MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2018 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	5th May 2018
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Find general solution of the differential equation

$$(1+y)\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y}\sin x\,.$$

Find also the particular solution for which $y = 0$ when $x = 0$.

[8, 2 marks]

2. (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{(x^2+1)^{\frac{3}{2}}}\,.$$

[4 marks]

(b) Let $y = \cos(\sqrt{x^2+1})$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$x(x^2+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + x^3y = 0\,.$$

[6 marks]

3. (a) The points A and B have position vectors $\mathbf{i}-5\mathbf{k}$ and $4\mathbf{i}+3\mathbf{j}+\mathbf{k}$, respectively. Find the vector equation of the line ℓ_1 that passes through A and B.

[2 marks]

(b) The line ℓ_2 with vector equation $\mathbf{r}=6\mathbf{i}+\mathbf{j}+a\mathbf{k}+\mu(2\mathbf{i}+b\mathbf{j}-\mathbf{k})$ intersects ℓ_1 and is perpendicular to it. Find a and b and the position vector of the point of intersection C of ℓ_1 and ℓ_2.

[5 marks]

(c) Let D be the point with position vector $2\mathbf{i}+5\mathbf{j}-2\mathbf{k}$. Find the angle $\angle ADC$.

[3 marks]

4. (a) Solve for x the equation $3\cdot 2^{x+1}+7-5\cdot 2^{-x}=0$.

[5 marks]

(b) Solve the following simultaneous equations,

$$\log_2 3 = \log_2 5 - \log_2(x-y+1)$$
$$\frac{\ln x}{\ln y} = \frac{\ln 16y - \ln 3}{\ln y} - 2.$$

[5 marks]

5. (a) Consider the function $f(x)=2+\sqrt{x-4}$ which is defined for all real $x\geq 4$.

(i) Find the range of $f(x)$.

(ii) Find an expression for the inverse function $f^{-1}(x)$ and state its domain and range.

[1, 4 marks]

(b) The function $g(x)=\dfrac{5}{x-1}$ is defined for all real $x\neq 1$ and the function $h(x)=\dfrac{4}{3x-2}$ is defined for all real $x\neq\frac{2}{3}$. Find an expression for the composite function $(g\circ h)(x)$ and find its domain (as a composite function).

[3, 2 marks]

6. (a) Solve the inequality $3u^2-4u+1>0$. Hence, find the range of values of $0\leq\theta\leq 360^\circ$ satisfying

$$3\sin^4\theta-4\sin^2\theta+1>0.$$

[4 marks]

(b) If α and β are the roots of the quadratic equation

$$x^2+2x\cos\theta+2\sin^2\theta=0,$$

find (in terms of θ) the quadratic equation which has roots α^2 and β^2. For what values of θ does the equation have real and distinct roots?

[3, 3 marks]

7. (a) Evaluate the definite integral

$$\int_1^2 \frac{2x^5 - x + 3}{x^2}\,\mathrm{d}x\,.$$

[3 marks]

(b) (i) Express $\dfrac{1}{x^3+x}$ into partial fractions.

(ii) Using the substitution $u = \sin\theta$, or otherwise, find

$$\int \frac{\cos\theta}{\sin^3\theta + \sin\theta}\,\mathrm{d}\theta\,.$$

[3, 4 marks]

8. The circle $\mathscr{C}_1$ having centre C with coordinates (p, q) passes through the points A with coordinates $(4, 5)$ and B with coordinates $(4, -3)$.

(a) Find the equation of the locus of the centre C of the circle $\mathscr{C}_1$.

[2 marks]

(b) Write down, in terms of the constant p, the general equation of the circle $\mathscr{C}_1$.

[3 marks]

(c) Find the equation of the circle $\mathscr{C}_2$ which touches the y-axis and has centre A.

[2 marks]

(d) Determine the equations of two circles of the same radius as $\mathscr{C}_2$ which touch the circle $\mathscr{C}_2$ externally and have the locus of C as a tangent.

[3 marks]

9. (a) Find two real numbers a and b satisfying

$$\frac{3+7i}{(2+i)(1-3i)} = a + ib\,.$$

[2 marks]

(b) Show that

$$\frac{2x^3+3x^2+x+1}{2x^2+3x-2} = x + \frac{1}{x+2} + \frac{1}{2x-1}\,.$$

Hence, show that when $2|x| < 1$

$$\frac{2x^3+3x^2+x+1}{2x^2+3x-2} \approx -\frac{1}{2} - \frac{5}{4}x - \frac{31}{8}x^2\,.$$

[4, 4 marks]

10. Anna and Bernard are two of 15 participants taking part in a lottery to win one of five different cash prizes. Any participant can win at most one prize and prizes are given in decreasing order of value.
    (a) In how many ways can the prizes be distributed among the participants?

[2 marks]

    (b) What is the probability that:
        (i) either Anna or Bernard (or both) win one of the prizes?
        (ii) Anna wins a larger prize than Bernard (noting that it is possible that Bernard does not win anything)?

[3, 5 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2018 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	7th May 2018
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the differential equation

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-y),$$

given that $y = 0$ when $x = 0$.

[8 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} = 5e^x - \sin 2x,$$

given that $y = -\frac{1}{10}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{5}$ when $x = 0$.
[When finding the particular integral, use $Axe^x + B\cos 2x + C\sin 2x$ as a trial solution.]

