1$ and $\mathscr{C}_2$.

[3 marks]

(d) Find the area enclosed between the two curves.

[4 marks]

6. Let $I_n = \displaystyle\int_0^{\ln 2} (e^x + e^{-x})^n \, \mathrm{d}x$.

   (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(e^x - e^{-x})(e^x + e^{-x})^{n-1}\right] = n(e^x + e^{-x})^n - 4(n-1)(e^x + e^{-x})^{n-2}.$$

   (b) Hence show that

$$nI_n = 4(n-1)I_{n-2} + \frac{3}{2}\left(\frac{5}{2}\right)^{n-1}.$$

   (c) The region bounded by the axes, the line $x = \ln 2$ and the curve $y = (e^x + e^{-x})^2$ is rotated by 2π radians about the x-axis to form a solid of revolution. Find the volume of this solid of revolution, giving your answer correct to *three* decimal places.

[6, 5, 4 marks]

7. (a) Show that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$.
   Hence, or otherwise, show that $\tan\dfrac{5\pi}{12} = 2 + \sqrt{3}$.

[2, 2 marks]

   (b) Let $w = 1 + (2+\sqrt{3})i$, where $i = \sqrt{-1}$.

      (i) Find the modulus and argument of w^n, where n is a positive integer.
      (Give $|w^n|$ in surd form and $\arg w^n$ as a fraction of π.)

[4 marks]

      (ii) Let $P(x, y)$ be the point on an Argand diagram representing the complex number $z = x + iy$ and satisfying

$$\operatorname{Re} z + \operatorname{Im} z = \frac{1}{4}\operatorname{Re} w^4.$$

      Express the equation of the locus of $P(x, y)$ in Cartesian form and sketch the locus.
      Hence, show that the least value of $|z|$ on this locus is $7\sqrt{2} + 4\sqrt{6}$.

[4, 3 marks]

8. (a) Given that $e^y = e^x + e^{-x}$, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 1 = 0.$$

   Find the values of y, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ when $x = 0$. Hence determine the Maclauren series for y in ascending powers of x up to and including the term in x^4.

[8 marks]

   (b) Prove by the principle of mathematical induction that for every integer $n \geq 1$,

      (i) $n(n+1)$ is divisible by 2;

      (ii) $n(n+1)(n+2)$ is divisible by 6.

[7 marks]

9. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 5 & 1 & 7 \end{pmatrix},$$

find $\mathbf{A}^{-1}$. Hence, solve the equation $\mathbf{Ax} = \mathbf{y}$, when $\mathbf{y} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

[5, 2 marks]

(b) Let $\mathbf{T} = \begin{pmatrix} 17 & 9 \\ 9 & -7 \end{pmatrix}$ and $P(a, ka)$ be a point on the line $y = kx$.

(i) If $P'(x', y')$ is the image of $P(a, ka)$ under the transformation $\mathbf{T}$, find the coordinates of P' in terms of a and k.

(ii) $\mathbf{T}$ maps the line $y = kx$ onto another straight line. Using your answer to part (i), find the equation of this straight line.

(iii) Find the Cartesian equation of the straight lines, passing through the origin, that $\mathbf{T}$ maps onto themselves.

[2, 2, 4 marks]

10. (a) Attila and Beatrice play a table-tennis match in which whoever wins two sets wins the match (note that no set can end in a draw). Initially, both players have the same probability of winning the first set, but the probability of winning a subsequent set depends on the result of the preceding set. The probability that Beatrice wins the subsequent set if she has won the preceding set is $3/4$, but her probability of winning the subsequent set decreases to $1/5$ if she has lost the preceding one. By using a tree diagram, or otherwise, determine the probability that:

(i) the match ends after only two sets;

(ii) Attila wins the match.

[3, 4 marks]

(b) (i) Given two events A and B neither of which is impossible, show that

$$P(A) = \frac{P(A|B)}{P(B|A)} P(B).$$

(ii) In a group of girls, some own a smartphone, some own a tablet, some own both, and the rest own neither of the two. A person is chosen at random from the group. The probability that she owns a smartphone if it is known that she owns a tablet is of 52%, while the probability that she owns a tablet if it is known that she owns a smartphone is 26%. Given that 25% of the girls in the group own a tablet, find the percentage of the group who own neither a smartphone nor a tablet.

[2, 6 marks]