MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2019 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	2nd September 2019
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Find the general solution of the differential equation

$$(1+\cos 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2(2y+1)\sin 2x\,.$$

Find also the particular solution for which $y=0$ when $x=0$, giving your final answer in terms of $\sec x$.

[8, 2 marks]

2. A curve is given by the implicit equation

$$x^2 - xy + y^2 = 12\,.$$

(a) Find the coordinates of the stationary points on the curve.

(b) Find the value of k if at the stationary points, $(x-2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = k$.

(c) Hence determine the nature of the stationary points.

[4, 4, 2 marks]

3. Two lines ℓ_1 and ℓ_2 are given by $\mathbf{r} = -5\mathbf{i} + \mathbf{j} + 10\mathbf{k} + \lambda(-3\mathbf{i} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 9\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 7\mathbf{k})$, respectively.
   (a) The lines ℓ_1 and ℓ_2 intersect at P, find the coordinates of P.
   [2 marks]
   (b) Show that $Q(5, 2, 5)$ lies on ℓ_2.
   [2 marks]
   (c) Find the coordinates of point M on ℓ_1 such that QM is perpendicular to ℓ_1.
   [4 marks]
   (d) Find the area of the triangle PQM.
   [2 marks]

4. (a) Show that $(x+2)$ is a factor of $4x^3+4x^2-7x+2$ and factorise $4x^3+4x^2-7x+2$ completely.
   [2 marks]
   (b) Let
   $$f(x) = \frac{10-17x+14x^2}{4x^3+4x^2-7x+2}.$$
   (i) Express $f(x)$ into partial fractions.
   (ii) Hence, or otherwise, obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. State the restrictions which must be imposed on x for the expansion in ascending powers of x to be valid.
   [4, 4 marks]

5. The transformation T_1 is an anticlockwise rotation by an angle of θ about the origin, whereas the transformation T_2 is a reflection in the line $y = x\tan\alpha$.
   (a) **Derive** the matrices representing the transformations T_1 and T_2.
   (b) Given that T_1 maps the point $(2, 3)$ to the point $\left(\frac{1}{2}(-3+2\sqrt{3}), \frac{1}{2}(2+3\sqrt{3})\right)$, find the value of θ.
   (c) Given that T_2 maps the point $(2, 3)$ to the point $\left(\frac{1}{2}(2+3\sqrt{3}), \frac{1}{2}(-3+2\sqrt{3})\right)$, find the value of α.
   [6, 2, 2 marks]

6. (a) Sketch the graph
   $$y = \sin\left(\theta - \frac{\pi}{4}\right)$$
   for $0 \le \theta \le 2\pi$, indicating clearly the intercepts that the graph makes with the lines $y = 0$ and $y = -\frac{\sqrt{2}}{2}$.
   [5 marks]
   (b) Solve the inequality
   $$\frac{\sin\theta + 1}{\cos\theta} \le 1,$$
   where $0 \le \theta < 2\pi$ and $\cos\theta \ne 0$.
   [5 marks]

7. (a) Use integration by parts to find

$$\int (3x^2 - x + 1)\sin x \, \mathrm{d}x\,.$$

[4 marks]

(b) (i) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(2^x) = 2^x \ln 2$;

(ii) Using a suitable substitution evaluate

$$\int_0^3 2^x \sqrt{2^x + 1}\, \mathrm{d}x\,.$$

[2, 4 marks]

8. (a) The function $f(x) = \dfrac{6 - 10x}{8x + 7}$ is defined for all real $x \neq -\frac{7}{8}$. Find an expression for $f^{-1}(x)$, giving its domain, where f^{-1} is the inverse function.

[4 marks]

(b) The function $g(x) = \dfrac{3x}{x-1}$ is defined for all real $x \neq 1$ and the function $h(x) = \dfrac{2}{x}$ is defined for all real $x \neq 0$. Find an expression for the composite functions $(h \circ g)(x)$ and $(g \circ h)(x)$ and find their domains (as composite functions).

[6 marks]

9. (a) Consider the set of all 6-digit positive integers. Find the probability that an integer chosen at random from this set begins and ends in an even digit.

