MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION CERTIFICATE EXAMINATION
ADVANCED LEVEL
SEPTEMBER 2016

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	I
DATE:	2^{nd} SEPTEMBER 2016
TIME:	09.00 to 12.05

Directions to Candidates

Answer **ALL** *questions.*

Each question carries 10 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. (a) Let $y = \ln\left(\dfrac{1}{1+\cos^2 x}\right)$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.
Show that y satisfies the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (1+3\cos 2x)e^{2y} = 0.$$

[5 marks]

(b) A curve has equation $y^2 + x^3 y + 2x = 4$. Find the equation of the normal to the curve at the point $(-1, 3)$.

[5 marks]

2. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x)e^{y+\sin x},$$

given that $y = 0$ when $x = 0$.

[10 marks]

3. (a) Evaluate the integrals

$$\int_0^{\pi/2} \sin 2\theta \cos\theta \, \mathrm{d}\theta \quad \text{and} \quad \int_3^4 \frac{1}{(x-1)(x-2)} \, \mathrm{d}x\,.$$

[6 marks]

(b) By using the substitution $x = \sin^2\theta$, evaluate

$$\int_0^{1/2} \left(\frac{x}{1-x}\right)^{1/2} \mathrm{d}x\,.$$

[4 marks]

4. The points A and B have coordinates $(\sqrt{2}\,, 1)$ and $(2\sqrt{2}\,, \sqrt{2}-1)$, respectively.
(a) Find the acute angle between the lines OA and OB, where O is the origin. Give your answer correct to $0.1°$.
The straight line ℓ_1 passes through the points A and B.
(b) Find the equation of ℓ_1 in the form $y = f_1(x)$.
A second straight line ℓ_2 has equation $y = f_1\big((1+\sqrt{2}\,)x\big) + \sqrt{2} - 3$.
(c) Write down the equation of ℓ_2 in terms of x and y.
(d) Show that the perpendicular distance from the origin to the line ℓ_1 is $\frac{1}{2}\sqrt{10-\sqrt{2}}$.
[3, 2, 2, 3 marks]

5. (a) Find values of θ for $0 \le \theta \le 180°$ which satisfy the equation

$$\tan\theta \tan 2\theta = 2\,.$$

[5 marks]

(b) Show that the series

$$\log_2 x + \log_4 x + \log_{16} x + \ldots$$

is geometric and find the sum of the series for infinite terms.

[5 marks]

6. The function $f(x) = 3x^2 + 12x + 8$ is defined for all real x.
   (a) Express $f(x)$ in the form $f(x) = a(x-h)^2 + k$ and state the values of a, h and k.
   (b) State the range for $f(x)$.
   (c) Explain why the inverse of $f(x)$ is not a function.
   (d) Restrict the domain of $f(x)$ in such a way that the domain is as large as possible but so that the inverse of $f(x)$ will be a function. State the *new* restricted domain for $f(x)$.
   (e) For $f(x)$ having the domain stated in (d), find $f^{-1}(x)$.

[3, 1, 1, 2, 3 marks]

7. (a) The transformation T maps the point A with coordinates $(1,0)$ to the point A' with coordinates $(c,0)$, and the point B with coordinates $(0,1)$ to the point B' with coordinates $(0,\frac{1}{c})$, where c is a positive constant.
      (i) Write a matrix $\mathbf{T}$ to represent the transformation T.
      (ii) Show that T maps the curve with equation $y = \frac{1}{x}$ onto itself.

   The transformation T is followed by the transformation S represented by the matrix $\mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
      (iii) Write down the matrix representing the compound transformation described above.

[2, 2, 2 marks]

   (b) A group of twelve people are to make a short journey by using three different taxis. Each taxi can hold four people. Two of the people in the group refuse to travel in the same taxi. Find the number of different ways in which the group can be divided to make the journey.

[4 marks]

8. (a) Given that $z = 15 - 8i$, express $\sqrt{z}$ and $\dfrac{1}{\sqrt{z}}$ in the form $a + ib$, where a and b are real numbers.

[5 marks]

   (b) Show that $z = 3 - \sqrt{2}\,i$ is a root of the equation

$$2z^3 - 13z^2 + 28z - 11 = 0.$$

   Find the other roots of the equation.

