MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

L-Università ta' Malta

ADVANCED MATRICULATION LEVEL
2024 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	4th May 2024
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL TEN** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Express $\dfrac{1}{y^2-4}$ into partial fractions. **[3 marks]**

   (b) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2 + 3y^2 - 8x - 12,$$

   given that $y = -1$ when $x = 0$. Give your final answer in the form $y = f(x)$. **[7 marks]**

2. (a) Let $y = \ln\left(\dfrac{1}{x^2+1}\right)$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

[5 marks]

   (b) A curve is given by the parametric equations

$$x = \cos t + 5\cos 2t,$$
$$y = \sin t - 5\sin 2t,$$

   where t takes values between 0 and 2π.

   (i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in terms of t.

   (ii) Find the equation of the line that is tangent to the curve at the point where $t = \frac{\pi}{2}$.

[3, 2 marks]

3. Consider the two functions $f(x) = x^2 + 4x$ and $g(x) = 3x - 1$, both defined for all $x \in \mathbb{R}$.

   (a) Determine the composite function $(f \circ g)(x)$ giving its domain and range. **[3, 1, 2 marks]**

   (b) Find the values of k for which the equation $(f \circ g)(x) = k$ has real solutions. **[4 marks]**

4. (a) Use the substitution $u^2 = 8 - x$, to find the integrals

$$\int \sqrt{8-x}\,\mathrm{d}x \qquad \text{and} \qquad \int (8-x)^{3/2}\,\mathrm{d}x\,.$$

[2, 2 marks]

(b) Hence show, using integration by parts, that

$$\int (x-2)\sqrt{8-x}\,\mathrm{d}x = -\frac{2}{5}(8-x)^{3/2}(x+2)+c\,,$$

where c is a constant. **[4 marks]**

(c) Evaluate the integral

$$\int_4^7 (x-2)\sqrt{8-x}\,\mathrm{d}x\,.$$

[2 marks]

5. The points A and B have position vectors $\mathbf{i}+6\mathbf{j}-5\mathbf{k}$ and $5\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, respectively.

(a) Find the equation of the line ℓ_1 that passes through A and B. **[3 marks]**

(b) Let C be the point with position vector $\alpha\mathbf{i}+6\mathbf{j}-3\mathbf{k}$. Find α given that the line ℓ_2 passing through C and the midpoint of the line segment AB is perpendicular to ℓ_1. **[4 marks]**

(c) Find the area of the triangle ABC. **[3 marks]**

6. (a) If $f(t) = t^3 - 3t^2 + t + 1$ show that $t = 1$ is a root of $f(t)$. **[1 marks]**

(b) Show that if $\tan^2\theta = 2\tan\theta + 1$ then $\tan 2\theta = -1$. **[2 marks]**

(c) Prove that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\,.$$

[3 marks]

(d) Hence find all solutions of the equation

$$\tan\theta = 2 + \tan 3\theta$$

which satisfy $0 < \theta < 2\pi$. **[4 marks]**

7. In Morse code, a sequence of dashes and dots is used to represent a symbol. For example, the letter A is represented by "·—" and the number 1 is represented by "· ————". How many different symbols can be represented:

(a) by sequences of four or fewer dashes and dots? **[4 marks]**

(b) by sequences of five dashes and dots if the sequence needs to have at least one pair of dots next to each other but cannot have three or more dots next to each other? **[6 marks]**

8. (a) Solve the simultaneous equations

$$\log_y x = 2,$$
$$5y = x + 12\log_x y\,.$$

[3 marks]

(b) (i) Factorize $1-x+x^2-x^3$ and resolve into partial fractions the function $f(x)$ given by

$$f(x) = \frac{1}{1-x+x^2-x^3}\,.$$

[4 marks]

(ii) Expand $f(x)$ in ascending powers of x, giving the first four terms. State the necessary restrictions on the values of x. **[3 marks]**

