MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

**ADVANCED MATRICULATION LEVEL**
**2022 SECOND SESSION**

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|---|---|
| SUBJECT: | **Pure Mathematics** |
| PAPER NUMBER: | I |
| DATE: | $29^{\text{th}}$ August 2022 |
| TIME: | 09:00 to 12:05 |

**Directions to Candidates**

Answer **ALL** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. Solve the differential equation
$$\cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = e^{-2y}, \qquad \text{for } 0 \le x < \frac{\pi}{2},$$
given that $y = \dfrac{1}{2}$ when $x = \dfrac{\pi}{4}$. Give your answer in the form $y = f(x)$.

**[10 marks]**

2. (a) Let $y = e^{-\frac{1}{x^2}}$. Show that
$$x^4 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

**[3 marks]**

(b) A curve is given by the following parametric equations
$$x = 2\tan t,$$
$$y = 5\cos^2 t,$$
where $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in terms of $t$.

(ii) Find the equation of the line that is normal to the curve at the point where $t = \frac{\pi}{4}$.

**[7 marks]**

3. The points $A$ and $B$ have position vectors $\mathbf{i}+2\mathbf{j}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$, respectively. The line $\ell_1$ has vector equation
$$\mathbf{r}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{k}).$$
   (a) Find the point $P$ on $\ell_1$ that is equidistant from both $A$ and $B$.
   **[4 marks]**
   (b) Find the distance from $A$ to $P$.
   **[2 marks]**
   (c) Find the angles of the triangle $APB$.
   **[4 marks]**

4. (a) In a triangle $ABC$, $|AC|=\sqrt{3}|BC|$ and $2\angle A=\angle B$.
   (i) Show that triangle $ABC$ is right-angled.
   (ii) Find the area of this triangle when $|BC|=20$.
   **[5 marks]**
   (b) The complex number $w$ is given by
$$w=\frac{3+\sqrt{2}i}{(\sqrt{3}+i)(1-\sqrt{2}i)}.$$
   Show that $4\sqrt{2}\,\mathrm{Re}\,w-\mathrm{Im}\,w=11/4$.
   **[5 marks]**

5. (a) A circle $\mathscr{C}_1$ has centre at the point $C$ with coordinates $(3,1)$ and passes through the point $A$ with coordinates $(6,5)$.
   (i) Find the radius of the circle $\mathscr{C}_1$ and deduce its equation.
   (ii) Find the equation of the tangent to the circle at $A$.
   **[5 marks]**
   (b) Another circle $\mathscr{C}_2$ is orthogonal to $\mathscr{C}_1$ at $A$. The distance between the centres of the two circles is 13. If $\mathscr{C}_2$ has its centre $D$ in the second quadrant, find the coordinates of $D$.
   **[5 marks]**

6. (a) Use integration by parts to find the integral $\displaystyle\int_0^1 xe^x\,\mathrm{d}x$.
   **[3 marks]**
   (b) Let $I=\displaystyle\int_0^1 \frac{e^x}{1+x}\,\mathrm{d}x$.
   (i) Show that $\dfrac{x^2}{1+x}=x-1+\dfrac{1}{1+x}$ and hence, evaluate $\displaystyle\int_0^1 \frac{x^2e^x}{1+x}\,\mathrm{d}x$ in terms of $e$ and $I$.
   (ii) Use the substitution $x^2=u+1$ to evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{e^{x^2}}{x}\,\mathrm{d}x$ in terms of $e$ and $I$.
   **[7 marks]**

7. (a) The cubic equation $x^3 - kx^2 + 3k = 0$, where $k > 0$, has roots $\alpha$, $\beta$ and $\alpha + \beta$.
   (i) Show that $3\alpha\beta(\alpha+\beta) = 2k\alpha\beta + 3k$ and $\alpha\beta(\alpha+\beta) = -3k$.
   (ii) Given that $4\alpha\beta = -k^2$, find the value of $k$.