[5 marks]

9. The points A and B have position vectors $\mathbf{i}-3\mathbf{j}-3\mathbf{k}$ and $4\mathbf{i}-4\mathbf{j}+2\mathbf{k}$, respectively.
   (a) Find the equation of the line ℓ_1 that passes through A and B.
   [2 marks]
   (b) The point C has position vector $2\mathbf{i}-5\mathbf{j}+\alpha\mathbf{k}$. Find α given that $\overrightarrow{BC}$ is orthogonal to ℓ_1.
   [2 marks]
   (c) Find the position vector of the point D which is the midpoint of the line segment $\overrightarrow{AC}$.
   [2 marks]
   (d) Show that the triangle $\triangle ADB$ is an isosceles triangle, and find the $\angle ADB$.
   [4 marks]

10. (a) Factorise $x^3-2x^2-31x-28$.
   [2 marks]
   (b) Find the remainder when x^4+4x^2+3x-7 is divided by $x+3$.
   [2 marks]
   (c) A chord divides a circle, centre O, into two regions whose areas are in the ratio 2 : 1. Show that the angle θ (in radians) subtended by the chord at O satisfies the equation
   $$\theta-\frac{2\pi}{3}-\sin\theta=0.$$
   Show that this equation has a solution between $\theta=2.5$ and $\theta=2.7$. Hence find θ correct to 1 decimal place.
   [6 marks]

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MATRICULATION CERTIFICATE EXAMINATION
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SEPTEMBER 2016

SUBJECT:	PURE MATHEMATICS
PAPER NUMBER:	II
DATE:	3rd SEPTEMBER 2016
TIME:	09.00 to 12.05

Directions to Candidates

Answer **SEVEN** *questions. Each question carries* 15 *marks.*

Graphical calculators are **not** *allowed however scientific calculators can be used but all necessary working must be shown.*

1. Let $\mathbf{M} = \begin{pmatrix} 0 & 2 & a \\ 3 & 1 & -3 \\ 1 & 2a & 1 \end{pmatrix}$.
   (a) Determine the value(s) of a for which the matrix is singular.
   (b) Find the inverse of the matrix $\mathbf{M}$ in the case when $a = 1$.
   (c) Consider the system of equations given by $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ b \\ 2 \end{pmatrix}$, when $a = 3/2$. Find the value of the constant b such that the given system of equations is consistent, and solve the system in this case.
   (d) By using the inverse matrix found in (b), or otherwise, solve the following system of equations:
$$\begin{aligned} 6y + 3z &= 3, \\ 6x + 2y - 6z &= 6, \\ x + 2y + z &= 3. \end{aligned}$$

[3, 4, 5, 3 marks]

2. A curve $\mathscr{C}_1$ is given by the parametric equations $x = t^2 - 2$ and $y = t(t^2 - 2)$.
   (a) Find the Cartesian equation of $\mathscr{C}_1$.
   (b) Show that there are no points on $\mathscr{C}_1$ for which $x < -2$.
   (c) Sketch the curve $\mathscr{C}_2$ given by the equation $y = x^2(x + 2)$.
   (d) Find the coordinates of the points where the two curves $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect.
   (e) Hence, on the same diagram used in (c), sketch the curve $\mathscr{C}_1$.

[2, 2, 3, 4, 4 marks]

3. The point A has position vector $5\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and the line ℓ_1 has vector equation $2\mathbf{i}-8\mathbf{j}-17\mathbf{k}+\lambda(\mathbf{i}-3\mathbf{j}-4\mathbf{k})$.
   (a) Find the equation of the plane Π_1 that passes through the point A and the line ℓ_1.
   (b) Find the points B and C on the line ℓ_1 that are at a distance of 10 units from A.
   (c) The point D has position vector $2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$. Find the equation of the plane Π_2 that contains B, C and D.
   (d) Find the volume of the parallelepiped that has the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ as three of its edges.

[4, 4, 4, 3 marks]

4. (a) Show that the equation $\sqrt{x}\ln(x+1)=1$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do *two* iterations and give your working to *four* decimal places.