[3 marks]

(b) Find the number of 6-digit positive integers that:

(i) contain the digit 7 only once;

(ii) contain the digit 7 only once and the digit 8 only once.

[3, 4 marks]

10. (a) Find the values of p and q if $x^4 + 12x^3 + 46x^2 + px + q$ is the square of a quadratic expression.

[4 marks]

(b) Let a and b be distinct, non-zero real numbers and let c be a real number not equal to 1. Show that the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c \tag{1}$$

has exactly one real solution if $c^2 = -\dfrac{4ab}{(a-b)^2}$. Show further that $c^2 = 1 - \left(\dfrac{a+b}{a-b}\right)^2$, and hence deduce that if equation (1) has exactly one real solution, then $0 < c^2 \leq 1$.

[6 marks]

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2019 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	3rd September 2019
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Solve the following first order linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+(2x+1)y-e^{-x^2}=0.$$

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+4y=8x^2,$$

given that $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=0$.

[7, 8 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Let $f(x)=\sin(2x)+\ln x$, where $x>0$. Show that the equation $f(x)=0$ has a root in the interval $[0.2,1]$. Use the Newton-Raphson method to find an approximate value of this solution, taking 0.6 as a first approximation. Do **two** iterations and give your working to **four** decimal places.

[8 marks]

(b) Estimate $\displaystyle\int_{-1}^{3}\frac{1}{\ln(x+4)}\,\mathrm{d}x$ by (i) Trapezoidal Rule, and (ii) Simpson's Rule, with an interval width of $h=1$, giving your answers to **three** decimal places. If $\displaystyle\int_{-1}^{3}\frac{1}{\ln(x+4)}\,\mathrm{d}x\approx 2.593$, which of the two is the better estimate?

[7 marks]

3. (a) Let $I_n = \displaystyle\int \frac{\mathrm{d}x}{(1+x^2)^n}$.

   (i) Show that $I_{n+1} = \dfrac{1}{2n}\dfrac{x}{(1+x^2)^n} + \dfrac{2n-1}{2n} I_n$.

   (ii) Find $\displaystyle\int \frac{\mathrm{d}x}{(1+x^2)^3}$.

[3, 5 marks]

   (b) The curve C has parametric equations

$$x = t - \ln t\,, \quad y = 4\sqrt{t}\,, \quad \text{where } 1 \le t \le 4\,.$$

   (i) Show that the length of C is $3 + \ln 4$.

   (ii) The curve is rotated by 2π radians about the x-axis. Find the area of the generated surface of revolution.

[2, 5 marks]

4. (a) Solve for $z \in \mathbb{C}$ the equation $z^3 + 2z^2 + 2z + 1 = 0$. Show the roots on an Argand diagram.

[4 marks]

   (b) Let $i = \sqrt{-1}$. Prove that

$$(\cos\vartheta + i\sin\vartheta)(\cos\varphi + i\sin\varphi) = \cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)\,.$$

   Use mathematical induction to show that for every positive integer n,

$$(\cos\vartheta + i\sin\vartheta)^n = \cos n\vartheta + i\sin n\vartheta\,.$$

[3 marks]

   (c) Using the results of parts (a) and (b), find **all** real numbers $x \ge 0$ and y that satisfy the following simultaneous equations

$$x^3\cos 3y + 2x^2\cos 2y + 2x\cos y = -1$$
$$x^3\sin 3y + 2x^2\sin 2y + 2x\sin y = 0\,.$$

[8 marks]

5. (a) Prove by the principle of mathematical induction that $n^2 > 7n + 1$ for every integer $n \ge 8$.

[7 marks]

   (b) (i) Find the first five terms of the sequence given by

$$u_1 = 1 \quad \text{and} \quad u_{n+1} = \frac{2u_n - 1}{3}\,.$$

   (ii) Prove by the principle of mathematical induction that $u_n = 3\left(\dfrac{2}{3}\right)^n - 1$ for every integer $n \ge 1$.