**[5 marks]**

(b) (i) Show that $x - 2$ is a factor of $x^3 - x^2 - 4$. Deduce that $x = 2$ is the only real root of the equation $x^3 - x^2 - 4 = 0$.
   (ii) In a geometric sequence with real and non-zero terms, the sum of the seventh term and four times the fifth term equals the eighth term. Does the sequence have a sum to infinity? Explain your answer.

**[5 marks]**

8. (a) A box contains 4 white discs and 12 black disks. The disks are taken out one by one at random from the box and placed on top of each other. How many different arrangements are possible such that the bottom 3 disks have alternating colours?

**[5 marks]**

(b) Find the image of the point with coordinates $(p, q)$ after it undergoes a clockwise rotation by an angle of $\frac{\pi}{6}$ radians about the origin, followed by a reflection in the line $y = x$.

**[5 marks]**

9. (a) Resolve into partial fractions $f(x)$, where

$$f(x) = \frac{3x^2 - 2x + 2}{(x+2)(x-1)^2}.$$

Show that

$$f(x) \approx 1 + \frac{x}{2} + \frac{9x^2}{4},$$

when $|x| < 1$.

**[5 marks]**

(b) Consider the expansion of $(4 + x^2)^{n+1}$, where $n$ is a positive integer. Find the value of $n$ given that the coefficient of $x^4$ is $7 \times 4^n$.

**[5 marks]**

10. (a) Consider the functions $f(x) = \sqrt{x-2}$, for $x \geq 2$, and $g(x) = x^2 + x$. Find the composite function $(f \circ g)(x)$, giving its domain.

**[5 marks]**

(b) Consider the real function $f(x) = \dfrac{3x-5}{x-k}$, for $x \neq k$.
   (i) Find the inverse function $f^{-1}(x)$, giving its domain.
   (ii) Find the value of the constant $k$ so that $f(x) = f^{-1}(x)$.

**[5 marks]**

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

**ADVANCED MATRICULATION LEVEL**
**2022 SECOND SESSION**

| | |
|---|---|
| SUBJECT: | **Pure Mathematics** |
| PAPER NUMBER: | II |
| DATE: | 30th August 2022 |
| TIME: | 09:00 to 12:05 |

**Directions to Candidates**

Answer **SEVEN** questions. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) Solve the first order linear differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}-2xy=\frac{x^4}{x-3},$$

given that $y=16$ when $x=4$. Give your answer in the form $y=f(x)$.

**[7 marks]**

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+8y=32x^2,$$

given that $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=0$. Give your answer in the form $y=f(x)$.

**[8 marks]**

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $e^{\cos x}=1+2\ln(x+1)$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do **two** iterations and give your working to four decimal places.

**[8 marks]**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} e^{\cos x}\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h=\dfrac{\pi}{8}$. Give your answer to four decimal places.

**[7 marks]**

3. Consider the following system of linear equations in $x$, $y$ and $z$, where $a$ and $b$ are constants.

$$\begin{array}{rcrcrcl} -2x & + & y & + & z & = & 7 \\ ax & + & 2y & - & z & = & 2 \\ -3x & + & y & + & 2z & = & b \end{array}$$

(a) Show that the system of equations has a unique solution exactly when $a \neq -1$.

**[4 marks]**

(b) Giving your answer in terms of $b$, solve the system of equations when $a = 3$.

**[6 marks]**

(c) Assume that $a = -1$. Find the value of $b$ for which the system has a solution. Solve the system of equations for this value of $b$.

**[5 marks]**

4. (Note that angles should be taken in radians throughout this question.)

(a) (i) Use the substitution $x = \sin\theta$ to show that

$$\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{6} - \frac{\sqrt{3}}{8}.$$

[Hint: You may need to use the identity $2\cos^2\theta = 1+\cos 2\theta$.]