9. (a) Find the range of values of k for which

$$(k-1)u^2 + (2k-3)u + k \geq 0$$

for every $u \in \mathbb{R}$. **[3 marks]**

(b) Let

$$f(x) = kx - \frac{x^3}{1+x^2}\,.$$

Show that if $f'(x) \geq 0$ for every $x \in \mathbb{R}$ then $k \geq 9/8$. **[7 marks]**

10. Two points A and B have coordinates $(-1,2)$ and $(3,-1)$, respectively.

(a) Find the coordinates of the midpoint D of the line AB. **[2 marks]**

(b) Find the equation of the circle $\mathscr{C}_1$ with centre D and diameter AB, writing it in the form $ax^2 + by^2 + cx + dy + e = 0$, for integers a, b, c, d and e. **[4 marks]**

(c) The circle $\mathscr{C}_1$ is reflected in the line $y = -x$ to obtain a circle $\mathscr{C}_2$.

(i) Write down the 2×2 matrix representing this linear transformation.

(ii) Find the equation of $\mathscr{C}_2$, writing it in the same form as that of $\mathscr{C}_1$.

[2, 2 marks]

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ADVANCED MATRICULATION LEVEL
2024 FIRST SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	6^{th} May 2024
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions from **TEN**. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Find the particular solution to the differential equation

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} - y \sin x = 2\cos^2 x\,,$$

given that $y = 1$ when $x = 0$. **[7 marks]**

(b) Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 3e^{2x}\,,$$

given that $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$. Give your answer in the form $y = f(x)$.
[When finding the particular integral, use ax^2e^{2x} as a trial solution.]

[8 marks]

2. (a) Let $f(x) = \ln\left(\dfrac{1-2x}{1+x}\right)$, where $-1 < x < \frac{1}{2}$.

(i) Express $\dfrac{3}{(2x-1)(x+1)}$ into partial fractions. Hence, or otherwise, obtain the first four non-zero terms in the Maclaurin's series for $f(x)$.

(ii) By taking $x = \frac{1}{5}$, use the expansion to find an approximate value of $\ln 2$.

[7, 2 marks]

(b) The curve $\mathscr{C}$ has Cartesian equation given by $(x+y)^2 = \left(x^2+y^2\right)^{\frac{3}{2}}$.

(i) Write the equation of $\mathscr{C}$ in polar form.

(ii) Sketch the curve $\mathscr{C}$ for $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$.

[3, 3 marks]

3. (a) (i) Let $I_n = \int \sin^n x \, \mathrm{d}x$. Show that, for $n \geq 2$,

$$I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}\,.$$

(ii) Show further that $\int \cos^2 x \sin^n x \, \mathrm{d}x = I_n - I_{n+2}$. **[6, 1 marks]**

(b) (i) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x$ for $n = 2, 4$ and 6.

(ii) The region bounded by the curve $y = \cos x \sin^2 x$ and the x-axis between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation. **[5, 3 marks]**

4. (a) How many 4-digit numbers contain the digit 7 exactly once? (Note that the first digit cannot be zero.) **[5 marks]**

(b) A *palindrome* is a finite sequence of characters that reads the same backwards and forwards (for example, "1234321"). How many different numbers of 7-digit palindromes can be formed if no digit can appear more than twice in the number? (Note that the first digit cannot be zero.) **[5 marks]**

(c) A binary sequence of length n is a sequence of n elements where each element is either 0 or 1 (for example, 01101 and 10110 are two binary sequences of length 5).

(i) How many binary sequences of length 7 can be formed?

(ii) What is the probability that one of these sequences chosen at random has more 1's than 0's?

[3, 2 marks]

5. The line ℓ_1 is given by the vector equation

$$\mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

and the plane Π_1 is given by the equation $2x - y + z = 6$.

