MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2019 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	4th May 2019
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y-x}(1+x^2)\sec y\,.$$

Find also the particular solution for which $y=0$ when $x=0$.

[8, 2 marks]

2. (a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x = \sin 2x$.

[2 marks]

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2\ln x)$.

[2 marks]

(c) Let $y=\sin^2(\ln x)$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2\,.$$

[Hint: Use the identity $\sin^2 A = \frac{1}{2}(1-\cos(2A))$.]

[6 marks]

3. (a) The points A and B have position vectors $4\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $-2\mathbf{i}-3\mathbf{j}+3\mathbf{k}$, respectively. Find the position vector of M, the midpoint of the line segment AB.

[2 marks]

(b) The line ℓ_1 with vector equation $\mathbf{r}=a\mathbf{k}+\lambda(b\mathbf{i}+2\mathbf{j}+c\mathbf{k})$ passes through M and is perpendicular to the line segment AB. Find a, b and c.

[4 marks]

(c) The line ℓ_2 has vector equation $\mathbf{r}=2\mathbf{i}+d\mathbf{j}+\mu(e\mathbf{i}+\mathbf{j}+2\mathbf{k})$. Find the positive number e such that the lines ℓ_1 and ℓ_2 make an angle of 60^o with each other. Find d for this value of e given that ℓ_1 and ℓ_2 intersect.

[4 marks]

4. (a) The function $f(x)=\dfrac{1-x}{3x}$ is defined for all real $x\neq 0$ and the function $g(x)=\dfrac{1}{1+3x}$ is defined for all real $x\neq -\frac{1}{3}$.

(i) Find an expression for the composite functions $(f\circ g)(x)$ and $(g\circ f)(x)$, and the domain of each (as composite functions).

(ii) Is $(f\circ g)(x)$ and $(g\circ f)(x)$ the same function? Give reasons for your answer.

[4, 1 marks]

(b) Consider the function $h(x)=\ln(-x-1)$ which is defined for all real x in the interval $-2\leq x<-1$.

(i) Draw a sketch of the graph of h.

(ii) Find an expression for the inverse function $h^{-1}(x)$ and state its domain.

[2, 3 marks]

5. Let $i=\sqrt{-1}$.

(a) Find the values of the real numbers a and b such that $(a+ib)^2=\sqrt{3}i-1$. Hence, or otherwise, solve the quadratic equation

$$4z^2+4z+2-\sqrt{3}i=0,$$

giving your answers in the form $p+qi$, where p and q are real numbers given in surd form.

[5 marks]

(b) If α and β are the roots of the quadratic equation

$$(1+i)z^2-2iz+3+i=0,$$

express each of $\alpha+\beta$ and $\alpha\beta$ in the form $x+iy$, where x and y are real numbers. Find the quadratic equation whose roots are $\alpha+2\beta$ and $2\alpha+\beta$.

[5 marks]

6. (a) Considering the cases $x > 0$ and $x < 0$ separately, solve the inequality

$$1 + 2x - x^2 > \frac{2}{x} \qquad (x \in \mathbb{R},\ x \neq 0).$$

[5 marks]

(b) The second, fourth, and eighth terms of an arithmetic progression are in a geometric progression, and the sum of the third and fifth terms is 20. Find the first four terms of the arithmetic progression.

[5 marks]

7. (a) Using an appropriate substitution, evaluate

$$\int_0^1 (x^2 + x)\cos(x^3 + \frac{3}{2}x^2)\,\mathrm{d}x\,.$$

[5 marks]

(b) Use integration by parts to find the rational values of p and q such that

$$\int_1^9 \sqrt{x}\ln x\ \mathrm{d}x = p\ln 3 + q\,.$$

[5 marks]

8. Four lines are given by the equations:

$$\ell_1:\ y - 2x - 7 = 0$$
$$\ell_2:\ 2y - 4x - 6 = 0$$
$$\ell_3:\ 2y + x + 4 = 0$$
$$\ell_4:\ 6y + 3x = 6$$

(a) Determine which lines are parallel and which are perpendicular.
(b) Find the length of a diagonal D of the rectangle enclosed by the four lines.
(c) Find the equation of the circle $\mathscr{C}$ having D as one of its diameters.

[2, 5, 3 marks]

9. (a) Eight posters are to be distributed in twelve classrooms so that no classroom receives more than one poster. Find the number of different ways how this can be done if:
   (i) the posters are all different;
   (ii) the posters are all the same.