[3, 5 marks]

6. The function f is given by $f(x) = \dfrac{x-2}{2x^2+4x-6}$.
   (a) Find the range of values of f.

   [4 marks]

   (b) Hence, deduce the coordinates and the nature of the stationary points of the curve $y = f(x)$.

   [3 marks]

   (c) Find where the curve $y = f(x)$ cuts the coordinate axes.

   [2 marks]

   (d) Determine the equations of the asymptotes of the curve $y = f(x)$.

   [2 marks]

   (e) Sketch the curve of $y = f(x)$.

   [3 marks]

   (f) Give an estimate (correct to the nearest whole number) of the real root of the equation $f(x) = x - 1$.

   [1 marks]

7. (a) (i) Obtain the Maclaurin's series of the function $(1+e^{-x})^2$, up to and including the term in x^4.
   (ii) Give the coefficient of x^n of this expansion, where n is any positive integer.

   [4, 2 marks]

   (b) The curve $\mathscr{C}$ has Cartesian equation $\left(x^2+y^2\right)^{3/2} = x^2+5y^2$.
   (i) Show that the polar equation of $\mathscr{C}$ is $r = 3-2\cos 2\theta$.
   (ii) Show that $\mathscr{C}$ has **no** tangents at the pole.
   (iii) By taking values of θ at intervals of $\frac{\pi}{6}$, sketch the curve of $\mathscr{C}$ for $0 \leq \theta \leq 2\pi$.
   (iv) Given that the area enclosed by the curve is 11π, determine

$$\int_0^{\pi/2} (3-2\cos 2\theta)^2 \, \mathrm{d}\theta \,.$$

   [3, 1, 3, 2 marks]

8. (a) Calculate the vector product of $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}-\mathbf{k}$.
   (b) Plane Π_1 has normal vector $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and contains point $A(3,4,-2)$. Find the Cartesian equation of the plane.
   (c) Plane Π_2 has equation $3x+y-z=15$. Show that Π_2 contains point A.
   (d) Write down the vector equation of the line of intersection of Π_1 and Π_2.
   (e) A third plane, Π_3, has equation $\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+2\mathbf{k}) = 12$. Find the coordinates of the point of intersection of **all** three planes.
   (f) Find the angle between Π_1 and Π_3 in degrees.

   [2, 2, 1, 3, 4, 3 marks]

9. In this question x, y and z are real numbers and $i = \sqrt{-1}$.
   (a) Show that if u and v are nonzero complex numbers, then

$$\frac{y}{u} + \frac{z}{v} = \left(y \frac{\operatorname{Re} u}{|u|^2} + z \frac{\operatorname{Re} v}{|v|^2} \right) - i \left(y \frac{\operatorname{Im} u}{|u|^2} + z \frac{\operatorname{Im} v}{|v|^2} \right).$$

[3 marks]

   (b) Consider the simultaneous equations

$$x(3+i) + y(1+2ki) - z(3-i) = \frac{-7+11i}{2}, \tag{1}$$

$$\operatorname{Im}\left(\frac{y}{(1+k^2)(3+2i)} + \frac{z}{13(1+ki)} \right) = \frac{-5}{13(1+k^2)}, \tag{2}$$

   where $k \in \mathbb{R}$.
   (i) Show that when $k \notin \left\{ -\frac{4}{3}, \frac{3}{2} \right\}$ these equations have a unique solution for x, y and z.
   (ii) Solve equations (1) and (2) when $k = 2$.

[7, 5 marks]

10. (a) A very important painting in a museum is protected by three independent alarm systems. The probability that any alarm system fails is of 1 in 100. What is the probability that all the three alarm systems fail together?

[2 marks]

   (b) A game is played with balls coloured white and black and a box. We start by placing one black ball and one white ball in the box and then repeatedly do the following: choose a ball at random from the box, check its colour, and put it back in the box together with another ball of the same colour. This is repeated until there are four balls in the box. Show that the number of black balls is equally likely to be any number from 1, 2 and 3.

[6 marks]

   (c) Peter has five coins, four are fair and the other is two-headed. He chooses one of the five coins at random and flips it. Given that the coin shows a head, what is the probability that he chose the two-headed coin?

[7 marks]