(a) Find the point of intersection A of Π_1 and ℓ_1. Find the equation of the line ℓ_2 that lies on the plane Π_1, passes through A, and is perpendicular to ℓ_1. **[6 marks]**

(b) Find the equation of the plane Π_2 that contains ℓ_1 and is perpendicular to Π_1. **[3 marks]**

(c) Find the equation of the line ℓ_3 where Π_1 and Π_2 intersect. Find also the angle between ℓ_1 and ℓ_3. **[6 marks]**

6. Let $w = \dfrac{10 - i\,2\sqrt{3}}{1 - i\,3\sqrt{3}}$, where $i = \sqrt{-1}$.

   (a) Express w in the form $a + i\,b$, where a and b are real. **[2 marks]**

   (b) Given that $w = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$, obtain the values for r and θ. Hence, solve the equation

   $$z^3 = \frac{40 - i\,8\sqrt{3}}{1 - i\,3\sqrt{3}},$$

   giving your answers in the form $z = x + i\,y$, where x and y are real and rounded up to two decimal places. **[6 marks]**

   (c) Sketch on an Argand diagram the locus of the points representing the complex number z satisfying $|z - w| = 2|z + 1 - \sqrt{3}i|$. Find the maximum value of $|z|$. **[7 marks]**

7. The function f is given by $f(x) = \dfrac{2x^2 - 5x - 3}{x^2 - x - 2}$.

   (a) Find where the curve $y = f(x)$ cuts the coordinate axes. **[2 marks]**

   (b) Show that the curve $y = f(x)$ does **not** have any stationary points. **[4 marks]**

   (c) Determine the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. **[3 marks]**

   (d) Find where the curve cuts the line representing the horizontal asymptote. **[2 marks]**

   (e) Sketch the curve of $y = f(x)$. **[4 marks]**

8. (a) Show that the equation $\ln(1 + e^{-x}) = 3\cos x - 1$ has a solution between 1 and 2. Use the Newton-Raphson method to find an approximate value of this solution, taking 1 as a first approximation. Do TWO iterations and give your working to four decimal places.
   [Note that angles should be taken in radians in this question.] **[7 marks]**

   (b) (i) Evaluate the integral $\displaystyle\int_1^2 \ln(1 + e^{-x})\,\mathrm{d}x$ by Simpson's Rule with an interval width of $h = 0.25$. Give your answer to four decimal places.

   (ii) By using the series expansion of $\ln(1 + t)$ and taking $t = e^{-x}$, show that $\ln(1 + e^{-x})$ may be approximated by the following expression

   $$e^{-x} - \frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x} - \frac{1}{4}e^{-4x} + \frac{1}{5}e^{-5x},$$

   for $x > 0$.
   Use this expression to find another estimate of the integral evaluated in part (i). **[4, 4 marks]**

9. (a) Prove by the principle of mathematical induction that for $n \geq 1$,

$$(1+x)^n \geq 1+nx\,,$$

where x is any real number greater than -1; i.e. $(1+x) > 0$. **[6 marks]**

(b) (i) Find A, B, C if

$$\frac{r-1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}\,.$$

(ii) Hence show by the method of differences that

$$\sum_{r=1}^{n} \frac{r-1}{(r+1)(r+2)(r+3)} = \frac{n(n-a)}{6(n+b)(n+c)}\,,$$

where the positive integers a, b, c are to be determined. **[3, 6 marks]**

10. Let $\mathbf{A}$ denote the following matrix:

$$\begin{pmatrix} 3 & 1 & -3 \\ 1 & 2k & 1 \\ 0 & 2 & k \end{pmatrix}.$$

(a) Find the values of k for which $\mathbf{A}$ is singular. **[3 marks]**

(b) Solve the equation

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3\frac{1}{2} \\ 5\frac{1}{2} \\ 5 \end{pmatrix}, \qquad (\star)$$

in the case $k = 2$. **[6 marks]**

(c) For which of the values found in (a) does equation $(\star)$ have a solution? Solve equation $(\star)$ for this value of k. **[2, 4 marks]**