MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2021 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	4th October 2021
TIME:	16:00 to 19:05

Directions to Candidates

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Find the general solution of the differential equation

$$y(e^x+1)\mathrm{d}y-(y+2)e^x\mathrm{d}x=0.$$

Find also the particular solution for which $y=0$ when $x=0$.

[10 marks]

2. (a) Let $y=x\sin(2\ln x)$. Show that

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+5y=0.$$

[5 marks]

(b) A curve has equation $y^2-5xy+2x^2+8=0$. Find the equation of the line that is tangent to the curve at the point $(3,2)$.

[5 marks]

3. The points A, B and C have position vectors $4\mathbf{i}+\mathbf{j}+\mathbf{k}$, $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\alpha\mathbf{i}-\mathbf{j}+4\mathbf{k}$, respectively.
   (a) Find α given that the angle $\angle ABC$ is a right angle.
   [4 marks]
   (b) Find the lengths of the sides of the triangle ABC, and verify that they satisfy Pythagoras' theorem.
   [3 marks]
   (c) Find the angle $\angle CAB$.
   [3 marks]

4. (a) Express $3\cos\theta+\sqrt{3}\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R>0$ and $0°\leq\alpha\leq 90°$, stating the exact values of R and α. Hence, determine the greatest and least possible values of

$$\frac{1}{(3\cos\theta+\sqrt{3}\sin\theta)^2+7},$$

   as θ varies.
   [5 marks]
   (b) Find the modulus and argument of z given that

$$\frac{1}{z}+\frac{1}{2(\cos\theta+i\sin\theta)}=\cos\theta\,.$$

   [5 marks]

5. The points A and B have coordinates $(2,3)$ and $(-1,-4)$, respectively.
   (a) Find the midpoint M of the line AB.
   [1 marks]
   (b) Determine the equation of the line ℓ_1 perpendicular to AB and passing through M.
   [2 marks]
   (c) Calculate the acute angle θ between ℓ_1 and the line ℓ_2 having equation $7x+3y-1=0$.
   [2 marks]
   (d) Find the equation of the line which bisects the angle θ.
   [5 marks]

6. (a) Use partial fractions to find

$$\int\frac{x+3}{(x-1)^3}\,\mathrm{d}x\,.$$

   [5 marks]
   (b) Use integration by parts to find the integral

$$\int(x+1)^2\ln 3x\,\mathrm{d}x\,.$$

   [5 marks]

7. (a) If α and β are the roots of the quadratic equation

$$x^2 + \sqrt{2}\cos\theta\, x + \sin^2\theta = 0,$$

find the quadratic equation with roots α/β and β/α.

[5 marks]

(b) Show that the quadratic equation in x

$$(e^{\omega} - 1)x^2 + 3e^{\omega}x + e^{\omega} = 0$$

has real and distinct roots for every real value ω.

[5 marks]

8. (a) The matrices **P** and **Q** are given by

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Find the values of the constants a, b and c given that the matrix **Q** is the inverse of the matrix **P**.

[4 marks]

(b) In a room there are 16 wooden chairs and 10 plastic chairs. Except for the colour, the wooden chairs are identical, and the same holds for the plastic chairs. Of the wooden chairs, 5 are red, 5 are blue and 6 are green. Of the plastic chairs, 4 are red, 2 are blue and 4 are green.

(i) In how many different ways can 9 chairs be chosen from the total number of 26 chairs in the room such that there are 3 of each colour?

(ii) What is the probability that **only** one of the 9 chosen chairs is wooden?

[4, 2 marks]

9. (a) Express $\dfrac{3}{1+x-2x^2}$ into partial fractions. Hence, expand $(1+x-2x^2)^{-1}$ as far as the term in x^4.

[6 marks]

(b) Solve the following simultaneous equations:

$$5^{x+2} + 7^{y+1} = 3468$$
$$7^y = 5^x - 76.$$

[4 marks]

10. (a) The real functions f and g are defined by

$$f(x) = 3x + k \quad \text{and} \quad g(x) = \frac{x-4}{3}.$$

For what value of k is $(f \circ g)(x) = (g \circ f)(x)$.

[4 marks]

(b) A function f is defined by $f(x) = 3 - \sqrt{\frac{x+2}{5}}$ for real values of $x \geq -2$.

(i) Find the range of f.

(ii) Define the inverse function f^{-1} stating its domain and range.

[2, 4 marks]

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L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2021 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	5th October 2021
TIME:	16:00 to 19:05

Directions to Candidates

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the first order linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+3y=2x-1,$$

given that $y=3$ when $x=0$. Give your answer in the form $y=f(x)$.

[8 marks]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+y=4e^{-x},$$

given that $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=0$.
[When finding the particular integral, use Ax^2e^{-x} as a trial solution.]

[7 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $\cos e^x = x$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 0 as a first approximation. Do **TWO** iterations and give your working to four decimal places.

[8 marks]

(b) Evaluate $\int_0^1 1-\sin x^2\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h=0.25$. Give your answer to four decimal places.

[7 marks]

3. Let T be the linear transformation on $\mathbb{R}^3$ mapping the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ into $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $2\mathbf{i}+\mathbf{j}$ and $\mathbf{i}$, respectively.
   (a) Write down the matrix $\mathbf{T}$ that allows for the linear transformation T to be expressed in matrix form as:

$$T : \mathbf{x} \mapsto \mathbf{Tx}.$$

[3 marks]

   (b) Show that

$$\mathbf{T}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

   Hence, or otherwise, find the image of the plane $x+y+z=0$ under the transformation T.