[2, 2 marks]

(b) (i) Show that the number of positive integers with **at most** 4-digits such that all the digits are distinct is

$$9(1+9(1+8(1+7(1)))) .$$

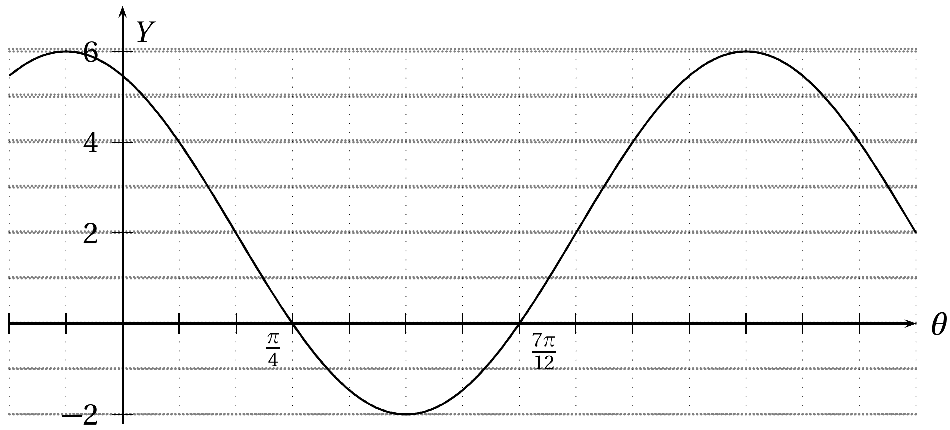
   (ii) Hence, or otherwise, find the number of all possible positive integers with distinct digits.

[3, 3 marks]

10. The diagram shows part of the graph

$$Y = Y_0 + R\cos(\omega\theta + \alpha) \qquad \theta \in \left[-\frac{\pi}{6}, \frac{7\pi}{6}\right],$$

where Y_0, R, ω and α are constants.



(a) Find the values of Y_0, R, ω and α given that $R > 0$ and $\alpha \in (0, \pi)$.

(b) Show that $Y = Y_0 + A\cos(\omega\theta) + B\sin(\omega\theta)$ for some constants A and B. Find A and B and hence find the general solution of the equation

$$Y + \left(2\sqrt{3} + 4\sin\theta\right)\cos\theta = 2 .$$

[4, 6 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2019 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	6^{th} May 2019
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the following first order linear differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x - \cos x = 0.$$
(b) Solve the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = e^{3x},$$
given that $y = \frac{1}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.
[When finding the particular integral, use Cx^2e^{3x} as a trial solution.]

[7, 8 marks]

2. (Note that angles should be taken in radians throughout this question.)
(a) Show that the equation $\sin(x^2+1) = 2e^x - 3$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 0.5 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[7 marks]

(b) (i) Evaluate $\int_0^1 \frac{81}{9+x^5}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to **four** decimal places.
(ii) Write down the series expansion of $\frac{81}{9+x^5}$ up to and including the term in x^{10}. Obtain an approximate value of the integral in part (i) by integrating this series expansion. Give your answer to **four** decimal places.

[4, 4 marks]

3. (a) (i) Show that $\int x^2 \ln x \, \mathrm{d}x = x^3\left(\frac{\ln x}{3} - \frac{1}{9}\right) + C$, where C is a constant.

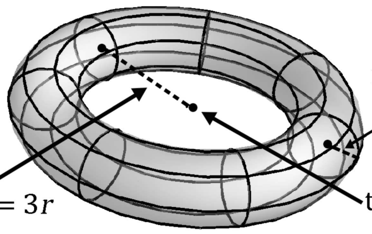
(ii) Let $I_n = \int x^2 (\ln x)^n \, \mathrm{d}x$. Show that

$$I_n = \frac{1}{3}x^3(\ln x)^n - \frac{n}{3}I_{n-1}.$$

(iii) The region bounded by the curve $y = 9x(\ln x)^2$ and the x-axis between $x = e^{-1}$ and $x = 1$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation.

[2, 3, 5 marks]

(b) A tyre/doughnut shaped balloon is being inflated, such that its shape retains the proportions as shown in the diagram.



Air is pumped into the balloon at a rate of 2 cm^3/sec. Find the rate of change of the surface area of the balloon when the inner radius r is 16 cm.
[Note that the balloon with inner radius r has volume given by $6\pi^2 r^3$ and surface area given by $12\pi^2 r^2$.]

[5 marks]

4. The function f is given by $f(x) = \frac{3x^2 - 7x - 6}{x+1}$.

(a) Determine the equations of the asymptotes of the curve $y = f(x)$.

[3 marks]

(b) Find where the curve $y = f(x)$ cuts the coordinate axes.

[2 marks]

(c) Find the stationary points of the curve $y = f(x)$ and determine their nature.

[4 marks]

(d) Sketch the curve of $y = f(x)$.

[3 marks]

(e) Deduce:
(i) the range of values of f;
(ii) the nature of the roots of the equation $f(x) + 4x + 12 = 0$.