[7 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $e^{\sin x} = 2\cos x$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 0.5 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[7 marks]

(b) (i) Evaluate $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.5$. Give your answer to **four** decimal places.

(ii) Evaluate the integral in part (i) by using the substitution $x = 2\sin t$. Give your answer to **four** decimal places.

[4, 4 marks]

3. (a) (i) Let $I_n = \int \sec^n x \, dx$. Show that

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}.$$

[Hint: Write $\sec^n x$ as $\sec^{n-2} x \sec^2 x$, and integrate by parts using $\frac{d}{dx} \tan x = \sec^2 x$.]

(ii) The region bounded by the curve $y = \sec^3 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[5, 5 marks]

(b) A curve is given parametrically by $x = e^t \cos t$ and $y = e^t \sin t$. Find the length of the arc of the curve from the point where $t = 0$ to the point where $t = \ln 2$.

[5 marks]

4. The function f is given by $f(x) = \frac{(2x+3)(x-4)}{x+1}$.

(a) Find where the curve $y = f(x)$ cuts the coordinate axes.

[2 marks]

(b) Determine the equations of the asymptotes of the curve $y = f(x)$.

[3 marks]

(c) Show that the curve has no stationary points.

[2 marks]

(d) Sketch the curve of $y = f(x)$.

[4 marks]

(e) Let a be a real number. Explain why the equation $f(x+a) = f(x)$ has no real roots for $a = -1$ but has two real and distinct roots for $a = -5$.

[4 marks]

5. The curve $\mathscr{C}$ has polar equation given by $r = \sqrt{3} + \tan\theta$, where $\pi \leq \theta \leq \frac{23\pi}{16}$.

(a) Sketch the curve $\mathscr{C}$.

[5 marks]

The tangent to the curve $\mathscr{C}$ at P is perpendicular to the initial line.

(b) Find the polar coordinates of the point P.

[4 marks]

The point Q is the point of intersection of the curve $\mathscr{C}$ and the half line $\theta = \frac{4\pi}{3}$.

(c) Given that O is the pole, find the area bounded by OP, OQ and the curve $\mathscr{C}$, giving your answer in exact form.

[6 marks]

6. The points A, B and C have position vectors $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $3\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}-7\mathbf{j}-3\mathbf{k}$, respectively.
   (a) Find the equation of the plane Π_1 that passes through A, B and C.
   [4 marks]
   (b) The equation of the plane Π_2 is $x-5y+z=5$. Explain why the planes Π_1 and Π_2 intersect, and find the vector equation of the line of intersection.
   [6 marks]
   (c) Find the equation of the plane Π_3 that is perpendicular to both Π_1 and Π_2, and passes through the point D with position vector $16\mathbf{i}+10\mathbf{j}+6\mathbf{k}$. Find the position vector of the point E where the three planes intersect.
   [5 marks]

7. Let $P(x,y)$ be the point on the Argand diagram representing the complex number $z=x+iy$ and satisfying the equation

$$\operatorname{Re}\left(\frac{z+ia}{z+b}\right)\leq k,$$

where a, b and k are real numbers.
   (a) Show that the locus of $P(x,y)$ is a disc when $k<1$. Find the radius and the coordinates of the centre of this disc in terms of a, b and k.
   Find the maximum value of $|\operatorname{Re} z|$ when $\operatorname{Re}\left(\frac{z+i}{z+1}\right)\leq 0$.
   [12 marks]
   (b) Describe the locus of $P(x,y)$ when $k=1$ and $a,b>0$.
   [3 marks]

8. (a) By direct application of the Maclaurin expansion formula, find the first **three** terms of Maclaurin series for $\ln(1+e^x)$. Hence, or otherwise, determine the value of

$$\lim_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2}.$$

   [8 marks]
   (b) The function f is defined by $f(x)=xe^{-x}$. Prove by the principle of mathematical induction that for every integer $n\geq 1$,

$$f^{(n)}(x)=(-1)^n(x-n)e^{-x},$$

   where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.
   [7 marks]

9. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \qquad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find $\mathbf{A}^{-1}$ and solve for $\mathbf{u}$ and $\mathbf{v}$ the equations $\mathbf{A}\mathbf{u} = \mathbf{a}$ and $\mathbf{A}\mathbf{v} = \mathbf{b}$.

[7 marks]

(b) Let ℓ denote the line passing through the points with position vectors **a** and **b**. Find the Cartesian equation of the line that is transformed by **A** into the line ℓ.

[3 marks]

(c) A line ℓ', lying in the xy-plane and passing through the origin, is such that its image under the transformation **A** is perpendicular to it. Find the vector equation of ℓ'.

[5 marks]

10. The principal of a school decided to assess what proportion of the student population has ever made use of illegal drugs. To find an answer to this question, the principal chose at random an equal number of male and female students. Each student was asked to toss a coin and hide the result from the interviewer. If the result was a **head** they were to answer the question "Are you female?", whereas if the result was a **tail** they were to answer the question "Have you ever made use of illegal drugs?" In total, 35% of the respondents gave a **yes** answer.

(a) Estimate the percentage of students who made use of illegal drugs?

[8 marks]

(b) If three students are chosen at random, what is the probability that:

(i) at least one of them made use of an illegal drug?

(ii) at least two of them made use of an illegal drug?

[3, 4 marks]