[6 marks]

   (c) Evaluate $\mathbf{T}^2$ and show that $\mathbf{T}^2+\mathbf{T}^{-1}=4\mathbf{T}+2\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. Deduce that $\mathbf{T}^3-4\mathbf{T}^2-2\mathbf{T}+\mathbf{I}=\mathbf{0}$ and use this equation to find $\mathbf{T}^3$.

[6 marks]

4. (a) (i) Let $I_n = \displaystyle\int x^n(e^x-e^{-x})\,\mathrm{d}x$. Show that

$$I_n = x^n(e^x+e^{-x}) - nx^{n-1}(e^x-e^{-x}) + n(n-1)I_{n-2}.$$

   (ii) The region bounded by the curve $y = x^2(e^x-e^{-x})^{1/2}$ and the x-axis between $x=1$ and $x=2$ is rotated through 2π radians about the x-axis. Show that the volume of the solid generated by this rotation is $\pi(8e^2-9e-65e^{-1}+168e^{-2})$.

[3, 5 marks]

   (b) (i) Evaluate $\left(\frac{1}{2}e^{x/2}+\frac{1}{2}e^{-x/2}\right)^2$.

   (ii) The portion of the curve given by $y=e^{x/2}+e^{-x/2}$ between $x=-1$ and $x=1$ is rotated through 2π radians about the x-axis. Find the area of the surface of revolution generated by this rotation.

[2, 5 marks]

5. The curve $\mathscr{C}_1$ is given by the equation $r=3+2\cos\theta$, and the curve $\mathscr{C}_2$ is given by the equation $r=2$.
   (a) On the same diagram, sketch the **TWO** curves $\mathscr{C}_1$ and $\mathscr{C}_2$ for $0\le\theta\le2\pi$.

[6 marks]

   (b) Find the polar coordinates of the points of intersection of the two curves.

[2 marks]

   (c) Determine the total area enclosed by the two curves.

[7 marks]

6. (a) Find the complex numbers z satisfying simultaneously the equations

$$|z+i| = 2|z| \qquad \text{and} \qquad |3z-1-2i| = 3|z-1|.$$

[8 marks]

(b) (i) Show that for $n \in \mathbb{Z}$

$$\cos\frac{2n\pi}{3} = \begin{cases} 1, & \text{if 3 divides } n \\ -\frac{1}{2}, & \text{otherwise.} \end{cases}$$

(ii) Hence, or otherwise, prove that for any integer n the sum

$$(-1+i\sqrt{3})^n + (-1-i\sqrt{3})^n$$

is either equal to 2^{n+1} or -2^n.

[5, 2 marks]

7. The function f is given by $f(x) = \dfrac{(x-1)(2x-1)}{x^2-4}$.

(a) Determine the equations of the asymptotes of the curve $y = f(x)$.

[3 marks]

(b) Find where the curve $y = f(x)$ cuts the coordinate axes.

[2 marks]

(c) Find the stationary points of the curve $y = f(x)$ and determine their nature.

[4 marks]

(d) Sketch the curve of $y = f(x)$.

[4 marks]

(e) Hence, deduce that the equation $f(x) = -1-x$ has **TWO** negative real roots and **ONE** positive real root. Mark these roots on your graph.

[2 marks]

8. The planes Π_1 and Π_2 have equations $x-y-z=1$ and $3x-y+2z=8$, respectively.

(a) Find the equation of the plane Π_3 that is perpendicular to both Π_1 and Π_2, and passes through the point A with position vector $-\mathbf{i}+\mathbf{j}-\mathbf{k}$.

[5 marks]

(b) Find the point B which lies on both Π_2 and Π_3, and has y-coordinate equal to 6. Find the vector equation of the line ℓ where Π_2 and Π_3 meet.

[6 marks]

(c) Hence, or otherwise, find the position vector of the point C where the three planes meet.

[4 marks]

9. (a) Use the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1\,.$$

[7 marks]

(b) If $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, use the principle of mathematical induction to prove that

$$\mathbf{M}^n = \begin{pmatrix} 2n+1 & -n \\ 4n & 1-2n \end{pmatrix},$$

for any positive integer n.

[8 marks]

10. (a) (i) Show that $(r+2)! + (r+1)! - 2(r!) = r!(r^2 + 4r + 1)$.

(ii) Hence, determine $\sum_{r=1}^{n} r!(r^2 + 4r + 1)$.

(iii) Evaluate $\sum_{r=1}^{100} \left(\frac{100r!(r^2 + 4r + 1) + 4}{100 \cdot 101!} \right)$.

[2, 4, 2 marks]

(b) Let $a > b$, such that powers above the third of $\frac{b}{a}$ can be neglected.

(i) Show that

$$\frac{1}{\sqrt{a+b}} - \frac{1}{\sqrt{a-b}} = -\frac{1}{\sqrt{a}} \left(\frac{b}{a} + k \frac{b^3}{a^3} \right),$$

where k is a constant to be determined.

(ii) Hence, find the values of p and q in the approximation of $\sqrt{5} - \sqrt{3}$ given by $\frac{p\sqrt{15}}{q}$.

$\left[\text{Hint: Consider } \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{3}}.\right]$

[5, 2 marks]