[1, 2 marks]

5. (a) (i) Show that

$$\frac{1}{r(r+1)} - \frac{1}{(r+2)(r+3)} \equiv \frac{2(2r+3)}{r(r+1)(r+2)(r+3)}.$$

(ii) Find $S_n = \sum_{r=1}^{n} \frac{2r+3}{r(r+1)(r+2)(r+3)}$ and hence, deduce S_∞.

[2, 5 marks]

(b) The curve $\mathscr{C}$ has polar equation $r = f(\theta)$ for $0 \le \theta \le \pi$, where $f(\theta) = 1 - 2\sin 3\theta$.

(i) Show that $f(\frac{\pi}{2} - t) = f(\frac{\pi}{2} + t)$, and hence deduce the equation of a line of symmetry of $r = f(\theta)$.

(ii) Find the equations of the tangents to the curve at the pole.

(iii) By taking values of θ at intervals of $\frac{\pi}{6}$, sketch the curve $\mathscr{C}$.

[2, 2, 4 marks]

6. The point A has position vector $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, and the line ℓ_1 has a vector equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - 4\mathbf{k} + \lambda(5\mathbf{i} - 8\mathbf{j} + \mathbf{k})$.

(a) Find the equation of the plane Π_1 that contains both the point A and the line ℓ_1.

[4 marks]

(b) Find the equation of the plane Π_2 that passes through ℓ_1 and is perpendicular to Π_1. Find the distance of A from Π_2.

[6 marks]

(c) The plane Π_3 has equation $2x + \alpha y + \beta z = -6$. Find α and β given that Π_3 passes through A and is parallel to ℓ_1. Explain why there is no point that lies on all three of the planes Π_1, Π_2 and Π_3.

[5 marks]

7. In this question $i = \sqrt{-1}$.

(a) Let $f(x) = (k^2-1)x^2 - 2ak^2x + k^2a^2$ where $k \ne \pm 1$. Show, by the method of completing the square, that

$$f(x) = (k^2-1)\big((x-\lambda a)^2 - a^2\lambda(\lambda-1)\big), \quad \text{where } \lambda = \frac{k^2}{(k^2-1)}.$$

(b) Let $P(z)$ be the point on the Argand diagram representing the complex number z and satisfying the equation

$$|z| = k\,|z-w|,$$

where w is a fixed complex number and k is a fixed positive real number not equal to 1. Use part (a) to show that the locus of $P(z)$ is a circle having centre at λw and radius equal to $|w|\sqrt{\lambda(\lambda-1)}$.

(c) Making the substitution $z = u - 2 - 3i$, use part (b) to find the maximum value of $|u|$, given that $u \in \mathbb{C}$ satisfies

$$\sqrt{2}\,|u-2-3i| = \sqrt{3}\,|u+2|.$$

[4, 5, 6 marks]

8. (a) Use the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1).$$

(b) Hence find an expression for $3^2 + 6^2 + \cdots + (3n)^2$.

(c) Given that

$$A_n = 1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2, \quad \text{and}$$
$$B_n = 2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2,$$

show that $A_n + B_n = 6n^3 - n$ and $A_n - B_n = -3n^2$. Hence find A_n and B_n in terms of n.

[6, 2, 7 marks]

9. Let

$$\mathbf{A}_t = \begin{pmatrix} t & 2 & 1 \\ 2 & t+2 & 1 \\ 5 & 8-t & 4 \end{pmatrix} \qquad \mathbf{u}_t = \begin{pmatrix} 1 \\ 2 \\ t \end{pmatrix} \qquad \mathbf{n}_t = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix},$$

where t is a real number.

(a) (i) For which values of t does the equation $\mathbf{A}_t\mathbf{x} = \mathbf{u}_t$ have a unique solution?

(ii) Show that if the equation $\mathbf{A}_t\mathbf{x} = \mathbf{u}_t$ has a solution, then it is unique.

(iii) Find $\mathbf{A}_1^{-1}$.

[9 marks]

(b) Let Π_t denote the plane passing through the point with position vector $\mathbf{u}_t$ and normal to the vector $\mathbf{n}_t$.

(i) Find the Cartesian equation of the plane Π_t.

(ii) Show that $4x+6y+3z+2=0$ is the Cartesian equation of the plane that is transformed into the plane Π_1 by the transformation $\mathbf{A}_1$.

[6 marks]

10. A game is played with three standard six-sided fair dice. The player wins if all the three dice show the same number. The game starts with the player rolling all three dice. After the first roll, the player decides to stop or re-roll one or three dice according to the following strategy:

- if the three dice show the same number, the player stops and wins;
- if exactly two of the dice show the same number, the player rolls the other die;
- if all the dice show a different number, the player rolls all three dice.

Find the probability that:

(a) all three dice show the same number on the first roll;

(b) exactly two dice show the same number on the first roll;

(c) the player wins the game given that on the first roll exactly two of the dice showed the same number;

(d) the player loses the game.

[3, 4, 4, 4 marks]