[8 marks]

   (b) Evaluate $\displaystyle\int_0^1 \frac{1}{1-x^3+x^4}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h=0.25$. Give your answer to *four* decimal places.

[7 marks]

5. (a) Use De Moivre's Theorem to show that if $z=\cos\vartheta+i\sin\vartheta$ and n is a positive integer then

$$2\cos n\vartheta = z^n+\frac{1}{z^n} \quad \text{and} \quad 2i\sin n\vartheta = z^n-\frac{1}{z^n}.$$

   Hence show that

$$\cos^4\vartheta\sin^2\vartheta = \frac{1}{32}(2+\cos 2\vartheta-2\cos 4\vartheta-\cos 6\vartheta).$$

[7 marks]

   (b) Find the points of intersection of the loci given by the following two equations:

$$|z-1|=2|z+i| \quad \text{and} \quad |z-i|=|z-2+i|.$$

[8 marks]

6. The circle $\mathscr{C}_1$ has equation $x^2+y^2-2x=0$ and the circle $\mathscr{C}_2$ has equation $x^2+y^2+6x=0$.
   (a) Find the polar equations of $\mathscr{C}_1$ and $\mathscr{C}_2$.

[2 marks]

   (b) Sketch the two curves on the same diagram.

[4 marks]

   The curve $\mathscr{C}_3$ has polar equation $r=4-2\cos\theta$ for $0\le\theta\le\pi$.
   (c) Sketch $\mathscr{C}_3$ on the same diagram used in (b).

[3 marks]

   (d) Find the area between the three curves.

[6 marks]

7. (a) Show that if A, B and C are three events, then

$$P[A \cap B \cap C] = P[A]P[B|A]P[C|(A \cap B)] = P[A|(B \cap C)]P[B]P[C|B].$$

[6 marks]

(b) The administrators of a dog sanctuary calculate that 30% of their dogs are brought to the sanctuary within one week from being born. Statistical evidence shows that the probability that these dogs are subsequently adopted before turning one year old is of 0.8, and the probability that such a dog remains with the adoptive family for the rest of its life is of 0.75. Find the probability that a dog from the sanctuary:

(i) Was taken there during its first week of life and was subsequently adopted within one year to remain with the adoptive family for the rest of its life;

(ii) Was either taken to the sanctuary when it was more than one week old, or if it was taken during its first week of life, then it was not adopted before it turned one year old.

[4, 5 marks]

8. (a) The function f is defined by $f(x) = \dfrac{5-x}{(1+x^2)(1-x)}$.

(i) Express $f(x)$ in partial fractions.

(ii) Find the expansion of $f(x)$ as a series of ascending powers of x up to and including the term in x^4.

[3, 4 marks]

(b) Prove the following statements by the principle of mathematical induction.

(i) $\displaystyle\sum_{m=1}^{n} \ln m = \ln(n!)$, for all integers $n \geq 1$.

(ii) $2^{2n-1} + 1$ is divisible by 3, for all integers $n \geq 1$.

[4, 4 marks]

9. (a) Solve the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = (x+1)^4,$$

given that $y = 16$ when $x = 1$, giving your answer in the form $y = f(x)$.

[5 marks]

(b) Solve the differential equation

$$3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 9\cos 2x - 23\sin 2x,$$

given that $y = -2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{5}{3}$ when $x = \dfrac{\pi}{2}$.

[10 marks]

10. (a) The region bounded by the curve $y = x^2 \ln x$ and the x-axis between $x = 1$ and $x = e$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[5 marks]

(b) A funnel has the shape of an inverted cone with a right angle at the vertex. Water is being poured into the funnel at a rate of 5 cm^3/s, and drains out of the funnel at a rate of 3 cm^3/s. Find the rate of increase of the surface area of the water in contact with air when the depth of the water is 8 cm.

[5 marks]

(c) (i) Expand $\left(x^2 + \frac{1}{4x^2}\right)^2$.

(ii) A curve is given by $y = \frac{x^3}{3} + \frac{1}{4x}$. Find the length of the arc of the curve from the point where $x = 1$ to the point where $x = 5$.

[1, 4 marks]