(ii) Use the substitution $x = e^{-t}$ to evaluate

$$\int_{0}^{\ln 2} e^{-t}\sqrt{1-e^{-2t}}\,\mathrm{d}t\,.$$

**[7 marks]**

(b) Show that $\displaystyle\int_0^1 \frac{1+x}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4} + \frac{\ln 2}{2}$.

**[4 marks]**

(c) A curve is given parametrically by $x = 5t^6$ and $y = 15t^4$. Find the length of the arc of the curve from the point where $t = 0$ to the point where $t = 2$.
[Hint: You may need to use the substitution $u = t^4 + 4$.]

**[4 marks]**

5. (a) On the same diagram, sketch the curves

(i) $\mathscr{C}_1$ given by the polar equation $r = 3 + 3\cos\theta$, and

(ii) $\mathscr{C}_2$ given by the polar equation $r = -\frac{6\theta}{\pi}$,

for $0 \leq \theta \leq \pi$.

**[6 marks]**

Hence, determine the area enclosed by the two curves.

**[5 marks]**

(b) Write down the first three non-zero terms of the expansion of $\ln(1+2x) - 2xe^x$, stating the range of values of $x$ for which the series converges.

**[4 marks]**

6. (a) Find the modulus and the argument of $\left(\dfrac{1-i}{1+i}\right)^3$. Hence, solve the equation

$$(z-2)^3 - \left(\frac{1-i}{1+i}\right)^3 = 0\,.$$

**[8 marks]**

(b) Find the greatest and least value of:
   (i) $|z_1 + z_2|$, given that $|z_1| = 6$ and $z_2 = 3 + 4i$;
   (ii) $\operatorname{Re} z$, given that $|z - 3 - 4i| \leq 1$.

**[7 marks]**

7. The function $f$ is given by $f(x) = \dfrac{x^2 - 4}{2x - 5}$.

(a) Determine the equations of the asymptotes of the curve $y = f(x)$.

**[3 marks]**

(b) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature.

**[4 marks]**

(c) Find where the curve $y = f(x)$ cuts the coordinate axes.

**[2 marks]**

(d) On the same diagram, sketch the curves of $y = f(x)$ and $y = \left|f(x)\right|$.

**[4 marks]**

(e) Hence, deduce the number of real roots of the equation $|f(x)| = \frac{1}{4}(2x + 5)$.

**[2 marks]**

8. The points $A$, $B$ and $C$ have position vectors $\mathbf{i}+7\mathbf{j}+4\mathbf{k}$, $-\mathbf{i}+\mathbf{j}+6\mathbf{k}$ and $-4\mathbf{i}+3\mathbf{j}+20\mathbf{k}$, respectively.

(a) Find the area of the triangle $ABC$.

**[4 marks]**

(b) Find the equation of the plane $\Pi_1$ passing through $A$, $B$ and $C$.

**[3 marks]**

(c) Let $\ell_1$ be the line that passes through $A$ and $B$. Find the vector equation of the line $\ell_2$ that lies in $\Pi_1$, is perpendicular to $\ell_1$ and passes through $C$.

**[3 marks]**

(d) Find the point of intersection of $\ell_1$ and $\ell_2$. Does it lie between $A$ and $B$?

**[5 marks]**

9. (a) Use the principle of mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{(a+i-1)(a+i)} = \frac{n}{a(a+n)},$$

where $a$ is a positive integer.

**[8 marks]**

(b) Prove by the principle of mathematical induction that $4^n + 6n - 1$ is divisible by 9 for every integer $n \geq 1$.

**[7 marks]**

10. (a) Write down the first three non-zero terms of the Maclaurin's series of $\ln(1 + \sin x)$. Hence, use this expansion to find an approximate value of $\ln 2$ in terms of $\pi$.

**[7 marks]**

(b) (i) Express $\dfrac{5x+11}{(x+1)(x+2)(x+3)}$ into partial fractions.

(ii) Determine $S_n = \displaystyle\sum_{r=0}^{n} \frac{5r+11}{(r+1)(r+2)(r+3)}$.

(iii) Deduce $S_\infty$.

**[8 marks]**